PRINCIPLES OF
LECTRICAL ENGINEERING
SERIES

ectric Circuits

ST COURSE IN CIRCUIT ANALYSIS
'OR ELECTRICAL ENGINEERS

By Members of the Staff of the
)epartment of Electrical Engineering
Massachusetts Institute of Technology

A PUBLICATION OF
THE TECHNOLOGY PRESS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

JOHN WILEY & SONS, INC.
NEW YORK
LONDON CHAPMAN & HALL, LTD.

SEVENTH PRINTING, JUNE, 1948

PRINTED IN THE UNITED STATES OF AMER

Table of Symbols

For space vectors, a bold-face italic or script letter is used to represent the vector, and an ordinary italic or script letter is used to represent merely the magnitude of the vector, for example: $\boldsymbol{T}$, T, or $\boldsymbol{\mathcal{E}}$, $\mathcal{E}$. For complex quantities, a Roman letter is used to represent the complex quantity, and an italic letter is used to represent merely the magnitude of the complex quantity, for example: E, E. Ordinary italic or script letters represent ordinary scalars, for example: t, m. For voltage, current, charge, and time, capital letters generally represent fixed quantities, and lower-case letters generally represent variable quantities, for example: E, e. To some extent this system of capital and lower-case letters is carried out for other quantities. It should be recognized that the number of symbols required in a treatment of this sort is so large that it is neither feasible to be entirely consistent, nor possible to avoid duplications.

The symbols used for quantities are very often given special meanings by the addition of subscripts or superscripts. In such cases, the discussion in the text defines the meaning intended, and no attempt is made to include all these special designations in the list of symbols. However, certain subscripts appear in many places with the same meaning, as for example, Z_{11}; such symbols are listed in this table.

Abbreviations used in the text are, in general, in accordance with the *American Tentative Standard Abbreviations for Scientific and Engineering Terms*, approved by the American Standards Association in 1932, wherein such abbreviations are listed.

English Letter Symbols

Symbol			Description	Defined or First Used Page
	$[A]$		Acceleration [dimensional]	746
A	A		Coefficient in transient solution	411, 176
A	A		General circuit constant	454
	a		Transformer turns ratio	386
	a		Coefficient in transient solution	236
	a		Integrodifferential operator	371
	a	$\boldsymbol{a}$	Acceleration	746
B	B		Coefficient in transient solution	210, 186
B	B		General circuit constant	454

Symbol			Description	Defined or First Used Page
	B		Susceptance	287, 290
	$\mathfrak{B}$	$\boldsymbol{\mathfrak{B}}$	Magnetic flux density	4
	b		Number of branches	126
C	C		General circuit constant	454
	C		Capacitance	20
	C_{aa}		Self-capacitance of node a	392
	C_{ab}		Mutual capacitance of nodes a and b	393
	C_{eq}		Capacitance of equivalent electric circuit	632
	C_M		Mechanical capacitance	612
	C_t		Time-varying capacitance; compliance	662
	C_u		Capacitance of a unit length	40
	c		Magnitude of the velocity of propagation of electromagnetic waves in free space when expressed in cm/sec	749
D	D		General circuit constant	454
D	D		Determinant	434, 141
	$\mathfrak{D}$	$\boldsymbol{\mathfrak{D}}$	Electric flux density; displacement	5
	E		Constant source voltage	4
E	E		Effective value of alternating source voltage	280
E_m	E_m		Amplitude of alternating source voltage	257
	E_y		Effective value of balanced 3-phase line-to-neutral source voltage	529
	E_Δ		Effective value of balanced 3-phase line-to-line source voltage	549
E_0	E_0		Effective value of zero-sequence component of source voltage	585
E_1	E_1		Effective value of positive-sequence component of source voltage	588
E_2	E_2		Effective value of negative-sequence component of source voltage	588
	$\mathcal{E}$	$\boldsymbol{\mathcal{E}}$	Electric field intensity	4
	e		Instantaneous source voltage	167
	$[F]$		Force [dimensional]	746
F	F		Magnetomotive force	22
	f		Frequency of a periodic function	13
	$\mathfrak{f}$		Force	611
	G		Conductance, reciprocal of resistance	94
	G		Conductance, real part of complex admittance	290
	G_{aa}		Self-conductance, node a	128, 392
	G_{ab}		Mutual conductance, nodes a and b	128, 393
	G_{eq}		Conductance of equivalent electric circuit	632
	G_u		Conductance of a unit length	40

	Symbol		Description	Defined or First Used Page
	g_{11}		Short-circuit self-conductance	143
	g_{12}		Short-circuit transfer conductance	143
	$\mathcal{H}$	$\mathcal{H}$	Magnetic-field intensity	6
	I		Constant current	19
I	I		Effective value of alternating current	278, 280
	I_L		Effective value of line current, balanced 3-phase system	555
I_m	I_m		Amplitude of alternating current	261
	I_s		Continuous steady-state component of current	170
	I_y		Effective value of phase current, balanced Y-connected 3-phase system	549
	I_Δ		Effective value of phase current, balanced Δ-connected 3-phase system	555
I_0	I_0		Effective value of zero-sequence component of current	585
I_1	I_1		Effective value of positive-sequence component of current	588
I_2	I_2		Effective value of negative-sequence component of current	588
	$\mathcal{J}$		Imaginary part of	206
	i		Instantaneous current	5
	i_s		Instantaneous steady-state component of alternating current	261
	i_t		Instantaneous transient component of current	171
	$\imath$		Current per unit length	36
	$\mathcal{J}$	$\mathcal{J}$	Current density	4
	j		$\sqrt{-1}$	192
	K		Dielectric constant; relative permittivity	103
	K		Constant of proportionality between parameters of equivalent mechanical and electrical systems	631, 638
	k		Coefficient of coupling	386
	$[L]$		Length [dimensional]	747
	L		Self-inductance	6
	L'		Fictitious inductance corresponding to linear $\varphi(i)$ relation	698
	L_{av}		Average inductance	700
	L_M		Mechanical inductance	611
	L_t		Time-varying inductance	662
	L_u		Self-inductance of a unit length	41
	L_w		Self-inductance of winding	631
	L_{11}		Self-inductance of loop 1	369
	L_{12}		Inductance common to loops 1 and 2	370

Symbol	Description	Defined or First Used Page
ℓ $\boldsymbol{\ell}$	Length	4
ℓ	Number of loops	126
$[M]$	Mass [dimensional]	746
M	Mutual inductance	72
M_{jk} M_{jk}	Cofactor of jth row and kth column of a determinant	434, 141
m	Number of equipotential surface intervals	27
$\boldsymbol{m}$	Mass	746
N	Number of turns	6
n	Number of nodes	126
n	Number of flow lines	27
P	Constant power	135
P_{av}	Average active power	275, 301
P_M	Average power absorbed in mechanical system	639
$\mathcal{P}$	Permeance	61
p p	Exponential coefficient in transient solution	191, 186
p	Instantaneous power	111
Q	Quality of a coil, $\omega L/R$	321
Q	Reactive volt-amperes	296
Q	Fixed electrostatic charge	20
Q Q	Effective value of alternating charge	280
Q_m Q_m	Amplitude of alternating charge	261
Q_s	Constant steady-state component of charge	182
q	Instantaneous charge	5
q_s	Instantaneous steady-state component of alternating charge	261
q_t	Instantaneous transient component of charge	183
$\mathfrak{q}$	Electrostatic charge per unit length	43
R	Resistance	5
R	Static resistance	666
R_{ab}	Resistance of branch a–b	123
R_{av}	Fixed component of time-varying resistance	664
R_{eq}	Resistance of equivalent electric circuit	633
R_M	Mechanical resistance	612
R_m	Amplitude of variation of time-varying resistance	664
R_t	Time-varying resistance	662
R_u	Resistance of a unit length	41
R_w	Resistance of winding	631
R_{11}	Self-resistance of loop 1	127
R_{12}	Mutual resistance, loops 1 and 2	127
$\mathcal{R}$	Reluctance	61
$\mathcal{R}e$	Real part of	204

	Symbol		*Description*	*Defined or First Used Page*
	r		Dynamic resistance	666
	r_{aa}		Open-circuit self-resistance	144
	r_{ab}		Open-circuit transfer resistance	144
	r_p		Dynamic plate resistance	669
	S		Elastance	5
	S_{eq}		Elastance of equivalent electric circuit	633
	S_M		Mechanical elastance	637
	S_t		Time-varying elastance	662
	S_{11}		Self-elastance of loop 1	369
	S_{12}		Mutual elastance, loops 1 and 2	370
	s		Number of source branches	126
	s	$\mathbf{s}$	Area	4
	$[T]$		Time [dimensional]	747
	T	$\mathbf{T}$	Torque	283, 636
	T		Period	204
	T		Time constant	175
	T'		Time constant of fictitious linear circuit	698
	t		Time	3
	t		Temperature	75
$\mathbf{U}$	U		Difference of magnetic potential	61
	$\mathfrak{U}$		Magnetic potential function	22
	V		Constant voltage; difference of electric potential	19
$\mathbf{V}$	V		Effective value of alternating voltage	278, 280
$\mathbf{V}_m$	V_m		Amplitude of alternating voltage	268
	V_s		Constant steady-state component of voltage	247
	V_y		Effective value of balanced 3-phase line-to-neutral voltage	549
	V_Δ		Effective value of balanced 3-phase line-to-line voltage	534
$\mathbf{V}_0$	V_0		Zero-sequence component of system of 3 complex numbers	580
$\mathbf{V}_0$	V_0		Effective value of zero-sequence component of voltage	585
$\mathbf{V}_1$	V_1		Positive-sequence component of system of 3 complex numbers	581
$\mathbf{V}_1$	V_1		Effective value of positive-sequence component of voltage	590
$\mathbf{V}_2$	V_2		Negative-sequence component of system of 3 complex numbers	581
$\mathbf{V}_2$	V_2		Effective value of negative-sequence component of voltage	590
	$\mathfrak{V}$		Electric potential function	17
	v		Instantaneous voltage	5

Symbol		Description	Defined or First Used Page
	v $\mathbf{v}$	Velocity	13
	W	Reciprocal mutual inductance	109
	W	*Work; energy*	*180*
	W_e	Energy stored in electric field	618
	W_M	Mechanical work	622
	W_m	Energy stored in magnetic field	22
	X	Reactance	264
	X_C	Elastive or capacitive reactance	264
	X_L	Inductive reactance	264
$\mathbf{Y}$	Y	Admittance	287, 289
$\mathbf{Y}_{eq}$	Y_{eq}	Admittance of equivalent electric circuit	633
$\mathbf{Y}_{aa}$	Y_{aa}	Self-admittance, node a	393
$\mathbf{Y}_{ab}$	Y_{ab}	Mutual admittance, nodes a and b	393
$\mathbf{Y}_{1a}$	Y_{1a}	Apparent input admittance	394
$\mathbf{y}_{11}$	y_{11}	Short-circuit input admittance of two-terminal pair	449
$\mathbf{y}_{12}$	y_{12}	Short-circuit transfer admittance of two-terminal pair	449
$\mathbf{Z}$	Z	Impedance	263
$\mathbf{Z}_{ab}$	Z_{ab}	Impedance of branch a b	293
$\mathbf{Z}_{eq}$	Z_{eq}	Impedance of equivalent electric circuit	633
$\mathbf{Z}_q$	Z_q	Charge impedance	263
$\mathbf{Z}_w$	Z_w	Impedance of winding	639
$\mathbf{Z}_y$	Z_y	Line-to-neutral impedance, balanced 3-phase system	538
$\mathbf{Z}_\Delta$	Z_Δ	Phase impedance of balanced Δ-connected 3-phase system	538
$\mathbf{Z}^0$	Z^0	Impedance to zero-sequence component of current	586
$\mathbf{Z}^+$	Z^+	Impedance to positive-sequence component of current	598
$\mathbf{Z}^-$	Z^-	Impedance to negative-sequence component of current	598
$\mathbf{Z}_0$	Z_0	Zero-sequence component of impedance	590
$\mathbf{Z}_1$	Z_1	Positive-sequence component of impedance	590
$\mathbf{Z}_2$	Z_2	Negative-sequence component of impedance	590
$\mathbf{Z}_{11}$	Z_{11}	Self-impedance, loop 1	373
$\mathbf{Z}_{12}$	Z_{12}	Mutual impedance, loops 1 and 2	373
$\mathbf{Z}_{1a}$	Z_{1a}	Apparent or input impedance	379
$\mathbf{z}_{11}$	z_{11}	Open-circuit input impedance of two-terminal pair	451
$\mathbf{z}_{12}$	z_{12}	Open-circuit transfer impedance of two-terminal pair	451

Greek Letter Symbols

Symbol	Description	Page
α Alpha	Damping constant	224
α	Temperature coefficient of resistance	75

Symbol		*Description*	*Defined or First Used Page*
β	Beta	Supplementary damping constant for over-damped circuit	224
Γ	Gamma	Reciprocal self-inductance	109
Γ_{aa}		Self-reciprocal inductance of node a	392
Γ_{ab}		Mutual reciprocal inductance, nodes a and b	393
Γ_{eq}		Reciprocal self-inductance of equivalent electric circuit	632
γ		Conductivity	3
γ		Angle of $-\alpha + j\omega_d$ in transient solution	228
δ	Delta	Angle of B in transient solution	210
δ		$(\omega - \omega_0)/\omega$ in resonant circuit solution	320
ϵ	Epsilon	Naperian base of logarithms (2.71828...)	168
ε		Permittivity	3
ε_0		Permittivity of free space	104
θ	Theta	Angle between voltage and current vectors	263, 291
θ		Angular deflection	636
Θ_m θ_m		Amplitude of angular oscillation	639
θ_s		Constant steady-state component of angular deflection	643
λ	Lambda	Wavelength	13
λ		Flux linkages	55
λ_u		Flux linkages per unit length	55
λ_s		Constant steady-state component of flux linkages	247
μ	Mu	Permeability	3
μ_0		Permeability of free space	61
ν_0	Nu	Reluctivity of free space	61
v		Volume	22
π	Pi	Ratio of circumference to diameter of circle (3.14159...)	5
ρ	Rho	Volume charge density	29
ρ		Resistivity	87
σ	Sigma	Surface charge density	20
ϕ	Phi	Phase angle of current	261
ϕ		Constant magnetic flux	22
Φ_m ϕ_m		Amplitude of alternating magnetic flux	700
ϕ_r		Residual magnetic flux	700
ϕ_s		Constant steady-state component of magnetic flux	700
ϕ_t		Coefficient in transient solution	700
ϕ_u		Magnetic flux per unit length	45
φ		Instantaneous magnetic flux	624
φ_{max}		Maximum instantaneous magnetic flux	701
φ_s		Instantaneous steady-state component of alternating flux	700
φ_t		Transient component of magnetic flux	700

Symbol	Description	Defined or First Used Page
ψ Psi	Phase angle of voltage	258
ψ	Electric flux	20
ω Omega	Angular frequency	257
ω_d	Angular frequency of damped oscillations	225
ω_0	Undamped or resonant angular frequency	192, 313

Other Symbols

$\bar{A}$	Conjugate of complex number A	258
$\sim$	Cycles per second	293
$\angle$	At an angle of	199
$\measuredangle_A^B$	The angle measured from A to B	549
$\equiv$	Identically equal to; equal by definition	94
$\approx$	Approximately equal to	51
$\propto$	Varies as	321
▶ ◀	Marks used for emphasis	3

Introduction

This is an age which has seen electricity become universally significant for the conveniences it has added to man's everyday practical life. These are today's outward expression of the accomplishments of a great body of men of the past who have made possible and who symbolize the field of electrical engineering.

Today, by the mere closing of a switch, a humble dwelling may have light, a factory may operate as if by daylight, a baseball diamond may be floodlighted for night games, or a great structure may be caused to take on a pageantry of color and shadow against the dark background of night. These things could not be were electric light not available as a utilitarian servant and an aesthetic medium. Today, the human voice may be electrically transported almost instantly to the extremities of the earth's surface and into its surrounding atmosphere farther than man has dared to venture. Where wires cannot be strung or buried as guides, the signal is carried by radio waves to complete the link. The electric light, the telegraph, the telephone, the electric motor with its vast applications, the radio, and the electric cars and trains are commonplace in our everyday life.

Equally important are other applications which, although not so obvious, nevertheless, are deeply insinuated into our existence today. For instance, the copper electroplates without which the printing industry would be greatly handicapped are made through use of the phenomenon of electrolysis. In electrometallurgy, the recovery of aluminum and magnesium are instances of the application of this same basic phenomenon, which has other wide applications, including electroplating. Electroplating may be used to protect metal surfaces, to fabricate thin sheets of metal, to embellish objects, or as a part of the process of transforming into a rugged metal matrix to be used in the stamping of phonograph records, the delicate sound undulations first engraved, also electrically, into a wax master.

In traction, the train is led along its way by an electric guide, the signalling system. It is stopped automatically by an independent electric monitor should a danger signal be passed. The transport airplane carries its passengers cross-country in bee-lines set by radio ranges. A ship may be given its position by radio bearings. The quality or forms of a product may be watched over by an electric eye. A printing press may be controlled so that margins are exact and color plates in precise register. In the textile mills a loom is stopped by electrical means when a warp thread breaks. Alternating-current power systems have their frequencies so

precisely controlled that as a network they in effect are a rigid time shaft to which electric clocks may be coupled simply by plugging into a receptacle. In geophysical studies, strata beneath the earth's surface are examined indirectly through their reflection of elastic waves detected by electrical means. Mathematical formulations of physical phenomena, too complex for manual solution, are handled by electrical machines. Mechanical problems are set up in the form of their electrical analogues for study. Application of the X-ray in industry may point out a flaw in a casting, or at the service station may indicate a potential puncture by showing on a fluorescent screen the shadow of an imbedded object. The X-ray in medicine is a proven instrument for diagnosis and treatment. High-frequency currents are applied in diathermy and electrosection. Amplifiers are used to magnify the delicate voltages appearing about the scalp as a result of brain action and also the voltages between the extremities caused by cardiac impulses that these signals may be recorded for diagnostic purposes.

This recitation of accomplishments of applied electricity might be extended much further. Manifestly, however, the student who plans to prepare himself for the profession of electrical engineering needs something more than a narrative cataloguing of the end results. More important to him is the understanding that electrical engineering as a profession is concerned fundamentally with the translation of energy, perhaps that stored in a central coal pile or a distant elevated body of water, into a more convenient form electricity; with the transmission and control of the energy in this more convenient form; and finally with the retranslation of this convenient electrical energy into other forms for use.

The history of electrical engineering makes clear that the art is built on a set of underlying principles which in their essence are not many. These ideas, which are physically represented in electrical machines, electric circuits, power-transmission and telephone lines, and appliances, express in compact form what we know about the process of converting chemical, mechanical, thermal, and radiant energy into electrical energy, and what we know about using electrical energy as such or reconverting it to one of the first four, or into other, forms for special application. The readiest approach to a survey of this material is a glance first at the devices by which primary conversions of energy from one form to another are made and then at devices useful in molding or controlling energy while it is in the convenient electrical form.

The primary battery is one of the oldest means of translating energy from chemical to electrical form. This it does by releasing the energy stored in the molecules of a chemical substance or substances, as in the pocket flashlight which carries its potential light in the form of a compact package of chemicals. In electroplating, conversely, electric energy is

translated into chemical energy — the attendant chemical reaction is utilized to deposit metals from the electrolyte. The secondary or storage battery is a more general example of electrochemical conversion, for electrical energy may be fed into it as a reservoir, to be stored in chemical form, and then at will released, to the accompaniment of an inverse chemical action, for use in electrical form. The translation of mechanical energy into electrical energy is usually performed by rotating mechanico-electrical machines termed generators, and the translation — commonly by identical machines — of electrical energy into mechanical energy is effected by rotating electromechanical machines termed motors. Unfortunately there happens to be no efficient means for converting thermal energy or heat directly into electrical energy. The thermocouple makes such a conversion, but owing to its low conversion efficiency it is utilized directly only as a means of measuring temperature and indirectly as a device to indicate radiant energy. Hence, to obtain electrical energy from coal, fuel oil, or gas — rich sources of thermal energy — the translation is a roundabout process. The energy, after it is converted into thermal, must then be changed into mechanical form, in order that the rotating generator may serve to convert it finally into electrical form. Radiant energy when in the visible or near-visible region of the spectrum may be translated directly into electrical energy by means of what are called photoelectric cells.* In some uses, these devices supply a current unaided and in others they control the current of a local battery.

The incandescent lamp is an important means of translating electric into radiant energy visible to man. In it, actually, energy is translated into both thermal and radiant forms, the thermal energy being, of course, an undesired concomitant in the process. The filament is heated hotter and hotter — that is, its constituent molecules are given more and more kinetic energy by the passage of more and more electric current — and at the same time, as the temperature of the filament rises, relatively more and more energy is radiated. The electric arc is an effective translator of electric energy into both thermal and radiant form and it has been used, therefore, both as a luminary and as a heater. Arc furnaces are examples of this last application. Another good practical translator of electric energy into thermal energy is a metallic conductor which has been given a comparatively high resistance and good antioxidizing properties by alloying. The flatiron is an application of this form of heater; another is the resistance furnace.

Such, briefly sketched, are the translations which, when utilized in combination with the idea and techniques of electrical transmission, are

* A typical application of these cells is to be found in certain types of photographic exposure-time indicators.

employed to make natural energy available for light, heat, and electrical and mechanical power at a distant useful point of application. It is desirable also to consider briefly a second class of manipulations of energy in electrical form — manipulations used either to mold a stream of that energy for special work or to control it for ease and effectiveness in transmission and general application. As will later appear, the same basic principles guide the electrical engineer in handling problems of this second as of the first class of transformations or manipulations.

Clearest example of the molding of a stream of electrical energy to perform a special task is the translating of sound waves into facsimile electrical waves or the transformation of light manifestations into facsimile electrical manifestations for the transmission, respectively, of sounds and pictures. Sound waves, longitudinal elastic waves, are made to mold electric waves into an identical pattern by means of a microphone. The sound wave may set a diaphragm into a forced mechanical vibration, which may, in turn, be caused to change the electrical resistance of a capsule of carbon granules and thus to modulate or control the current of a battery, as an electrical source. Practically every telephone contains a carbon-granule microphone or " transmitter." Conversely, electric energy may be translated into sound energy when waves of electric current are caused to set up vibrations in a mechanical system coupled to the air, and so to transfer the energy of the mechanical system to energy of vibration of the air particles by means of a diaphragm alone, or a diaphragm in the throat of a horn. An everyday device for making this translation is the ordinary telephone receiver, in which an electric-current facsimile of the speech wave at the input to the microphone varies the magnetic force acting on a diaphragm and thus sets the diaphragm into vibration. The vibrating diaphragm in turn now sets up the corresponding sound waves, which convey the original speech intelligence to the telephone listener. The radio loud-speaker is, of course, another example of this type of device, in which translation is carried out on a larger scale and in many instances on a more nearly exact basis.

Substitution of a translator which transforms light manifestations into electric manifestations and another which performs the inverse operation makes it possible to mold electrical energy to another kind of pattern, and so to use the electrical transmission link for transmission of pictures. The picture to be transmitted may be scanned by a small beam of light so that point by point each minute area of its surface passes an amount of light governed by the optical properties of the area. Impinging on the active surface of a photoelectric cell, the varying light intensity may be made to produce a correspondingly varying electric current. This current, or a current proportional to it for each and every point on the negative, is transmitted. At the receiving end it is caused to control the

intensity of a light source producing a correspondingly small beam, which, by tracing over a light-sensitive emulsion surface in a manner corresponding exactly to the point-by-point scanning process at the transmitting station, produces an image of the original picture. In television the same broad idea of translation is effected, though by rapid scanning many complete images are transmitted in one second to produce the illusion of continuous motion just as in the motion picture.

The engineering principles which find expression in these two manners of molding electrical energy for specific use are fundamentally those employed by the engineer in bringing about the primary transformations of energy earlier discussed. Similarly, they are utilized in the theory, design, construction, and operation of the various devices serving to control electricity. The transformer, a typical illustration of such devices, comprises in its simplest form two magnetically coupled coils by means of which an alternating voltage applied to one coil may be caused to appear in the other in multiplied form. Electricity of high voltage and low current may be changed to low voltage and high current by this apparatus. Even a transformer having a multiplication factor of unity is often useful, for example, in isolating a circuit to direct current — a procedure common in communications. Another representative device is the rectifier, which the engineer uses to transform alternating current into direct current, and which acts in effect as a valve to pass current in one direction only. Rectifiers may be mechanical, as in synchronous commutators; electromagnetic machines such as rotary converters; or static devices depending for their action on the behavior of electrons or ions or both, as in the mercury-arc rectifier and the two-electrode thermionic tube respectively. They are used to translate alternating-current energy of relatively large magnitude into direct-current energy for electric-railway system operation, for instance, and on the other extreme to furnish from the alternating-current power-supply circuit the small direct-current energy needed by the radio set. On an even smaller scale they perform on delicate radio waves a kind of rectification known as detection.

Direct current may, conversely, be translated into alternating current by means of an oscillator — a third characteristic appliance used in controlling electrical energy — an electron tube acting as an amplifier source. An electric amplifier, in essence, is a device which not only recreates a controlling signal of given energy but also endows this recreated signal with greater energy. The ordinary radio set vacuum-tube amplifier is an example. The same form of vacuum tube is utilized in the telephone repeater (amplifier) to rejuvenate the telephone signal. Through the use of an amplifier tube feeding energy into a tuned circuit, the energy fed to this circuit may be controlled by means of a signal taken from the circuit

and used to control the amplifier. If this energy supplies the losses in the tuned circuit, an "oscillator" results. The source of the energy is, of course, the battery or other direct-current source controlled or "valved" by the amplifier tube.

From this summary of characteristic machines and appliances used in converting energy to electrical form, in molding the electricity to specific purposes, and in controlling it for useful application, one may secure an idea of the complexity of the profession of electrical engineering in the present day. Emphasis has been laid on the fact that these three aspects of the work of the profession are unified by a central core of underlying scientific theory and principles, applicable with suitable change to the handling of the special problems engendered in each of the three branches of the work. It becomes desirable, therefore, that this discussion turn from summarizing present-day equipment and applications, and survey in brief compass the historical development of the profession, for the ideas of control and transmission of electric energy expressed tangibly in these devices have come to a high degree of refinement only through the continued repetition of discovery, analysis of the discovery, and synthesis. The electrostatic generator used in nuclear disintegration today, for example, is the rebirth in modern form of a machine first conceived by von Guericke almost three hundred years ago. The thermionic rectifier tube or diode, a common instrumentality in the radio-communications field of today, was created by Edison nearly sixty years ago, only a few years after his work in this country and Swan's in England had brought to the world a practical incandescent lamp, and before the experiments of Hertz which demonstrated the presence of electromagnetic waves. Radio communication had not then been thought of except perhaps in poetic prophecy. This practical incandescent light itself was the climax of an observation by Davy, early in the Nineteenth Century, that an electric current could heat a conductor to incandescence. And Davy had been able to perform this experiment and many others only because Volta had given the art the galvanic battery, which in turn depended on Galvani's observation of the twitching of the muscles of frogs' legs.

Through application of the battery by Davy, Oersted, Ampère, Arago, Ohm, Faraday, and Henry — to cite but a few investigators of the first half of the Nineteenth Century — came a host of basic ideas, devices, and theories, all within a period of a little over thirty years. By 1832, electrical incandescence of conductors, the electric arc, and electrolysis had been demonstrated. Primitive electric generators and motors had grown out of the quest for a relation between the electric current and magnetism. The relationship between current and voltage, that between a current and its magnetic field, and the force reaction between a current and a magnetic field had been developed. Induction of a voltage in one

circuit magnetically coupled to another in which a current is changing was observed and the transformer thus evolved. Shortly thereafter, Neuman gave mathematical expression to Faraday's and Henry's induction ideas just as Ampère earlier had succinctly formulated the implications of Oersted's discovery of the relation between a magnetic field and an electric current. Kirchhoff extended Ohm's work on conduction, establishing the fundamental topological relationships among branches, nodes, and independent loops in a network — relationships which are the foundation of circuit analysis today. In this period, Faraday set down the law of electrolysis relating the quantity of material decomposed and the quantity of electricity or current through a conducting solution. His work established the first principles of electrochemistry and the related field of electrometallurgy. Moreover his work was to be to the electromolecular theory of liquids, and thus to their atomicity, what the electron theory, over fifty years later, was to be to the atomicity of electricity.

During the first half of the Nineteenth Century, too, the older fields of magnetostatics and electrostatics were developed by Poisson, Gauss, and Weber, who followed up earlier experimental work of Priestley, Cavendish, Coulomb, and others. Just as Ohm in analyzing electric flow or current paralleled Fourier's earlier work on heat flow, so Poisson in his development of the field theory of electrostatics reasoned by analogy with LaGrange's field theory of gravitation. Not only were mathematical principles thus demonstrated as having parallel applications to the analysis of seemingly unrelated technical fields, but also the fields themselves were becoming related through successful "hunches." Currents produced by electrostatic and electrochemical means were found to produce the same effects in electrolysis. Magnetic fields were found to have their sources in electric currents just as elastic fields had been found to have their sources in electrical charges. The conception of energy, as visualized by Young, the demonstration of its universality by Rumford, Joule, and Mayer, and the doctrine of its conservation by Helmholtz had great influence in the unification of the interrelationships among mechanics, heat, and electricity. Contemporaneously, the theory of the energy of the electric and magnetic fields themselves was reduced to sound basis by Kelvin, who saw as the seat of their energies the whole space including the stationary or moving charge producing them.

As the second half of the Nineteenth Century got under way, then, the force relationship between electric and magnetic charges was known (Coulomb's laws); the relationship between electric and magnetic fields had been defined in terms of force on hypothetical unit positive electric charges and unit positive magnet poles respectively; a current and its magnetic field and the force relationship between two currents or between

a current and an independent magnetic field had been stated (Ampère); the law of induction had been discovered (Faraday, Henry) and formulated (Neuman). The idea of a dielectric and the idea of a limited motion or displacement of apparent charges in the dielectric when under the influence of an electric field had been proposed (Faraday). The laws governing the flow of charge and currents in conductors had been well developed (Ohm, Kirchhoff). Although conduction of electricity over wires had been treated analytically (Kelvin, Kirchhoff), the seat of the phenomenon was thought of as in the conductor itself. The Ampère, *Ohm, and Faraday laws applied to closed circuits. What seemed to be* needed now to explain the remaining puzzling observations and to complete the development was a theory of open circuits.

Maxwell met this need. He had before him not only the observations just outlined but also the theories and attempts to demonstrate that electrical effects are propagated. Kirchhoff had proved theoretically that an electric disturbance is propagated along a wire with a velocity c, a constant of proportionality between electrostatic and electromagnetic units. Weber and Kohlrausch had just determined the value of this constant. But this was only a partial answer to the speculations of Gauss, Riemann, and others as to the plausibility and proof of the propagation of electric energy through space.

Faraday's speculations and also those of Kelvin centered around the theory of a medium. Kelvin had investigated the analogies between electric and elastic phenomena, paralleling the idea of propagation in an elastic medium (elastic waves) with, possibly, a broadly similar propagation of electric energy in space. Faraday had postulated a dielectric as having within it charges limited in their motion or displacement upon the application of an electric field.

Taking all this, Maxwell accomplished his synthesis (1864) by adding to the displacement idea of Faraday the notion that a time variation of displacement or of charge density constituted a current and the idea that this displacement could occur wherever there is a time-varying electric field – between the plates of a condenser in a vacuum, for instance; or as we know the art today, in the space about an antenna or a transmission line. With this displacement current, just as with the conduction current of Ampère, there was associated a consequent magnetic field. Thus Maxwell's theory led to the conclusion that attending a time-varying electric field is always a consequent time-varying magnetic field and the converse -- one of these time-varying fields is always attended by the other.

Today this synthesis is known as Maxwell's theory or simply as Maxwell's equations. It is a complete set of expressions relating electric charges, current densities, magnetic and electric fields, and the rates of

change of these quantities. It explains light as an electromagnetic phenomenon and thus connects electricity and radiant energy. Through its formulation Maxwell was led to predict the existence of the electromagnetic waves which Hertz demonstrated in 1888. It is a tool in the analysis of radio waves and other wave phenomena such as the transmission of electric energy guided by wires, and the oscillation of electric energy back and forth from electric to magnetic in hollow (metal) electric resonators. Being basic, it thus forms the fundamental point of departure in the consideration of all problems* involving electromagnetic phenomena, including lumped-circuit theory, the theory of electric- and gaseous-conduction devices, etc.

When the Nineteenth Century opened, the one application of electrical engineering of any moment was Franklin's lightning rod. During the course of discovery, analysis, and synthesis just summarized, there naturally developed a body of applications, giving rise to the machines and devices sketched earlier and leading to the establishment of an electrical industry, with all that that means in terms of social and economic implication. Even at risk of repetition of details, it is wise for the student of the profession to survey it from this point of view.

The battery had appeared at the beginning of the Nineteenth Century; by the middle of the century, electrotyping and electroplating were industries and the electric telegraph was a successful commercial means of communication. Investigations of the incandescent light and the arc light already suggested the limitation of the battery as a source should they become full-fledged devices. This they did, being important commercially from about 1880 on. The demand for power at reasonable cost was met by the rotating generator, which grew from Faraday's simple copper-disk device to a slotted-drum armature, copper-commutator device with carbon brushes. Gramme, Siemens, Edison, Thomson, and many others were by this time important names in this applied art. Motors were evolving independently, and were used largely in electric-traction applications. Not until the demonstration of the reversibility of direct-current motors and generators at the Exposition in Vienna in 1873 did the primitive reciprocating motor of Henry and the early device of Davenport come to full development. The potentialities of electric lighting led to a great series of public-service developments here and abroad by 1890. Direct-current generation and distribution systems by this time were threatened by the alternating-current system of generation, transmission, and distribution. By the beginning of the Twentieth Century alternating-current systems had made their place. Today our

* A further development of field theory to include what are termed relativistic effects is necessary in order to handle certain problems occurring in engineering. This further development of the Maxwell theory is due primarily to H. A. Lorenz, Minkowski, and Sommerfeld.

great generating plants, hydroelectric and steam, are alternating-current sources. Some cities, where it has not proved economical to change, continue with direct-current distribution.

With the appearance of large electric-power sources came other applications of electricity, including the electrochemical industries, which, for their production of aluminum, magnesium, copper, cadmium, calcium carbide, graphite, and many other products, required a world-total energy of thirty-nine thousand million kilowatthours in 1939 or about one-third of the total energy for all purposes generated in this country.

On the other extreme in terms of power magnitudes but none the less significant in terms of capital invested and of importance are the telegraph and telephone systems. The Bell System of this country today has assets of the order of five billion dollars, the largest of any single concern in this country. No sooner had the wire telephone become established than radio communication was made available — at about the beginning of the Twentieth Century. Just as the telephone saw a prodigious development in its first twenty-five years so did radio. By 1925, radio communication by code had been supplemented by voice and had become an adjunct to the telephone. Broadcasting had become accepted. Today, three-quarters of a century after Maxwell's predictions of electromagnetic waves, not only voice but also facsimile and moving-image or television transmission are industrial activities of magnitude.

That this kind of growth, with all it implies in industrial, social, and economic senses, has by no means ceased is emphatically demonstrated by brief consideration of the development of the branch of electrical engineering which has most recently come to industrial importance — electronics. Evidence of the atomicity of electricity was inherent in Faraday's experiments on electrolysis. Later, through the work of Plücker, Hittorf, J. J. Thomson, and many others, the electron was cornered and its electrical measure was taken. By the turn of the century, the basic elements of the amplifier tube were at hand. The photoelectric cell was already here, and X-rays and some of the products of atomic disintegration were manifest. By 1907 the diode of Edison, long waiting, was to have a companion in the triode which DeForest termed an audion. The audion had, in addition to the hot filament and cold anode of the Edison tube a contro electrode called a grid. In the meantime Fleming had applied the Edison diode as a detector in radio reception and Lieben had a deflecting electron-beam type of amplifier tube. Thus it was that when a DeForest amplifier was demonstrated to telephone engineers in this country, they saw in it the means for doing well what could be done only partially well by the mechanical amplifiers which had been developing since the eighties. By 1915, stored away in the electronics art were many variations of simple vacuum tubes and many tube-circuit arrange-

ments to give not only amplification but also oscillation, modulation, demodulation, trigger control of arc discharges, and many other operations — terms we need not ponder here. Today electronics is a vital part of electrical engineering.

The social consequences of the development of electrical engineering are of equal importance and significance. Applied electrical engineering is vitally concerned with the service of mankind. A power company supplies electric energy, a telephone company furnishes communications service, an electrical manufacturer sells generators, vacuum tubes, or toasters. Thus human nature and human welfare become factors in the engineer's equations, and, therefore, something more than the mathematical formulation of inexorable physical facts in their application to inanimate materials must be in his training. To understand how this need arises one has only to consider a few simple illustrations.

The convenience with which light could be distributed and power could be applied locally through motors in the cities where they were first available rapidly changed ways of living. Gradually, as it became economical to run lines into rural communities, people earlier isolated were given something of the city's advantages. The introduction of electric power engendered quick urban transportation, of which a unique derivation is the subway. Vertical transportation made the modern tall office building feasible. In the majestic skyscrapers, epitomizing engineering triumphs, we have been given a new architecture a new beauty. The modern city is, in fact, a complex of engineering triumphs. And somehow the same electrical features which made these great concentrations of humanity possible, namely, transportation, illumination, and communications, may now with equal force lead men back to the hills, valleys, and plains whence it once drew them. Telephone, radio, sound and vision, and electric power can foster decentralization.

That electrical engineering was to be a community activity appeared earlier in this record. The comparatively heavy currents of electric traction caused electrolysis of water pipes and thus interfered with another type of public service. Power systems set up electrical interference which affected telephone conversations and thus raised problems of adjustment which were complex because their solution required more than technical analysis; they required a co-operative study and the recognition of individual rights. Power and telephone systems required franchises, which involved negotiations between companies and state. As in the case of rights of way for railroads, power transmission and telephone systems became so extensive as to require national or interstate consideration. These relationships between service companies and the public necessitated the establishment of city, state, and national regulatory bodies or commissions. Today radio broadcasting and other forms of communica-

tion, because of their extent, require not only national but also international regulation. In these complex relationships, where regulations and laws must reckon with technical consequences, and technical achievement with social consequences, the electrical engineer's role is clearly one of great responsibility.

In order to supply the apparatus necessary to make electricity useful, great new manufactures have been created. Necessarily, electrical engineers are vital elements in this commercial activity not only as technological experts but also as researchers to undertake to supply new methods, ideas, and things. Because of the way in which electrical engineering has thus involved itself in private and public enterprise technically, economically, and socially over the past one hundred years, there has been a steady growth in the number of electrical engineers in administrative posts, not only in electrical undertakings but also in the public service and in other fields in which the electrical engineer is useful because of the discipline he has had. Faced with these responsibilities the broad-gauged electrical engineer, along with other engineers, has acquired a *professional status involving a comprehensive and prime obligation to* serve the public welfare.

Hence there is heavy dependence on the education of electrical engineers, with one aspect of which — the technological principles of electrical engineering — this volume is concerned.

In their technological aspects, the systems dealt with in electrical engineering are divisible into two main parts. The first includes tangible things — generators, motors, loud-speakers, transformers, electron tubes and related devices, instruments, appliances — all devised to make use of intangible charges and fields. The second comprises abstractions — theories, analyses, formulations — devised to express these charges and fields. In the development of the art these two divisions are interdependent. The chapters which immediately follow are concerned with circuit theory, in which an electrical system may with sufficient precision be thought of in terms of combinations of elements characterized by resistance, inductance, and capacitance, together with voltage and current, that is, electric energy sources. For many purposes, for instance, the rotating generator may be considered a voltage source in series with a resistor. A battery and also the vacuum-tube amplifier may for many purposes be likewise represented. Again, a power or a telephone transmission line or a distribution network may be represented to a sufficient degree of approximation for many purposes by an arrangement of inductors, capacitors, and resistors. Many circuit elements — the tuned circuit of a radio set, for example — are in themselves combinations of coils (inductors) and capacitors. For some treatments, the transmission line may be thought of as a continuous distribution of elemental inductors,

capacitors, and resistors. Circuit theory is therefore a convenient starting point for a study of electrical engineering, because the tangible elements of so many electrical systems may be expressed by means of these five fundamental circuit components and because, inherently, circuit theory is more readily grasped at first than is the basic field theory from which it is derived.

Field theory as such must be invoked, nevertheless, when we are concerned with a problem that can not be expressed in these circuit terms — such problems as are encountered in the radiation and transmission of radio waves, the full consideration of electrical transmission over wires and of electrical machines, and the analysis of nearly all aspects of ultra-high frequency phenomena. It is toward this theory, which has classical epitome in Maxwell's equations and which is necessary for the full understanding of electrical processes and phenomena, that the student will progress.

CHAPTER I

Derivation and Evaluation of Circuit Parameters

1. CIRCUIT THEORY AND ITS RELATION TO FIELD THEORY

Though every problem in electricity and magnetism is basically a field problem, the engineer usually prefers not to deal with field quantities but rather with their integrated effects. That is, he prefers to deal with quantities such as voltage, current, charge, resistance, capacitance, inductance, rather than with field intensity, current density, charge density, resistivity, and so on. The possibility of characterizing a problem uniquely in terms of the integrated field effects alone depends to some extent upon the nature of the associated energy loss. Energy may be lost within the system itself because of dissipation as heat* (called an ohmic loss), or it may be lost by escaping from the system in the form of electromagnetic radiation. If the latter form of energy loss is a predominating feature of the problem, the seat of the major phenomenon is in the field, and the integrated effects are of merely collateral importance. If, on the other hand, the radiation is incidental and relatively negligible, the integrated effects are predominantly important. In such a situation the essential field relationships are uniquely determined in terms of the geometry and physical properties of the system configuration, together with the potential difference (integrated electric-field intensity) and the integrated conduction-current density at a finite number of equipotential surfaces called *terminals*. This statement does not imply that displacement current must be ignored or that the system must be so constituted that displacement current has no part in the field phenomena. Within the system there may well be displacement currents, but, because of the presence of conductors upon which the electric displacement or flux falls, the net effects are measurable in terms of the conduction currents in these conductors. The terminals of the system are particularly chosen conductor cross sections where displacement current from near-by conductor surfaces is negligible in comparison with the conduction current over the cross sections. Since voltage and conduction current are readily measurable quantities, it is effective, for the analysis of the electrical behavior, to formulate the equilibrium of such systems wholly in terms of these terminal quantities.

Problems to which this type of analysis applies are called *circuit prob-*

* Other internal losses such as magnetic and dielectric hysteresis losses are nonlinear and are not included in the present argument. They are customarily taken into account by approximate means and have no bearing upon the question at hand.

lems, and the particular method of treatment which the present chapter introduces is called *circuit analysis*.

It should be observed, however, that circuit analysis is an approximate form of field analysis and that it cannot in general be applied to problems in which radiation in the form of electromagnetic waves plays a major role, as it does, for example, in the behavior of antenna systems for radio communication. In the latter type of problem the behavior of the radiated field is only indirectly related to the voltages and currents of the antenna and its associated circuits. The wave-propagation phenomenon forms a major portion of the analysis and can be dealt with only by means of the field theory.

Of course some problems deal with the propagation of electromagnetic waves which are more specifically confined to, or guided by, the circuit producing them, as in transmission lines for power or communication systems. Here the connection between the circuit and the wave is sufficiently intimate to permit an adequate treatment in terms of circuit aspects. The engineer, therefore, commonly refers to such arrangements as circuits, although the analysis of their detailed behavior involves methods which deal largely with the field quantities rather than with their terminal effects.

It is thus rather difficult to define or circumscribe the circuit analysis as distinct from the more general field theory unless it is agreed to make the distinction on the basis of whether geometrical space co-ordinates are involved or not. The purpose of assumptions made in deriving the circuit relations from the field relations is to reduce the number of space dimensions of the system to one, or none, instead of three.

2. Reduction of field relations to circuit relations

Neither the simple circuit discussed in this article nor any other circuit can be solved rigorously by means of the field equations, even with the aid of the most advanced mathematics. In many situations, including the one to be considered, however, suitable assumptions make an approximate solution practicable. As a first step in the direction of simplification, the electromagnetic phenomena associated with a stationary* system are grouped according to the following classification:

(a) The electric fields in conductors, accompanied by pure conduction currents and the dissipation of energy in the form of heat.
(b) The electric fields in ideal dielectrics, accompanied by electric displacements and the storage of electric (potential) energy.
(c) The magnetic fields associated with conduction currents and accompanied by the storage of magnetic (kinetic) energy.

* By this is meant that the electric- and magnetic-field intensities do not vary with time.

When a time variation is involved, there is associated with (b) a displacement current whose integrated effect at the boundaries between the dielectric and metallic conductors may be measured in terms of a conduction current. In (c) the varying magnetic field induces, according to the Faraday relation, a voltage around a closed boundary which may be chosen so as to define with sufficient approximation the path of the associated conduction current.

In Fig. 1 is outlined a simple circuit which consists of a battery with terminals a and b; two lengths of conducting wire cfg and ha, long compared with their cross-section dimensions; two parallel conducting plates AF and HG having large surface dimensions compared with their separation, connected to the wires at g and h; and a switch K in the battery lead. The battery has a constant electromotive force E which, when K is closed, maintains a constant difference in potential between a and c, c being at the higher potential, or positive with respect to a. The wires are assumed to be made of homogeneous and isotropic material of conductivity γ. The entire circuit is assumed to be immersed in a homogeneous, isotropic, nonconducting medium of permittivity ε and permeability μ. A portion of the wire between c and f is formed into a coil in order to increase the energy stored in the magnetic field.

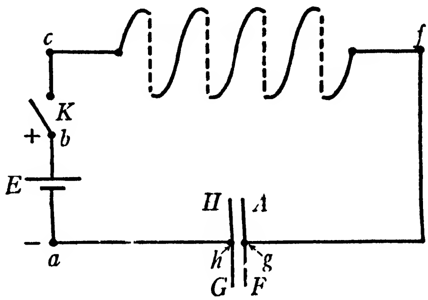


FIG. 1. Outline of a simple electric circuit.

With K open, there are no charges on plates AF and HG. If K is now closed, there is motion of charge — current — in the wires temporarily until a new condition of equilibrium is reached. If it is assumed that current in the wires is uniformly distributed over the cross section, and that displacement current is confined to the region between plates AF and HG, the elementary circuit relations can be developed by applying *Faraday's induction law*,

$$\oint \mathcal{E} \cdot d\ell = -\frac{d}{dt}\int_{s_{\mathcal{B}}} \mathcal{B} \cdot ds_{\mathcal{B}} \qquad ▶[1]$$

and the *principle of the conservation of charge*,

$$\int_{s_{\mathcal{J}}} \mathcal{J} \cdot ds_{\mathcal{J}} = -\frac{1}{4\pi}\frac{d}{dt}\int_{s_{\mathcal{D}}} \mathcal{D} \cdot ds_{\mathcal{D}}. \qquad ▶[2]$$

Application of Eq. 1 is made somewhat clearer by reference to Fig. 2, which enables the field relations to be visualized somewhat more in detail. The line integral of electric-field intensity $\mathcal{E}$ is taken in the arrow direction around a closed path of length ℓ coincident with the center lines of the wires and joining them through the battery, through the switch, and from g to h between the plates. Since the cross-section dimensions of the wire are small compared to the dimensions of the circuit loop or the cross-section dimensions of the coil, the actual position of the path of integration within the wire is not important. The magnetic flux density $\mathcal{B}$ directed downward across the hatched area s (bounded by the line around which $\mathcal{E}$ is integrated) is integrated over that area. This rule for

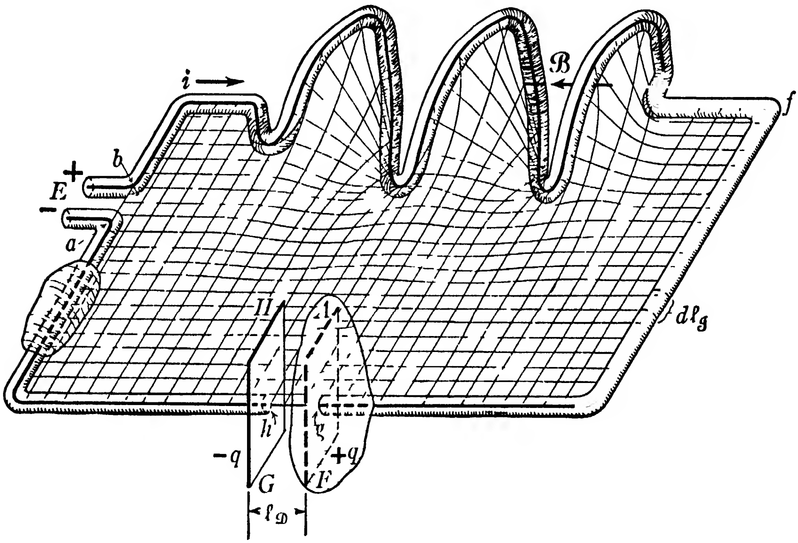


FIG. 2 Sketch of physical parts outlined in Fig. 1, showing space relations.

establishing the correct algebraic sign relations in Faraday's law, Eq. 1, is a form of *Lenz's law*.

The line integral $\oint \mathcal{E} \cdot d\ell$ can be divided into three parts. The first is the electromotive force E of the battery. The second arises from the electric-field intensity $\mathcal{E}_g$ accompanying and in the direction of the conduction-current density $\mathcal{J}$,

$$\mathcal{E}_g = \mathcal{J} \qquad [3]$$

The third arises from the electric-field intensity $\mathcal{E}_{\mathcal{D}}$ accompanying and in

the direction of the electric displacement $\mathcal{D}$ between the plates,

$$\mathcal{E}_{\mathcal{D}} = \frac{\mathcal{D}}{\varepsilon}. \qquad [4]$$

The result of $\int \mathcal{E} \cdot d\ell$ between any points along the path is *algebraically* a drop in potential in the direction of integration. This represents work done *by* the electric field *on* a unit charge. Outside the battery, as a charge moves in the direction from b to a, the field actually does work on the charge, which work in turn is dissipated or stored. Inside the battery, as a charge moves from a to b it actually does work on the electric field, which work it can do by virtue of the chemical energy received from the battery. Hence the first part of $\oint \mathcal{E} \cdot d\ell$, taken from a to b inside the battery, is $-E$. For the second part, integration is carried out in the wire only from b to f to g, and from h to a, and gives the result

$$v_{\mathcal{J}} = \int \mathcal{E}_{\mathcal{J}} \cdot d\ell = \int \frac{\mathcal{J}}{\gamma} \cdot d\ell = \int \frac{i}{\gamma s_{\mathcal{J}}}\, d\ell = \frac{\ell_{\mathcal{J}}}{\gamma s_{\mathcal{J}}}\, i = Ri, \qquad [5]$$

in which $\ell_{\mathcal{J}}$ is the total length of wire, $s_{\mathcal{J}}$ is the cross-section area of the wire, R is the resistance of the wire as computed for the steady state, and i is the conduction current. Since by assumption there is no displacement current except between the plates,* Eq. 2 reads

$$\int_{s_{\mathcal{J}1}} \mathcal{J} \cdot ds_{\mathcal{J}1} + \int_{s_{\mathcal{J}2}} \mathcal{J} \cdot ds_{\mathcal{J}2} = 0 \qquad [2a]$$

for any closed surface through which the wire passes. In Eq. 2a, since the current density is taken algebraically as *directed outward* all over the closed surface, the current at any wire cross section 1 is the same as at any other cross section 2, a fact which justifies the use of i in Eq. 5 as unique for the wire. For the third part, integration is carried out between plates from g to h and gives the result

$$v_{\mathcal{D}} = \int \mathcal{E}_{\mathcal{D}} \cdot d\ell = \int \frac{\mathcal{D}}{\varepsilon} \cdot d\ell_{\mathcal{D}} = \int \frac{4\pi q}{\varepsilon s_{\mathcal{D}}}\, d\ell_{\mathcal{D}} = \frac{4\pi \ell_{\mathcal{D}}}{\varepsilon s_{\mathcal{D}}}\, q = Sq, \qquad [6]$$

in which $\ell_{\mathcal{D}}$ is the spacing between plates, $s_{\mathcal{D}}$ is the area of a plate, S is the elastance (reciprocal of capacitance) as computed for the parallel-plate condenser in electrostatics, and q is the charge on plate AF.

The surface integral, $\int_{s_{\mathcal{B}}} \mathcal{B} \cdot ds_{\mathcal{B}}$, if taken over the shaded area as stated, must take into account the fact that magnetic flux within the coiled part of the wire crosses the surface repeatedly, essentially once for each turn,

* In order for displacement current to be about one-billionth as great as the conduction current in a copper conductor, a frequency of about 10^9 cycles per second is required.

if the coil is long and closely wound. Hence if the number of turns is large, and the surface area between the coil and the remainder of the circuit boundary is not great, substantially all the contribution to the integral comes from the coil. On the assumption that all the integral comes from the coil,

$$v_{\mathcal{B}} = \frac{d}{dt}\int_{s_{\mathcal{B}}} \mathcal{B} \cdot ds_{\mathcal{B}} = \frac{d}{dt}\int_{s_{\mathcal{B}}} \mu\mathcal{H} \cdot ds_{\mathcal{B}} = \frac{d}{dt}\int_{s_{\mathcal{B}}} \frac{4\pi\mu Ni}{\ell_{\mathcal{B}}}\, ds_{\mathcal{B}} = L\frac{di}{dt}, \qquad [7]$$

in which $\mathcal{H}$ is the magnetic-field intensity inside the coil, N is the number of turns in the coil, $\ell_{\mathcal{B}}$ is the axial length of the coil, $s_{\mathcal{B}}$ is the cross-section area of the coil, and L is the self-inductance for a long solenoidal coil as computed for the magnetostatic state. It is emphasized, however, that inductance requires a closed path for its definition, and that hence to speak of the concentration of inductance between two points on an electric circuit, such as cf, Fig. 1, involves a certain degree of fiction beyond the mere concentration of a distributed effect.

Combining the three parts of $\oint \mathcal{E} \cdot d\ell$ and using Eq. 7 give

$$\oint \mathcal{E} \cdot d\ell = -E + v_{\mathcal{G}} + v_{\mathcal{D}} = -v_{\mathcal{B}} = -E + Ri + Sq = -L\frac{di}{dt}. \qquad [1a]$$

If Eq. 2 is written for a closed surface inclosing only one of the pair of plates, for example, AF,

$$\int_{s_{\mathcal{G}}} \mathcal{G} \cdot ds_{\mathcal{G}} = -i = -\frac{1}{4\pi}\frac{d}{dt}\int_{s_{\mathcal{D}}} \frac{4\pi q}{s_{\mathcal{D}}}\, ds_{\mathcal{D}} = -\frac{dq}{dt}, \qquad [2a]$$

or

$$q = \int i dt + c, \qquad [2b]$$

in which c is zero if the condenser is initially uncharged. The minus sign is used before the current i, because Eq. 2 is taken to represent the total current *emerging from* the closed surface surrounding the plate AF. If the equation is taken to represent the total current *entering* the closed surface, the sign of the electric flux $4\pi q$ must be made negative and the final result is the same as obtained. The substitution of Eq. 2b into Eq. 1a gives (if c is zero)

$$L\frac{di}{dt} + Ri + S\int i dt = E. \qquad \blacktriangleright[1b]$$

This is the usual form of the fundamental circuit equation for a single loop. The term $\int i dt$ is usually understood to represent the total charge on the plate toward which i is directed, including the initial charge if any, in addition to the charge accumulated after i has started.

In the derivation of the fundamental circuit relations it is, of course, not essential to the validity of the result that the resistance take the form of a wire, that the condenser have large parallel plates, and that the coil be a long solenoid. These forms were chosen merely for simplicity. If due regard is given to the general assumptions and approximations set forth, the circuit parts may have any shapes.

3. Some circuit terminology and conventions

The quantities R, L, and S (or their reciprocals) are called the circuit *parameters*. They are merely mathematical quantities -- the coefficients in the corresponding terms of the differential equation -- and as such should be distinguished from resistance devices or *resistors*, inductance

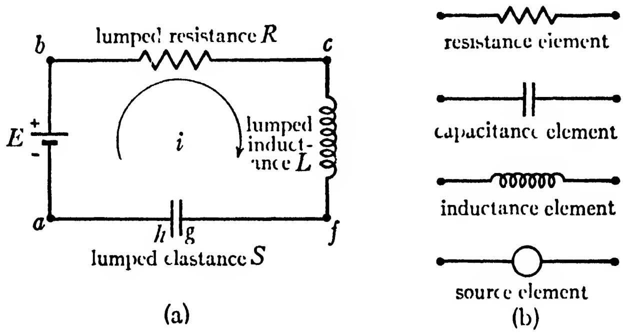


FIG. 3 Circuit representation by means of symbolic elements.

coils or *inductors*, and condensers or *capacitors*, which are physical apparatus. It is emphasized that the parameters defined as coefficients in the differential equation generally have different values in circuit theory from the corresponding values based upon static definitions. These differences may be inappreciable in some cases and extreme in others.

The parameters derived for the system of Fig. 1 are *lumped* parameters because they lead to a circuit representation such as that of Fig. 3a, in which each voltage component of Eq. 1b can be imagined to be concentrated or lumped between two particular points. It makes no difference in the terminal relations between voltage and current between points a and b, Figs. 1 or 3a, whether the resistance drop is distributed along the circuit or concentrated in one place. The same is true for the capacitance and inductance drops. In other words, in a lumped-parameter circuit, space considerations are unessential for the circuit analysis, except for the evaluation of the parameters themselves. However, in dealing with long transmission lines, for example, as is pointed out in Art. 5, it is neces-

sary to take into account the distribution of electromagnetic effects along the length of the line. If interest lies chiefly in the voltages and conduction currents rather than in the field aspects of the problem, a modified circuit representation may be used which is a one-space dimension or *distributed-parameter* representation.

A resistor, inductor, or capacitor idealized to have only resistance, inductance, or capacitance, respectively, is called a circuit *element*. A source idealized to have no internal resistance, inductance, or capacitance is also called an element. Just as parameters are mathematical symbols, *so are elements circuit symbols, not physical apparatus. Conventional* representation for the four types of elements is shown in Figs. 3a and 3b. The source representation of Fig. 3a is for a battery; that of Fig. 3b is more general, for any type of source.

In circuit theory it is customary to regard as source elements only those which are capable of sustained energy transfer into the electromagnetic system which the circuit elements represent. Source elements, hence, are sometimes called *active* elements. In contrast, resistance, inductance, and capacitance elements are called *passive* elements, because they serve merely to dissipate or store the energy which the source transfers to the system and when not acted upon by the source are of themselves inactive. The voltages of the passive elements in the circuit discussed may be viewed as analogous to the reaction forces in mechanics. For example, a spring cannot exert a force until some active agent deforms it and in so doing transfers energy to it; of itself it is inactive.

In the derivation of Eqs. 1a and 1b it is important to notice that the direction of the electromotive force or rise in potential of the source E is indicated by polarity marks + and − on the diagrams, Figs. 1, 2, and 3a; that the direction of current i the rate of flow of positive charge — is represented by an arrow; that the symbols $v_{\mathcal{G}}$, $v_{\mathcal{D}}$ and $v_{\mathcal{C}}$ represent voltage drops in the direction of the current; and that the symbol q represents the amount of positive charge on plate AF. In all such work the use of a consistent system of symbols associated with a workable system of indicating directions and polarities is of great importance.

Additional circuit terminology and further evolution of a system of symbols are presented subsequently as needed.

4. General approximations and limitations

In the study of circuit theory the active elements are taken for granted and accepted merely as means of sustaining a potential difference or a current by some mechanism the operation of which is not of immediate concern. The study of practical sources or generators of various kinds is

of itself a large division of the field of electrical engineering, a part of which is treated in this series in the volume on rotating electric machinery. In circuit theory the primary concern is with the passive elements – their evaluation by computation or measurement, the relations which associate voltage, current, charge, electric or magnetic flux, power, or energy within each element or within the circuit as a whole, and the practical limitations of the method of analysis.

Frequently a rather simple circuit representation suffices for the analytic study of the performance of a system. In problems to which this simplified point of view applies, it is unwise to set down a circuit representation which takes account of more detailed field relationships. The schematic circuit for a particular physical problem should be designed to meet the practical requirements. Overrefinement needlessly complicates the subsequent treatment. On the other hand, a circuit representation which is inadequate to account for all essential phenomena, or is inconsistent with the conditions under which the subsequent analysis is interpreted, is obviously also inappropriate. A circuit representation which is just adequate for the requirements of the problem in hand is sometimes difficult to set down at the outset and must occasionally be obtained by a process of successive approximations.

As an example, the circuit representation for a coil of insulated wire with a voltage applied to its terminals is considered. After the voltage is applied there is an electric field in the space surrounding the wire because different points along the wire are at different potentials. There is likewise an electric field within the wire, and with it a conduction current. The conduction current in turn has associated with it a magnetic field, partly within the conductor cross section but mainly within the surrounding medium.

If the problem is to determine the relation between a steady applied voltage and the ultimate conduction current, the adequate circuit arrangement is very simple. The voltage is given by a line integral of the electric-field intensity along a path which may be chosen to lie within the conductor from one terminal to the other. The determination of this line integral requires a knowledge of the electric field within the conductor, and this same field determines the conduction current also. Since the ultimate current is steady, there is no voltage of self-induction; and, since the electric field is stationary, there is no displacement current. Hence a knowledge of the resistance of the coil suffices to determine the conduction current. A resistance element alone then is the schematic representation for the coil. The determination of the electric-field distribution within the conductor and consequently the computation of the resistance depend entirely upon the conductor geometry, as is discussed more specifically in subsequent articles.

Matters are somewhat complicated, however, if the applied voltage varies with time. Then the electric and magnetic fields likewise vary with time. The varying electric field is accompanied by displacement current, and the varying magnetic field induces a voltage according to Faraday's law. Rough calculations show that, within the conductor, the displacement current ordinarily is entirely negligible in comparison with the conduction current and hence justify this commonly made assumption. For example, if in the conductor the electric-field intensity is varying sinusoidally, it is readily demonstrated that the frequency at which *the amplitude of the displacement-current density equals the amplitude of the conduction-current density* is $2\gamma/\varepsilon$. For copper, this corresponds to a frequency of about 10^{18} cycles per second if the permittivity of copper is about the same as the permittivity of free space. The values of permittivities for metals are largely speculative, but it is inconceivable that they can be sufficiently large to invalidate the assumption of negligible displacement current in conductors. Whether the displacement current due to the electric field outside the conductor is also negligible depends, among other things, upon the relative intensity of the field outside as compared to that inside the conductor, and hence upon the geometry of the coil. For example, the intensity of the external field depends upon the proximity of various turns of wire which are at different potentials with respect to each other. Such adjacent turns act like small capacitances. The displacement current due to these capacitances depends also upon the dielectric constant of the insulating material surrounding the wire and the time-rate of change of the applied voltage. An approximate calculation frequently suffices to show whether the net effect of this external displacement current is negligible compared to the conduction current within the wire in determining the resultant terminal current. For time variations ordinarily occurring in power engineering and for the usual coil configurations, the conductor current alone need be considered (though the inductive effect of the magnetic field in influencing the electric-field intensity within the conductor may not be negligible). In such a case an adequate schematic representation is given by a series combination of a resistance and an inductance element, while for other operating conditions or coil configurations a capacitance element may have to be placed in parallel with the resistance and inductance elements in order adequately to satisfy the physical requirements.[1]

In these and more complicated cases, the process of determining a suitable schematic representation is to a certain extent based upon experience gained through experimental measurement as well as upon calcula-

[1] D. B. Sinclair, "Residual Parameters in Resistance Standards," *Gen. Rad. Exp.*, XIII (1939), 6 11, gives interesting and useful data. (A table of abbreviations used for names of periodicals is in the bibliography, p. 757.)

tion. By assuming various circuit arrangements and determining their behavior analytically, and by comparing this with the experimentally observed behavior of physical systems, a facility for predicting adequate circuit arrangements is developed.

Through such combined analytic and experimental experience it is possible in circuit design to construct an inductor so that, under stated operating conditions, it will behave essentially like a series combination of resistance and inductance elements, or possibly as such a combination with a capacitance associated in parallel to account sufficiently well for displacement-current effects.

An entirely analogous line of thought may be used to investigate the ability of a pair of parallel conducting plates to simulate a lumped-capacitance element alone. Actually the associated conduction current within the plates and within the intervening dielectric may introduce an energy dissipation large enough to require the consideration of resistance elements both in series and in parallel with the capacitance. Furthermore, the inductive effects of the conduction or displacement currents may become sufficiently evident to require the consideration of inductance elements in order adequately to represent the observed behavior of such a capacitor. In the design of capacitors also, experience and calculation combine to dictate the physical configurations which can be represented adequately by a proposed arrangement of circuit elements.

Various combinations of circuit elements for the representation of coils and condensers are shown in Fig. 4. Depending upon circumstances, a coil may behave essentially like any of the three elements, and the same is true of a condenser.

Although experimental observation is an invaluable aid in the determination of the circuit representation of a given physical problem or in the inverse problem of designing a physical configuration to meet a desired circuit representation, at the same time the analytic aspects must be thoroughly appreciated.

5. Further consideration of the effect of time variation

For circuit analysis to be reasonably simple, it is essential that the terminal relations for the elements be independent of the nature of the terminal voltages, currents, or charges as functions of time. The resistance, inductance, and capacitance elements should, therefore, have values depending only upon the geometry and fixed physical properties — resistivity, permeability, and permittivity — of the various portions of the circuit which they represent. In other words, the circuit parameters should be essentially constants. Constant-parameter circuits are called *linear circuits* because their behavior can be described by linear differential equations. Their elements are called *linear* circuit elements. Simple

circuits for which the parameters are not constants *owing to the fact that the physical properties of the circuit parts are not constant* are analyzed in Ch. XIII. Such circuits are called *nonlinear circuits;* their elements are called *nonlinear* circuit elements. Complications under consideration in this article are apart from those which arise because of variable resistivity, permeability, or permittivity.

As indicated by the discussion in Art. 4, parameters are strictly constant only when the field conditions are stationary. When field conditions vary with time, as they do when the voltages and currents are not constant, the field distributions are only approximately determined by the resistivity, permeability, permittivity, and geometrical configuration of

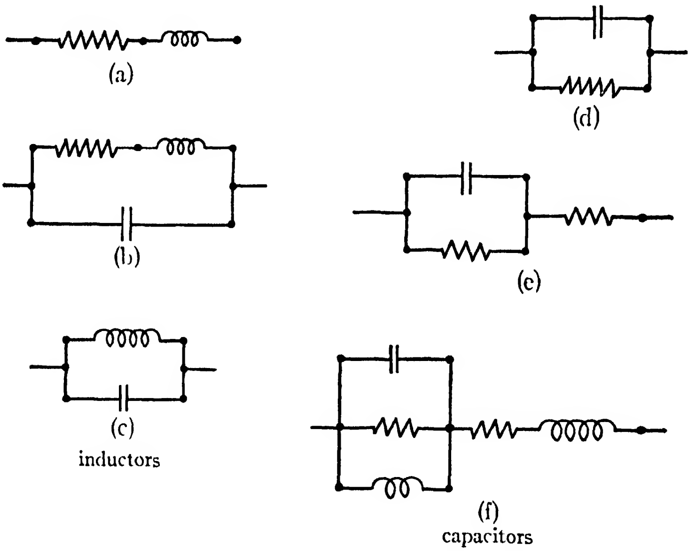


FIG. 4. Circuit representations for inductors and capacitors.

the system. The approximation is, however, very good unless the time variations are extremely rapid. In order to show why this is so, the discharge of a parallel-plate condenser* is considered. If the charges are somehow suddenly and completely removed, the electric-field distribution surrounding the plates is at the first instant not influenced at all, because the stored potential energy due to this field cannot instantly vanish. The only way it can eventually disappear is through some means of energy propagation, and the only known means in this instance is that by which light energy propagates. The electric field at first begins to collapse in the immediate vicinity of the plates, and this changing electric

* The medium is assumed to be free space so that polarization phenomena may be excluded from the present argument.

field causes a displacement current and hence a magnetic field to appear. The interaction of the two fields gives rise to an energy-propagation phenomenon known as an electromagnetic wave. The collapse of the electric field thus takes the form of electromagnetic wave motion, which travels with a finite velocity. Points in space remote from the condenser plates experience the collapse of the field at a later time. The rate of propagation of the wave equals the velocity of light, which in free space is about 3×10^8 meters per second, or about 186,000 miles per second.

Thus, while the electric field does not collapse simultaneously at all points in space, its disappearance at all points within distances comparable to the dimensions of the condenser plates (if these are of a size ordinarily employed) is practically complete within an extremely short time interval. Since the electric-field intensities of an appreciable magnitude are confined to the more immediate vicinity of the condenser plates, most of the stored energy disappears very quickly.

For some practical purposes it may, therefore, be legitimate to assume that the collapse is instantaneous, while in other cases this assumption may not be admissible. If the charges on the plates are removed and reapplied periodically at a very rapid rate so that the period of application and removal becomes comparable in duration to the short time it takes the electromagnetic wave to travel a distance equal to the plate size or less, then the finite rate of collapse and restoration of the field must certainly become noticeable in the behavior of the system. For a condenser plate having a largest dimension of about 20 centimeters, the period has to be of the order of magnitude of $20/(3 \times 10^{10})$, or $\frac{2}{3} \times 10^{-9}$, second, corresponding to a frequency of $\frac{3}{2} \times 10^9$, or 1,500 megacycles* per second, in order for such considerations to become important. On the other hand, if the conductor system forming the condenser consists of the parallel wires of a transmission line 100 miles in length, the wave-propagation phenomenon and its influence upon that system's behavior become important for applied frequencies of the order of magnitude of 186,000/100, or 1,860 cycles per second. A quite noticeable effect is experienced even at much lower frequencies.

These considerations may be expressed more simply in terms of the distance which the electromagnetic wave travels during one period. This distance is commonly referred to as the *wavelength*. If the applied frequency is denoted as f, the velocity of the electromagnetic wave by v, and the corresponding wavelength by the symbol λ,† the following relation evidently holds

$$\lambda = \frac{v}{f}, \qquad [8]$$

* Megacycles is an alternative for million cycles.

† This should not be confused with the same symbol as used to denote flux linkage.

in any consistent system of units. Roughly, then, it may be said that if this wavelength is comparable to the physical dimensions of the system, the propagation phenomenon becomes an important factor in the sense that the capacitance parameter cannot be considered a constant independent of the nature of the applied voltage function. Alternatively, when λ is large compared with the dimensions of the system, then the capacitance parameter can, for practical purposes, be considered a constant determinable from stationary-field calculations. In this case the time-rate of change of the field is slow enough to be practically synchronous throughout the region near the conductors — the region which contributes the major portion of the total potential energy.

An analogous situation exists with regard to the inductance parameter due to a magnetic field produced by a system of current-carrying conductors. The frequency of the time-varying magnetic field and the corresponding wavelength as determined from Eq. 8 together with the physical dimensions of the system enable one to estimate whether or not the calculation of this parameter from stationary-field considerations is admissible. The same applies to the calculation of the resistance parameter from the static electric-field distribution within a conductor and to the use of this value in cases where the field is nonstationary.

In applying Eq. 8 for determining the wavelength corresponding to a given frequency, one must know the velocity of propagation, which depends upon the properties of the medium in question.[2] In dielectrics, where the conduction current is negligible compared to the displacement current, the velocity of wave propagation is given by the relation

$$v = \frac{1}{\sqrt{\varepsilon\mu}}, \tag{9}$$

while in conductors where the displacement current is negligible compared to the conduction current

$$v = \sqrt{\frac{f}{\gamma\mu}}. \tag{10}$$

Thus *in nonconductors:*

$$\lambda = \frac{1}{f\sqrt{\varepsilon\mu}}, \tag{11}$$

while *in conductors:*

$$\lambda = \frac{1}{\sqrt{f\mu\gamma}}. \tag{12}$$

[2] Reference volume, Ch. XIV; N. H. Frank, *Introduction to Electricity and Optics* (New York: McGraw-Hill Book Company, Inc., 1940), Chs. viii, xiii.

In free space or in air $1/\sqrt{\varepsilon\mu}$ is about 3×10^8 meters per second; in the more commonly used dielectric substances this value may be one-half to one-third as large. In considering the capacitance between conductors, only the field in the intervening dielectric is essential. For inductance calculations, the magnetic field within the conductors, as well as in the dielectric surrounding them, has a contributing effect, although the geometrical configurations are usually such that the internal field adds only a small fraction to the total magnetic effect. For these parameters, then, the frequency must be high or the physical dimensions large before their calculation by stationary-field methods becomes appreciably inaccurate.

In the calculation of the resistance parameters, or of that contribution of the inductance parameter coming from fields within the current-carrying medium, Eq. 12 must be considered in estimating the degree of correctness to be exhibited from methods based upon stationary-field assumptions. Here the usual magnitudes for λ are quite small. For example, in the case of copper, Eq. 12 becomes

$$\lambda = \frac{0.415}{\sqrt{f}} \quad \text{m,} \quad \text{or} \quad \frac{41.5}{\sqrt{f}} \quad \text{cm.} \qquad [12a]$$

Even at 60 cycles per second, which is the frequency used in ordinary alternating-current power systems, this corresponds to a wavelength of a little more than five centimeters, or about two inches. For conductor sizes such as are used in large machines or power-transmission lines and auxiliary equipment, this wavelength is sufficiently near the physical dimensions to make the effect of wave propagation important in the process of parameter determination.

When the time variation of the fields is known to be simple harmonic, as it is in many practical applications, it is not very difficult to take these considerations into account in ordinary parameter calculations. With the resistance parameter, the effect is to increase its value over that which obtains for stationary fields. This corrected value holds only at the particular frequency considered, and is referred to as the *effective* resistance. The inductance parameter is usually only slightly affected, since the internal fields are a small portion of the total. The effect is a decrease in the value of the inductance since the time-varying field penetrates less deeply into the conductors.[3]

A greater difficulty arises when the nature of the time-varying fields is not simple harmonic and in general not known, as in problems where the transient response of a system is sought. If the system is *linear*, a formal process of solution in terms of effective parameter values may be

[3] Reference volume, Ch. XV.

set up utilizing Fourier integral methods,[4] but the labor in carrying this through is ordinarily so great that one is forced to be content with a much more approximate solution or else to have recourse to an experimental measurement of the behavior.

Whatever the conditions for a specific problem may be, and whatever refinement in circuit representation or parameter calculation may be found necessary, the point of departure in all matters pertaining to parameter determinations is the stationary-field consideration. This is true *because the fundamental definitions for the resistance, inductance,* and capacitance parameters relate to electrostatic or magnetostatic fields. Before an extension to nonstationary conditions can be intelligently contemplated, the methods of determining parameter values from their fundamental definitions must be understood adequately.

A word of caution seems desirable concerning the inductance parameter. It is pointed out in Art. 3 that the parameter definitions based on dynamic rather than static field conditions may depart from the static values quite substantially. The internationally adopted standard of inductance is, in fact, based upon the dynamic definition rather than upon the fundamental static definition. From what has already been said on this point it is necessary to interpret this definition of the inductance standard as the limit which this value approaches as the rate of time variation is decreased to zero. In many practical cases this refined interpretation may be wholly unnecessary; in others it may become of primary importance. It is well to understand clearly the point at issue in order not to be misled in circumstances requiring a careful distinction between fundamental and derived definitions. The present-day tendency toward the use of extremely high frequencies in communication circuits, for example, makes more common the occurrence of problems involving this distinction.

6a. The Calculation of Circuit Parameters from Their Static Definitions; the Resistance Parameter

The general problem of resistance computation is to determine the resistance of a conductor such as that sketched in Fig. 5, between the equipotential surfaces a–b. Unless the terminal surfaces of the conductor are equipotential surfaces for the given geometrical shape, the potential difference between them is not unique, and hence the resistance parameter cannot be defined. The surrounding medium is assumed to be nonconducting. Since static conditions are assumed, the electric field within the conductor is conservative and may be represented as the gradient

[4] *Id.* Ch. VIII; G. A. Campbell, and R. M. Foster, "Fourier Integrals for Practical Applications," *Bell Monograph* B-584.

of a scalar potential function $\mathcal{V}$. Displacement current is absent; therefore the conduction-current field and hence $\mathcal{E}$ must be parallel to the lateral surface s. Equipotential surfaces must then be normal to this lateral surface wherever they meet it.

The current density $\mathcal{J}$ at any point within the conductor is proportional to the electric-field intensity $\mathcal{E}$. There are no sources within the conductor. The electric field is hence determined as for free space in the absence of charges. By Gauss's theorem the number of $\mathcal{E}$ lines (or $\mathcal{J}$ lines) entering any closed internal volume must equal the number leaving this

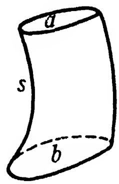


FIG 5. Portion of an electric conductor.

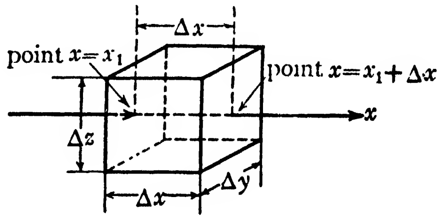


FIG 6 Differential volume used in derivation of divergence.

volume. The number of $\mathcal{E}$ lines leaving or diverging from a closed volume *per unit volume* is called the *divergence of* $\mathcal{E}$ for that volume, abbreviated div $\mathcal{E}$. For the conditions of the problem, the divergence of $\mathcal{E}$ for the volume and at all points in the volume is zero. Since

$$\mathcal{E} = -\operatorname{grad} \mathcal{V}, \tag{13}$$

the divergence of the gradient of $\mathcal{V}$ is zero. That is,

$$\operatorname{div} \operatorname{grad} \mathcal{V} = 0. \tag{14}$$

The problem is to find that function for $\mathcal{V}$ which satisfies Eq. 14, known as Laplace's equation, together with the stated boundary conditions. It is possible to express Eq. 14 as a differential equation. This is demonstrated with the aid of Fig. 6, which represents a differential volume in the form of a cube with the dimensions Δx, Δy, and Δz chosen to coincide in direction with a Cartesian system of co-ordinates. The x component of the electric-field intensity is denoted by $\mathcal{E}_x$, and the value of this component at the point x_1 by $\mathcal{E}_{x_1}$. At the point $x_1 + \Delta x$, the field intensity is expressible as

$$\left[\mathcal{E}_{x_1} + \frac{\partial \mathcal{E}_x}{\partial x}\Delta x\right], \tag{15}$$

and the x component of the total number of lines diverging from the cubical volume is given by the difference

$$\left(\mathcal{E}_{x_1} + \frac{\partial \mathcal{E}_x}{\partial x}\Delta x\right)\Delta y\Delta z - \mathcal{E}_{x_1}\Delta y\Delta z = \frac{\partial \mathcal{E}_x}{\partial x}\Delta x\Delta y\Delta z. \tag{16}$$

The y and z components have the analogous forms

$$\frac{\partial \mathcal{E}_y}{\partial y} \Delta x \Delta y \Delta z \tag{17}$$

and

$$\frac{\partial \mathcal{E}_z}{\partial z} \Delta x \Delta y \Delta z. \tag{18}$$

The total number of diverging lines is the sum of these three components. *The divergence for the differential cube, therefore, is this sum divided by the volume $\Delta x \Delta y \Delta z$, thus:*

$$\text{div}\ \mathcal{E} = \frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + \frac{\partial \mathcal{E}_z}{\partial z}. \tag{19}$$

The components of $\mathcal{E}$ can be expressed in terms of the components of the gradient of $\mathcal{V}$, as expressed by the relations

$$\mathcal{E}_x = -\frac{\partial \mathcal{V}}{\partial x}; \tag{20a}$$

$$\mathcal{E}_y = -\frac{\partial \mathcal{V}}{\partial y}; \tag{20b}$$

$$\mathcal{E}_z = -\frac{\partial \mathcal{V}}{\partial z}. \tag{20c}$$

Hence, in rectangular co-ordinates the Laplace equation 14 reads

$$\frac{\partial^2 \mathcal{V}}{\partial x^2} + \frac{\partial^2 \mathcal{V}}{\partial y^2} + \frac{\partial^2 \mathcal{V}}{\partial z^2} = 0, \tag*{▶[14a]}$$

which describes conditions within the conductor.

In simple situations a solution may be recognized by inspection. For instance, if in place of the shape given in Fig. 5 a rectangular parallelepiped is specified, and the x-y plane chosen parallel to the terminal planes a, b, then it is clear that both Eqs. 20a and 20b are zero or that $\mathcal{V}$ is a function of z alone. Equation 14a then becomes

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = 0. \tag{14b}$$

A solution to this equation is

$$\mathcal{V} = Mz + N. \tag{21}$$

in which M and N are arbitrary constants. It is also recognized that this simple solution can satisfy the stated boundary conditions ($\mathcal{V}$ constant

over the terminal surfaces a and b, and the gradient of $\mathcal{V}$ directed parallel to the lateral surface s). The field intensity is constant and equal to

$$\mathcal{E} = -\frac{\partial \mathcal{V}}{\partial z} = -M. \tag{22}$$

The potential difference between a and b is thus

$$V_{ab} = -M \times (\text{length from } a \text{ to } b), \tag{23}$$

and the total conduction current is

$$I_{ab} = -\gamma M \times (\text{area of cross section}). \tag{24}$$

The resistance between a and b is

$$R_{ab} = \frac{V_{ab}}{I_{ab}}. \tag{25}$$

The solution of Eq. 14a, carried out for the special case of Eq. 14b is very simple. On the other hand, when the geometry is not so simple, the solution generally demands a very elaborate method, detailed discussion of which is out of place here. In fact, the straightforward mathematical method of attack just outlined can be carried through only in relatively few cases where the geometrical configuration contains a certain degree of symmetry or uniformity which makes it possible to recognize at the outset the general character of the field configuration. For example, the conductor boundaries may have spherical or cylindrical or elliptic symmetry, in which case it may be possible to integrate Eq. 14a after transformation to polar or elliptic co-ordinates, that is, to such co-ordinate systems as fit the resulting field configuration. Otherwise the strictly rigorous mathematical procedure must be abandoned simply because, with the mathematical methods of analysis available at present, the formal process of integration of Laplace's equation cannot be carried through.

When the rigorous treatment fails, it is possible to resort to more approximate analytical, or to graphical methods, which are discussed in some detail in subsequent articles. In the meantime, the foregoing discussion suffices to show that the field-distribution problem is basic to the calculation of the resistance parameter and that the solution to this problem hinges upon the geometry of the configuration alone. The resistance parameter is thus fundamentally a geometric constant of the system, except for its dependence upon the physical conductivity of the medium involved.

6b. The capacitance parameter

In developing methods for evaluation of the capacitance parameter, the most general approach is to determine the capacitance between two conductors of arbitrary shape. When a potential difference is applied to these conductors, equal and opposite net charges are induced upon their surfaces. These charges ordinarily are distributed over the surfaces in a nonuniform manner which is determined by the geometrical shapes of the two conductors and their relative orientation.

If the applied voltage is constant, the resulting electric field surrounding the conductors is stationary and hence conservative. A potential function $\mathfrak{V}$ may again be assumed for the determination of the field. Each conductor surface is evidently an equipotential surface; hence the $\mathfrak{E}$ lines meet these surfaces at right angles everywhere. If in the surrounding space no charges exist, the potential function there again satisfies Laplace's equation. The *problem is thus essentially the same as the previous one and again depends entirely upon the geometrical configuration.*

Once the potential function is determined, the potential difference between the conductor surfaces is readily obtained. The surface density of charge σ can be found for any point on the surface from the relation

$$4\pi\sigma = \varepsilon\mathfrak{E}, \tag{26}$$

in which $\mathfrak{E}$ is the electric-field intensity at the point on the surface. The net charge on one surface then can be found by integration,

$$Q = \int_s \sigma ds. \tag{27}$$

The capacitance parameter is the ratio of this charge to the corresponding potential difference,

$$C = \frac{Q}{V}. \tag{28}$$

Alternatively the capacitance may, of course, be thought of as the total electric flux ψ streaming from one conductor to the other, divided by 4π times the potential difference,

$$C = \frac{\psi}{4\pi V}. \tag{29}$$

When the field distribution can be recognized by inspection of the given geometry, the capacitance is readily calculated. Otherwise one must again resort to approximations and graphical methods.

For systems having more than two conductors, the elementary con-

ception of capacitance between two bodies is by itself inadequate. Important examples of such systems are multiconductor transmission lines or cables, strings of suspension insulators on steel towers, the various arms and shields of bridge networks, and multielectrode vacuum tubes. By extensions of the idea of capacitance such systems can be represented by combinations of capacitance elements.[5]

6c. The inductance parameter

The general problem of inductance computation is to determine the inductance of a conductor system the current of which gives rise to a magnetic field within the conductors as well as in the surrounding medium. Since both fields must in general be considered, the present problem is more complicated in detail although fundamentally wholly analogous to the two foregoing. This added complexity can be eliminated from the present argument, and the analogies can thus be allowed to become more prominent, if the conductors are considered as having negligible cross sections compared to the surrounding space, for then the internal field plays an insignificant part. This is frequently the situation in practice.

Since a path carrying a steady current must be closed, it is helpful, for the present argument to think of the conductor system as a single closed loop of wire (with small cross section), although it is quite appropriate also to consider, for example, two short linear sections of wire carrying equal and opposite currents (a more closely analogous case to that considered for capacitance calculation), which are part of a longitudinally uniform system such as an electrical transmission line. The inductance of the short portion is then an increment of the total.

The basic problem again is to determine the field distribution in the region surrounding the current loop. If the current is constant, the magnetic field is stationary. The magnetic field, like the electric field in the previous problems, is conservative and may also be represented as the gradient of a scalar potential function — a magnetic potential. In connection with the conservative nature of the field, a caution must be mentioned which is peculiar to the present example. Conservatism of a field is tested by examining whether or not the line integral of the field inten-

[5] Representation in terms of *capacitance coefficients*, *partial* or *direct capacitances* is discussed in the reference volume, Ch. XVIII; also in James Clark Maxwell, *Treatise on Electricity and Magnetism* (3d ed.; Oxford: The Clarendon Press, 1892), I, 107–118; Alexander Russell, *A Treatise on the Theory of Alternating Currents* (2d ed.; Cambridge: at the University Press, 1914), Chs. v, vi; Edward Bennett and H. M. Crothers *Introductory Electrodynamics for Engineers* (New York: McGraw-Hill Book Company, Inc. 1926), Arts. 109 111; Leigh Page and N. H. Adams, Jr., *Principles of Electricity* (New York: D. Van Nostrand Company, Inc., 1931), Art. 19.

sity taken around a closed path vanishes. This process is identical with testing whether or not the potential difference between two points is independent of the path of the line integral used in its calculation. If this were not independent of the path, a potential function could not be defined uniquely. In the present example such a line integral does vanish except when the path incloses the current loop. In the latter case the value of the integral equals 4π times the inclosed current.

In spite of this fact, the problem involves the determination of a pure potential field in the surrounding space and can be attacked by defining a magnetic potential function $\mathcal{U}$ and writing Laplace's equation in terms of this function. The solution then again depends upon the success in finding functions which satisfy this equation together with the necessary boundary conditions.

With the magnetic field thus determined, the total flux ϕ inclosed by the loop is expressed in terms of a surface integral of the flux density $\mathcal{B}$ formed *over any convenient surface bounded by the loop, while the magnetomotive force* F causing the flux is given by a line integral taken around any path encircling the wire. These are, of course,

$$\phi = \int_s \mathcal{B} \cdot ds \qquad [30]$$

and

$$F = \oint \mathcal{H} \cdot d\ell = 4\pi I. \qquad [31]$$

The inductance parameter is the ratio of total flux to the current, or

$$L = \frac{\phi}{I} = \frac{4\pi\phi}{F}. \qquad [32]$$

An alternative method of calculating the inductance parameter, which is useful when the contribution from that portion of the field within the conductor is essential, is based upon energy considerations. The first step in this method again requires the determination of the magnetic-field distribution. Then by integrating the relation for the energy density over the entire volume occupied by the field the inductance can be calculated from

$$W_m = \int_v \frac{\mathcal{B}^2}{8\pi\mu} dv = \tfrac{1}{2}LI^2. \qquad [33]$$

This method avoids the necessity for calculating partial flux linkages (contributions from flux which links with only a portion of the total current),* correct formulation of which often requires extreme care.

* Article 10c, pp. 53–56.

7. A GRAPHICAL AID IN THE SOLUTION OF LAPLACE'S EQUATION

When the problem of parameter calculation is attacked from the basis of Laplace's equation, as discussed above, it is sometimes possible to obtain a solution graphically by plotting the field. By this is meant the drawing of lines (in suitably chosen places) which coincide in direction with the field intensity, and an orthogonal system of lines which represent the corresponding equipotential surfaces. This process is discussed in more detail with the aid of Fig. 7, which represents an enlargement of a differential portion of an electric field in a plane normal to an equipotential surface at some point.

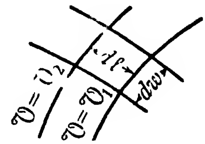


Fig. 7. Differential portion of a field map.

The lines marked $\mathcal{V}_1$ and $\mathcal{V}_2$ are equipotential lines in the plane of the paper and are traces of equipotential surfaces, since the field is three-dimensional. The orthogonal lines are lines of flux or current. The incremental distances dl between potential lines and dw between flow lines may be considered differential for the purpose of the present argument, although practically they must be finite. The differential increment in voltage between the potential lines is denoted by

$$dV = \mathcal{V}_2 - \mathcal{V}_1. \qquad [34]$$

On the equipotential surface the flow lines have spacings such as dw in the plane of Fig. 7 and other spacings, for example, dz, in the direction normal to the plane of the figure. That is, w and z may be considered as being orthogonal co-ordinates lying on the equipotential surfaces, and l the co-ordinate orthogonal to w and z in the flow direction.

The usefulness of the plot lies in letting each flow line represent a like increment in current, which may be denoted by dI. If the current density at the point in question is $\mathcal{J}$, then dI is expressible as

$$dI = \mathcal{J}dwdz = \gamma\mathcal{E}dwdz, \qquad [35]$$

in any consistent system of units. Since the magnitude of this current increment is quite arbitrary, it may be chosen equal to any convenient magnitude. Thus if it is chosen to let

$$dI = \gamma \text{ amp numerically}, \qquad [36]$$

it follows from Eq. 35 that

$$\mathcal{E} = \frac{1}{dwdz} \qquad \text{v/m numerically}, \qquad [37]$$

if the mks practical system of units is used;* that is, the electric-field intensity at any point is given by the density of flow lines at that point. This conclusion may be seen by recognizing that $dwdz$ equals the square meters of area on the equipotential surface allotted to a flow line, that is,

$$dwdz = \text{sq m/flow line}, \tag{38}$$

whence:

$$\frac{1}{dwdz} = \text{flow lines/sq m}. \tag{39}$$

Furthermore, since

$$\mathcal{E} = \frac{dV}{d\ell} \qquad \text{v/m} \tag{40}$$

it follows that

$$dV = \frac{d\ell}{dwdz} \qquad \text{v numerically}. \tag{41}$$

The increment in voltage dV between adjacent equipotential surfaces is, of course, constant over any such surfaces. In addition to this, it is very useful for the application of the field plot to parameter calculations to make the voltage increment between successive equipotential surfaces the same along the flow direction. In other words, dV is constant throughout the field plot.

In problems where the geometry is inherently uniform or symmetrical, Eq. 41 with dV constant leads to a uniformity in the ratios for the spacings $d\ell$, dw, and dz which greatly facilitates the sketching of the correct field plots. Such a plot, correctly carried through, constitutes the solution to Laplace's equation in conformance with the given geometrical boundaries and hence yields the field distribution from which a parameter calculation is readily made.

A longitudinally uniform system of linear conductors of cylindrical cross section is a commonly occurring example for which Eq. 41 may be further simplified. Two subcases must be distinguished according to whether the flow is longitudinal or transverse.

* In the following treatment, γ must be expressed either in units of a consistent system used throughout the problem or in units conforming with whatever mixed system of units is used. Otherwise it is necessary to apply a suitable conversion factor to obtain the final results in the units desired. In fact, in actual graphical work, conversion constants commonly are required anyhow in order to obtain a reasonable spacing of lines on the drawing. The same applies to ε and μ in analogous discussions regarding the computation of capacitance and inductance.

If the flow is longitudinal, this direction coincides with the ℓ co-ordinate. On account of the uniformity it is possible arbitrarily to let

$$d\ell = 1 \text{ m}, \tag{42}$$

$$dV = 1 \text{ v}, \tag{43}$$

and Eq. 41 then yields

$$dw\,dz = 1 \text{ sq m}. \tag{44}$$

Since there is no further relation to be satisfied, there is no reason why this last condition cannot be met by choosing

$$dw = dz = 1 \text{ m}; \tag{45}$$

that is, the flow lines may be spaced uniformly in both dimensions of any plane normal to the flow direction.

If the flow is transverse, then the z co-ordinate coincide. with the longitudinal direction. The spacing dz must evidently be a uniform one throughout the z direction and may be arbitrarily set equal to one meter for convenience. For such a system, Eq. 41 reads

$$dV = \frac{d\ell}{dw} = \text{constant} \tag{41a}$$

In the cross-sectional plane (the ℓ w plane) the field plot is hence carried out in such a way that the equipotential lines and flow lines intersect to form similar rectangles throughout. Unless the spacings $d\ell$ and dw are truly differential in magnitude, the rectangles in general have curved sides and are called *curvilinear rectangles*. They do, however, yield true rectangles after sufficient further subdivision. This fact is useful as a test in the making of freehand sketches of field maps. Since the constant voltage increment in Eq. 41a is arbitrary, the mechanism of sketching a field plot is further simplified by again choosing dV one volt, so that the condition on the spacing of lines becomes

$$\frac{d\ell}{dw} = 1. \tag{46}$$

This means that all rectangles become squares, or the larger ones become curvilinear squares; in other words, wherever oblongs instead of squares appear, the field plot is incorrectly drawn. The similarity of rectangles is much easier to recognize when they are squares. The guide in sketching the field is thus simple, although for complicated cross-sectional geometry the process of carrying through such a plot is still long and tedious.

When there is cylindrical symmetry with radial flow (normal to the

z axis), if n symmetrically spaced flow lines per unit length diverge

$$dwdz = \frac{2\pi r}{n}, \tag{47}$$

and Eq. 41 becomes

$$d\ell = dr = \frac{2\pi r}{n}. \tag{41b}$$

The spacing of concentric equipotential cylinders varies as their radii. As noted in the general derivation for longitudinal uniformity with transverse flow, it is possible to make the flow map in any cross-sectional plane consist of squares.

When there is spherical symmetry, with the flow in the radial direction, the flow lines diverge uniformly and are uniformly spaced on any spherical equipotential surface. The density of lines is then inversely proportional to the square of the radius of the surface, or if, altogether, n lines diverge,

$$dwdz = \frac{4\pi r^2}{n}, \tag{48}$$

and by again choosing dV one volt, Eq. 41 becomes

$$d\ell = dr = \frac{4\pi r^2}{n}. \tag{41c}$$

The spacing of equipotential surfaces varies as the square of their radii. For spherical symmetry it is not possible to make the flow map in a cross-sectional plane (any plane passing through the center of the spheres) consist of squares. There it is evident that dw and dz are equal; so in any cross-sectional plane the flow map consists of rectangles whose dimensions must fulfill the ratio

$$\frac{d\ell}{dw} = 2r\sqrt{\frac{\pi}{n}}, \tag{49}$$

which, for the choice of 4π lines for n, becomes

$$\frac{d\ell}{dw} = r. \tag{49a}$$

When the geometrical configuration exhibits rotational symmetry about an axial flow direction (for example, the electric-field map for two charged nonconcentric spheres), and r and θ are polar co-ordinates in a plane normal to this axis, then it is possible to set

$$dz = rd\theta, \tag{50}$$

and by again choosing dV one volt. Eq. 41 becomes

$$\frac{d\ell}{dw} = r d\theta. \quad [41d]$$

This may be further simplified by arbitrarily letting

$$d\theta = 1 \text{ radian}; \quad [51]$$

whence,

$$dz = r \quad [50a]$$

and

$$\frac{d\ell}{dw} = r. \quad [41e]$$

In any plane passing through the axis of rotation, the flow map consists of rectangles with varying ratio of length to width, this ratio depending upon the distance of the rectangle from the axis.

The result, Eq. 41e, is identical with Eq. 49a. This is as it should be, because the spherical symmetry with radial flow is a particular example of rotational symmetry. It should be observed, however, that these last two cases impose a condition on the drawing of the map which makes it almost hopeless to carry through a plot by means of a process of sketching alone. In other words, the usefulness of this graphical method of getting a solution to Laplace's equation is practically limited to situations involving longitudinal uniformity, for which

$$\frac{d\ell}{dw} = 1 \quad [46]$$

is the guiding principle. Any other configurations had best not be attacked by this method, even though it may be possible to formulate the necessary and sufficient conditions governing the line spacings.

8. Parameter calculation from a graphical flow map

When a current map has been drawn for a specific example according to the method of Art. 7, each flow line represents numerically γ amperes, while one volt is the increment between successive equipotential lines or surfaces. If between the given terminal surfaces there are m equipotential surface intervals and if over any cross section there are n current lines, the resistance parameter is

$$R = \frac{V}{I} = \frac{m}{\gamma n} \quad \text{ohms.} \quad [52]$$

When capacitance is being calculated, the flow lines represent electric

flux ψ and the equipotential surfaces are for electric potential, the same as when resistance is being calculated. Each line represents a flux of ε mks units numerically. Again, if there are n flow lines over any cross section, the total flux is

$$\psi = n\varepsilon = 4\pi Q, \tag{53}$$

where Q is the total terminal conductor charge. The capacitance parameter is then given by

$$C = \frac{Q}{V} = \frac{\psi}{4\pi V} = \frac{n\varepsilon}{4\pi m} \quad \text{farads.} \tag{54}$$

In the calculation of the inductance parameter, the flow lines represent *magnetic flux* ϕ *and the equipotential surfaces are for magnetic potential* $\mathcal{U}$. If each flux line represents μ webers numerically and if the map is constructed according to the methods of the preceding article (replacing $\mathcal{E}$ by $\mathcal{H}$ and $\mathcal{V}$ by $\mathcal{U}$), each increment between equipotential surfaces is one pragilbert. The total difference in magnetic potential is in this case usually taken around a closed path encircling the current which produces the flux. Hence the two " terminal " surfaces usually are the same surface. If there are m intervals between equipotential lines or surfaces as this closed path is traversed, the total magnetic potential difference is

$$U = \oint \mathcal{H} \cdot d\ell = m = 4\pi I = F, \tag{55}*$$

where I is the *total* current inclosed. If the total number of flow lines crossing any equipotential surface again is n, the inductance parameter is given by

$$L = \frac{\phi}{I} = \frac{4\pi\phi}{F} = \frac{4\pi n\mu}{m} \quad \text{h.} \tag{56}$$

If the total current is divided equally among N paths or turns and if the current per turn is denoted by I_t,

$$I = NI_t, \tag{57}$$

so that

$$F = 4\pi NI_t, \tag{58}$$

and

$$L = \frac{N\phi}{I_t} = \frac{4\pi N^2\phi}{F} = \frac{4\pi N^2 n\mu}{m} \quad \text{h.} \tag{59}$$

* The symbol $U = \int \mathcal{H} \cdot d\ell$ represents any difference of magnetic potential, analogous to V, which represents a difference of electric potential. The symbol $F \equiv \oint \mathcal{H} \cdot d\ell$ only when current is inclosed, termed magnetomotive force, is analogous to electromotive force E.

9. Analytical methods of obtaining the solution to field-distribution problems

When a space distribution of charge density is specified, the corresponding electric-field distribution is determined from the relation

$$d\mathcal{E} = \frac{1}{\varepsilon}\left(\frac{\rho_o}{r_{op}^3}\right) \mathbf{r}_{op} dv_o, \qquad [60]$$

in which ρ_o is the scalar charge density at some point o, dv_o is an element of volume at that point, and $\mathbf{r}_{op}$ the vector distance from o to a point p where the field intensity is desired.

Similarly from Ampère's rule, the magnetic-field distribution corresponding to a specified current-density distribution is determined from

$$d\mathcal{H}_p = \left(\frac{\mathcal{J}_o \times \mathbf{r}_{op}}{r_{op}^3}\right) dv_o, \qquad [61]$$

in which $\mathcal{J}_o$ is the vector current density at some point o, dv_o is an element of volume at that point, and $\mathbf{r}_{op}$ the vector distance from o to a point p where the field intensity is desired.

In order to be able to use these relations for the determination of an electric- or a magnetic-field distribution, the charge or current distributions must, of course, be known. In many practical problems of parameter calculation, these distributions can be determined exactly or with sufficient approximation by inspection. For example, in an electrostatic problem involving spherical or circular cylindrical symmetry of the conductor surfaces, the charge distribution over these surfaces is evidently uniform (except for short cylinders or near the ends of long cylinders). Similarly, in a magnetostatic problem involving straight homogeneous conductors of uniform cross section, the current density is uniform over this cross section and is uniform also in the axial direction in accordance with the principle of conservation of charge. In such problems or similar ones wherein advantage can be taken of symmetry, Gauss's theorem,

$$\oint_s \mathfrak{D} \cdot ds = 4\pi \int_v \rho dv, \qquad [62]$$

or Ampère's circuital law,

$$\oint \mathcal{H} \cdot d\ell = 4\pi \int_s \mathcal{J} \cdot ds, \qquad [63]$$

may be convenient to apply. In less simple cases, the charge or current distributions can sometimes be obtained by the use of one of several ingenious methods of reasoning, as illustrated by the examples on pp. 67 to 70.

Whenever the charge or current distribution is known, the problem of

determining the field distributions and the subsequent calculation of resistance, capacitance, or inductance parameters is preferably attacked by the use of Eqs. 60 or 61 or simpler variations suited to the particular problem. It should, however, be recognized that these relations may in themselves be shown to be consistent with general integrals of Laplace's equation* for the corresponding geometrical configuration, and that this method of obtaining the resulting field distributions is merely a short-cut way of solving Laplace's equation — not an independent alternative process.

10a. Illustrative examples of parameter calculation; introductory remarks

In the following discussion of examples illustrating graphical and analytic methods for the calculation of resistance, capacitance, and inductance, various typical problems occurring in practice are presented in more or less idealized form. A certain amount of idealization or simplification is always found necessary in the attack on any practical problem in order to put it in a form which yields to sufficiently simple treatment, and yet embodies the significant features unique to that problem. The necessary amount or degree of idealization is sometimes slight, but at other times it effects significant departures from the actual physical situations. Yet many such analyses are useful, and even the crudest approximations are frequently sufficient, provided that a proper interpretation is given to the results obtained.

The various practical illustrations are grouped or classified according to the type of geometrical symmetry or uniformity inherent in their associated field or flow pictures. Thus, for instance, the first group of problems comprises those which exhibit longitudinal uniformity with flow in the longitudinal direction. Among this group are such problems as the calculation of resistance of uniform linear conductors with arbitrary cross section, the calculation of self-inductance of solenoidal coils or of coils having toroidal form when the radius of the toroid is large enough to make the curvature negligible (an approximation commonly made), and the calculation of the capacitance between parallel plates. All these problems exhibit essentially the same general flow picture, and the solution to one of them is, with slight modifications, the solution to the others also. A knowledge of this fact is obviously useful. At the same time, this procedure very materially condenses the discussion of a large variety of examples.

* Or Poisson's equation within the region occupied by charge. For the determination of the magnetic field this situation is best treated in terms of a vector potential which is related to current density by Poisson's equation, just as the scalar electric potential is related to charge density. This is treated in the reference volume, Chs. IV and XIII.

10b. Longitudinally uniform system of arbitrary cross section with flow in the longitudinal direction

The geometry is illustrated in Fig. 8. The cross section is shown on the left and a typical longitudinal section on the right. As pointed out in Art. 7, the flow map for this case is readily drawn, since the spacings $d\ell$, dw, and dz are all equal.

If this representation is for a cylindrical conductor of arbitrary cross section and the resistance parameter for a length of ℓ meters is desired, then, with the mks system of units, each interval $d\ell$ is one meter and represents a potential difference of one volt, and each flow line represents

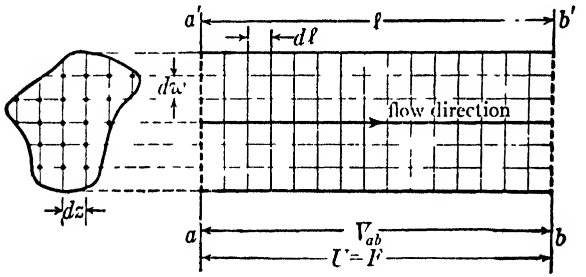


Fig. 8 Map representing longitudinal flow in longitudinally uniform system.

γ amperes. If the cross-sectional area is s square meters, there are s flow lines. Hence from Eq. 52 the resistance parameter is

$$R = \frac{m}{\gamma n} = \frac{\ell}{s\gamma} \qquad \text{ohms.} \qquad [52a]$$

The use of the mks system directly for carrying out actual plotting in a situation taking $d\ell$, dw, and dz each as one meter and using γ in mhos per meter cube obviously is rather absurd, because few practical conductors have a cross section of one square meter, and the plotting of fractional flow lines is rather hypothetical. Ordinarily it is more convenient, if actual plotting is carried out, to use the absolute electromagnetic system of units, involving centimeters and abmhos per centimeter cube, and giving the result in abohms; or to use the mixed practical system involving centimeters and mhos per centimeter cube, and giving the result in ohms. If it is still more convenient, arbitrary constants can be introduced to make the intervals $d\ell$, dw, and dz whatever is convenient for plotting. However, since in this simple problem the actual drawing of lines is superfluous anyhow, except for illustration, the further discussion of units and conversion constants is postponed for the examples wherein plotting

may actually be carried out as a practical means of obtaining a result. In this example, instead of actually drawing flow lines and equipotential lines, the obvious thing to do is to compute the length and cross-section area of the specimen. These same comments apply to the calculation of capacitance and inductance which follow immediately.

The longitudinal-flow picture, Fig. 8, may be thought of also as representative of an electric field within a dielectric having an arbitrary cross-sectional shape. *The equipotential surfaces a–a' and b–b' may be identified with the* surfaces of a pair of conducting plates. The problem is that of determining the capacitance parameter of this pair of plates. Unless the dielectric permittivity of the intervening medium is very large compared to that of the surrounding space, the field outside the medium cannot be neglected in contributing toward the value of the resulting capacitance. However, if the length ℓ is small compared to the cross-sectional dimensions, the field is principally confined to the intervening medium, even though its dielectric permittivity is not large compared to that of the surrounding space. The geometry of parallel-plate condensers used in practice conforms to this latter condition, and the flow picture is usually assumed to be uniform as pictured in Fig. 8, even when the mediums within and without are alike. Hence from Eq. 54 the capacitance parameter is

$$C = \frac{\varepsilon n}{4\pi m} = \frac{\varepsilon s}{4\pi \ell} \qquad \text{farads.} \tag{54a}$$

In practice, in order to obtain sufficiently large values of capacitance with reasonable sizes, the separation ℓ between conductor surfaces is made as small as possible considering the contemplated applied voltage and breakdown strength of the dielectric. Hence the approximation involved in this result is usually sufficiently good. It should not be forgotten, however, that the presence of an external field as well as a modification of the flow picture in the vicinity of the edges of the plates does modify the actual parameter value to some degree.

The capacitance parameter for the parallel-plate condenser also can be computed analytically readily. With the assumptions mentioned above, the charge distribution over the surfaces of the terminal plates may be considered uniform.

By Gauss's theorem, the flux at any cross section between the plates a–a' and b-b' is

$$\psi = \mathfrak{D}s = 4\pi\sigma s = \varepsilon\mathfrak{E}s. \tag{64}$$

Hence,

$$V_{ab} = \mathfrak{E}\ell = \frac{4\pi\sigma\ell}{\varepsilon}, \tag{65}$$

and

$$Q = \sigma s. \tag{66}$$

Therefore

$$C = \frac{Q}{V} = \frac{\varepsilon s}{4\pi l} \quad \text{farads.} \tag{54b}$$

The analytical computation can be carried out by integration of Eq. 60, using either rectangular or polar co-ordinates. The latter is carried out as an instructive example, because it gives some indication of the manner in which the field intensity is related to the charge distribution.

The condenser plates, portions of which are shown in Fig. 9, are assumed to carry the uniform surface charge densities $\pm\sigma$. Since the effects of the two plates linearly superpose, the contributions to the resulting field intensity at a point p may be calculated separately for each plate. The field intensity $d\mathcal{E}_p$ due to the differential surface area on the lower plate is

$$d\mathcal{E}_p = \frac{\sigma r dr d\phi}{\varepsilon(r^2 + h^2)}. \tag{60a}$$

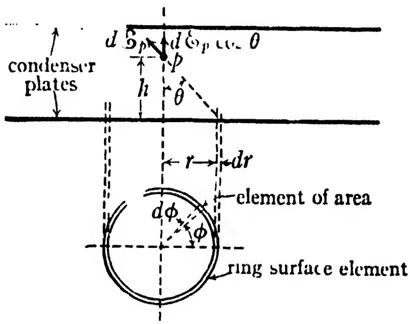


FIG. 9 For derivation of capacitance of parallel-plate condenser, using ring elements.

In integrating this with respect to ϕ, it should be noted that the horizontal components cancel, so the resultant component of field intensity due to the ring-surface element becomes

$$(d\mathcal{E}_p)_{ring\ surface} = \frac{\sigma r dr \cos\theta}{\varepsilon(r^2 + h^2)} \int_0^{2\pi} d\phi = \frac{\sigma 2\pi r dr \cos\theta}{\varepsilon(r^2 + h^2)}. \tag{60b}$$

But

$$\cos\theta = \frac{h}{\sqrt{h^2 + r^2}}; \tag{67}$$

so the resultant field intensity due to the uniform charge on the lower plate surface is

$$\mathcal{E} = \frac{2\pi\sigma h}{\varepsilon} \int \frac{r dr}{(r^2 + h^2)^{3/2}}, \tag{68}$$

in which the limits of integration are for the moment disregarded. In fact, the foregoing integral is capable of yielding the desired result only if the

plate is circular and the point p is in the center, whereas the actual plate shape is considered to be arbitrary and the section shown in Fig. 9 is any excised portion.

Meanwhile it is instructive to evaluate the integral Eq. 68 between limits zero and R where R is, for the moment, any finite value. To facilitate the discussion it is further useful to introduce the variable

$$x = \frac{r}{h}, \tag{69}$$

in terms of which Eq. 68 then becomes

$$\mathcal{E} = \frac{2\pi\sigma}{\varepsilon} \int_0^{R/h} \frac{x\,dx}{(1 + x^2)^{3/2}}. \tag{68a}$$

A plot of the integrand versus x is shown in Fig. 10. This shows the relative effect of contributions to the field strength $\mathcal{E}$ coming from charges

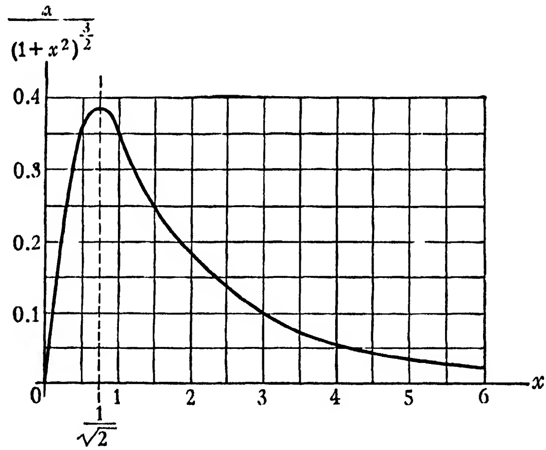


FIG. 10. Plot representing contribution of charged ring elements to field intensity at p, Fig. 9.

on ring elements at various radial distances measured from the point immediately below p. Thus the charge on the ring surface with radius $h/\sqrt{2}$ contributes most strongly, while the charge on more remote ring surfaces contributes in rapidly decreasing proportion. At larger distances (x greater than four or five), the relative contribution drops off very nearly as $1/x^2$ or h^2/r^2. This means that the field strength is determined essentially by the surface charge in the more immediate vicinity of the point in question; so for points which are at least several times the plate spacing distant from the boundaries of the plates, the field strength does

not depend upon the size or shape of the plates to any appreciable extent.

This may be made somewhat more precise by evaluating the integral of Eq. 68a, which yields

$$\mathcal{E} = \frac{2\pi\sigma}{\varepsilon}\left(1 - \frac{1}{\sqrt{1 + \frac{R^2}{h^2}}}\right), \tag{68b}$$

or, to a sufficient approximation,

$$\mathcal{E} \approx \frac{2\pi\sigma}{\varepsilon}\left(1 - \frac{h}{R}\right). \tag{68c}$$

At points near the edges of the plates where h/R is not a negligible quantity, the field strength is not uniform (independent of h) even for uniform surface distribution of charge on the plates. Actually the surface density of charge near the edges is not uniform. For this reason it is useless to attempt a more precise interpretation of the present analysis. Edge effect can be taken into account more precisely for a parallel-plate condenser by methods of flux plotting outlined in subsequent articles. If the plates actually are circular, rotational symmetry exists, but, as pointed out in Art. 7, the conditions which this imposes make plotting very laborious. If the plates are rectangular and if at least one plate dimension is large compared to the separation, a precise plot can be made except at the corners (or except at the short edges for a long, narrow rectangle) by taking the longitudinal direction coincident with one edge of a plate, and thus reducing the problem to one of longitudinal uniformity with transverse flow. Such a plot is illustrated in Art. 10c, Fig. 23.

If all edge effects are neglected, the field strength is again found to be uniform between the plates and, because of the effect of charges on *both* plates, is

$$\mathcal{E} = \frac{4\pi\sigma}{\varepsilon}, \tag{68d}$$

from which the capacitance as calculated from Eqs. 65 and 66 is readily obtained.

The longitudinal-flow picture, Fig. 8, may represent also a magnetic field if the cylindrical surface is thought of as an indefinitely long, thin conducting sheet carrying a uniform current in the circumferential direction. The flux density and hence the field intensity are uniform within the cylinder. The fact that there is no field outside the cylinder is explained by the condition of longitudinal uniformity; that is, if there is any field outside, it is uniform; and the map consists of regular squares like the map inside. But this means that the field intensity is the same outside as inside and is independent of the distance from the current

paths. Since only a zero field intensity can satisfy this last requirement, the flow is entirely within the current sheet. From Eq. 56 the inductance parameter is

$$L = \frac{4\pi\mu n}{m} = \frac{4\pi\mu s}{\ell} \quad \text{h.} \tag{56a}$$

In practice, a current sheet of this kind seldom occurs. Instead of actually existing as a current sheet of i amperes per unit of length, the current is commonly confined to the wire of a closely wound helix as sketched in longitudinal section in Fig. 11. This configuration, called a single-layer solenoid, may be thought of as very closely approximating a current sheet having circular cross section, so far as the net field is concerned. If there are N turns in the length ℓ, the total current for this length is divided into N equal parts. Hence from Eq. 59 the inductance parameter for this length becomes

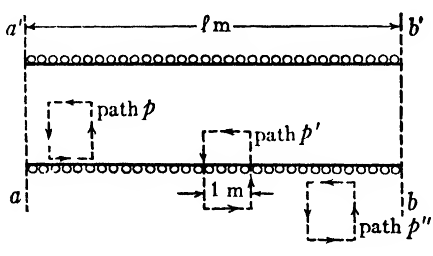


FIG. 11. Single-layer solenoid.

$$L = \frac{4\pi\mu s N^2}{\ell} \quad \text{h.} \tag{59a}$$

It should not be forgotten that in the derivation of the field map for this case, the system is assumed infinitely long, or in the form of a closed ring (torus) of radius so large that the portion under consideration is essentially straight. When a solenoidal winding of finite length is considered, the flow map is altered, principally in the vicinity of the ends of the solenoid. This alteration is considered in connection with a subsequent illustration. Equation 59a can be applied to such a case with only approximate results, the approximation becoming better as the ratio of length to diameter of the solenoid increases.

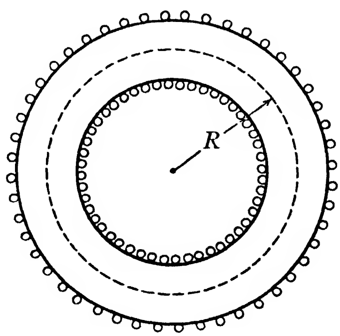


FIG. 12. Single-layer torus.

When the solenoid closes upon itself so as to form a toroidal winding of mean radius R with a total of N turns as sketched in Fig. 12, Eq. 59a (with ℓ equal to the mean circumferential dimension, $2\pi R$) applies with sufficient accuracy for many practical applications provided R is at least

several times the cross-sectional diameter. Otherwise the flow map for this case must be redrawn as discussed in Art. 7 for a system having rotational symmetry. The uniform squares of Fig. 8 no longer apply, and the inaccuracy of Eq. 59a, therefore, is apparent.

The inductance parameter for the configurations just discussed can very readily be determined also by analytic instead of graphical means, since in these instances the current distribution is assumed to be uniform and thus known beforehand. Instead of proceeding from the Ampère-rule relation, Eq. 61, however, the work in this case may be considerably shortened by recognizing the consequence of the longitudinal uniformity inherent in the system. Thus it may be said at the outset that the magnetic-field intensity $\mathcal{H}$ can be in the longitudinal direction only; that is, $\mathcal{H}$ has no transverse components anywhere. By consideration of the line integral of $\mathcal{H}$ formed around a path p which incloses no current, Fig. 11, it can be shown that the field intensity within the solenoid is the same at all points. It is the same because the transverse portions of the path contribute nothing, and since the value of the integral is zero, the longitudinal portions must contribute equal and opposite shares. Application of the same reasoning to the external path p'' leads to the identical conclusion, namely, that the field intensity is the same *at all points outside* the solenoid no matter how far away from the current-carrying conductors. Inspection of Ampère's rule shows that this constant value of the field intensity external to the solenoid must be zero. Finally, by application of the line integral of $\mathcal{H}$ to the path p', which is one meter in length and hence incloses i amperes, the field intensity within the solenoid is obtained:

$$\mathcal{H} = 4\pi i \qquad \text{oersteds.} \tag{70}$$

The total flux at any cross section is

$$\phi = \mu\mathcal{H}s = 4\pi\mu s i, \tag{71}$$

and the inductance parameter for a length of l meters (which carries li amperes) is

$$L = \frac{\phi}{li} = \frac{4\pi\mu s}{l} \qquad \text{h,} \tag{56b}$$

which is the same value as obtained from the flow map.

10c. Longitudinally uniform systems with flow in transverse directions

The first configuration to be considered, illustrated in cross section in Fig. 13, consists of two concentric circular cylindrical conductors which are longitudinally uniform. The flow lines may be taken as the equally

spaced radii, and the system of equipotential surfaces as the concentric cylinders, that is, concentric circles as viewed in the cross-sectional plane. This general character of the flow map is recognized by inspection because of the circular and concentric symmetry of the conductor boundaries and because of the well-known fact that concentric circles and their radii intersect at right angles everywhere.

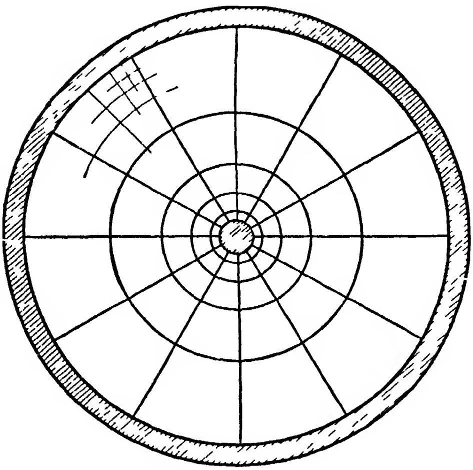


FIG. 13 Map of field between concentric cylinders

According to Art. 7, flow maps for systems having longitudinal uniformity and transverse flow are best drawn so as to yield squares. Though in drawing the flow map of Fig. 13, it may not be immediately obvious how to determine the relative spacing of radii and circles so as to form curvilinear squares, the condition of the spacings which yields true curvilinear squares in reality is readily derived. When the subdivision is thought of as carried to the limit of differential elements, the picture should be composed of true squares. In Fig. 13 one of the curvilinear squares in the upper left-hand portion is further subdivided to illustrate what may not be evident from the larger ones, namely, that true squares are ultimately approached. If polar co-ordinates are used to describe analytically the system of trajectories which is involved in the plot, using the symbols r for radius and θ for angle, differential radial and circular increments are expressed respectively by the quantities dr and $rd\theta$. If these are to be the sides of the squares, then the condition to be satisfied is

$$dr = rd\theta. \qquad [72]$$

Differential angular and radial increments are related by

$$d\theta = \frac{dr}{r}, \qquad [72a]$$

and hence larger increments are related by the corresponding integrals

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{r_1}^{r_2} \frac{dr}{r}, \qquad [72b]$$

which yields for the larger increments

$$(\theta_2 - \theta_1) = \ln \frac{r_2}{r_1}. \qquad [72c]$$

If there are to be n uniformly spaced flow lines, it follows that

$$(\theta_2 - \theta_1) = \frac{2\pi}{n}, \qquad [72d]$$

and the ratios of the radii of successive circles must satisfy the relation

$$\frac{r_2}{r_1} = \epsilon^{2\pi/n}. \qquad [73]$$

In Fig. 13, n is arbitrarily chosen equal to 12, so Eq. 73 becomes

$$\frac{r_2}{r_1} = \epsilon^{0.5236} = 1.6875. \qquad [73a]$$

The inner radius of the outer conductor in Fig. 13 is 7.60 centimeters. Hence the radius of the next smaller circle is

$$\frac{7.60}{1.6875} = 4.50 \text{ cm}; \qquad [74]$$

the next is

$$\frac{4.50}{1.6875} = 2.67 \text{ cm}, \qquad [75]$$

and so on, giving the series of values

7.60; 4.50; 2.67; 1.58; 0.938; 0.556 cm.

In the present illustration the radius of the inner conductor is arbitrarily chosen equal to that of the smallest of this series of values so as to make the number of intervals between equipotential surfaces come out a whole number. If the inner radius is not one of the series of values, a possible method of procedure is to choose n larger and obtain a new series of more closely spaced values. If n is made large enough, the radius of the inner conductor finally differs inappreciably from one of the members of the series radii. This is indeed the idea basic to the flow-map method of attack, namely, that the subdivision be carried to a point where any remaining fractional division may be dropped without exceeding an error consistent with the accuracy of the plot. Such continued subdivision is, however, cumbersome and time consuming.

In a situation like the present one it may easily be avoided. For example, if the radius of the inner conductor is 0.50 centimeter instead of

0.556 centimeter, according to Eq. 72c this last interval in radii implies an angular increment of

$$(\theta_2 - \theta_1) = \ln \frac{0.556}{0.500} = 0.1059 \text{ radian.} \qquad [76]$$

The uniform angular increment of Fig. 13 with 12 uniformly spaced flow lines, however, is

$$\frac{2\pi}{12} = 0.5237 \text{ radian.} \qquad [77]$$

Hence the last interval between the radii of 0.556 centimeter and 0.500 centimeter corresponds to the fraction

$$\frac{0.1059}{0.5236} = 0.2026 \qquad [78]$$

of a normal radial interval. The total number of intervals between equipotential surfaces for a number of flow lines is thus readily determined whether it comes out a whole number or not.

If the annular space between the cylindrical conductors of Fig. 13 is filled with a homogeneous conducting material of conductivity γ, the conductance between the cylindrical surfaces per unit of axial length is, from Eq. 52,

$$G_u = \frac{n\gamma}{m} = \frac{12\gamma}{5} = 2.40\gamma \qquad \text{mhos/m} \qquad [52b]$$

if γ is in mks units. However, in this example the units in which the squares are laid off have no influence on the result; only the ratio of numbers of lines is of consequence. Hence the units in which the result is expressed depend solely on the units of γ. That is, if γ is in abmhos per centimeter cube, the result is in abmhos per centimeter; if γ is in mhos per centimeter cube, the result is in mhos per centimeter. A similar comment applies to the calculations of capacitance and inductance which follow immediately.

If the annular space is occupied by a dielectric of permittivity ε, the capacitance parameter per unit length is given by the same flow map, in which the radial lines represent electric flux instead of current. From Eq. 54,

$$C_u = \frac{n\varepsilon}{4\pi m} = \frac{12\varepsilon}{4\pi \times 5} = 0.1910\varepsilon \qquad \text{farad/m} \qquad [54b]$$

if ε is in mks units.

Lastly, this same flow map may be for a magnetic field due to equal and opposite currents in the two conductors in the axial direction. The

concentric circles then are the flow lines and the equipotential surfaces are planes passing through the conductor axes, represented by the radial lines in the cross-sectional view of Fig. 13. The map is the same whether the flow is radial, as in the two preceding parameter calculations, or circular, as for the magnetic field associated with axial currents.

If the currents are assumed to be confined to the adjacent conductor surfaces (as they frequently are on account of skin effect),[6] the magnetic field is confined to the intervening space. From Eq. 56, the inductance parameter per unit length is

$$L_u = \frac{4\pi\mu n}{m} = \frac{4\pi\mu \times 5}{12} = 5.236\mu \qquad \text{h/m} \tag{56b}$$

if μ is in mks units. Here n is equal to the number of flow intervals and m equal to the number of intervals between radial lines taken around any flow line.

It is interesting that in this example

$$\frac{C_u}{G_u} = \frac{\varepsilon}{4\pi\gamma}, \tag{79}$$

$$\frac{R_u}{L_u} = \frac{1}{4\pi\mu\gamma}, \tag{80}$$

$$L_u C_u = \varepsilon\mu. \tag{81}$$

This condition is characteristic of all systems of conductors having longitudinal uniformity and transverse flow* as may be generally appreciated from a consideration of the fact that in the flow maps for such systems the current or flux lines for the electric field are the equipotential lines for the magnetic field. These relations are useful in practice, since systems of the kind discussed in the present article exhibit the type of configuration commonly used for transmission lines and cables.

The correctness of Eqs. 56b, 80, and 81, as pointed out, assumes that no current and hence no magnetic field penetrates inside the conductor surfaces. For stationary currents this assumption does not hold; these currents are uniformly distributed over the conductor cross sections. The modification in the inductance value is usually slight because the external field is by far the larger share of the total. The contribution to the inductance parameter due to internal flux linkages can be calculated analytically, from a straightforward application of Ampère's rule; but, since the usual practical application is to nonstationary currents, Eq. 56b

[6] This article, p. 52; Reference volume, Ch. XV; L. F. Woodruff, *Principles of Electric Power Transmission* (2d ed.; New York: John Wiley & Sons, 1938), Ch. iii.

* Equation 79 is true also for longitudinal flow.

sometimes gives a result nearer the correct value than that to which a correction due to a static internal field is added.

The parameter calculations for the present example can be made on an analytic basis which parallels the graphical procedure closely and yields formulas which are more generally useful than the numerical answers for a specific case.

For the calculation of the conductance parameter, it is again necessary at the outset to know the general character of the flow map and, in particular, to know that equipotential surfaces are circular concentric cylinders. In Fig. 14, r_2 is the inner radius of the outer conductor, and r_1 is the radius of the inner conductor. Since the equipotential surfaces are concentric cylinders, the net resistance of a unit length evidently can be obtained by integrating the differential contributions due to such shells as the one from r to $r + dr$, between the limits r_1 and r_2. The boundaries of such differential elements must be equipotential surfaces; otherwise the series combination is meaningless. Since the resistance of an irregularly shaped conductor can be calculated by an integration process only after the equipotential surfaces have been determined, the field must be mapped even for the starting point of an analytic method. The integral is

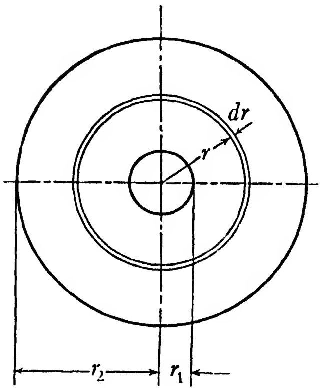


FIG. 14. Differential shell in solid cylinder.

$$R_u = \int_{r_1}^{r_2} \frac{dr}{2\pi\gamma r}; \qquad [82]$$

the integrand is the resistance of the differential shell, calculated as for the linear flow to which Eq. 52a applies. For a differential length, such as dr in the present instance, this is evidently correct in a limiting sense.

The evaluation of Eq. 82 yields

$$R_u = \frac{1}{G_u} = \frac{1}{2\pi\gamma} \ln \frac{r_2}{r_1} \qquad \text{ohms for one meter} \qquad [83]$$

if γ is in mks units. The analytic result of Eq. 83 is more useful, however, and shows, for example, that the resistance depends only upon the ratio of the radii and not upon their absolute values. By substitution of num-

bers for r_1 and r_2

$$G_u = \frac{1}{R_u} = \frac{2\pi\gamma}{\ln \dfrac{7.60}{0.556}} = 2.40\gamma \quad \text{mhos/m,} \tag{84}$$

as for *Eq. 52b.*

In calculating the capacitance parameter analytically, either of two methods can be used. One is to do for C what was just done for R, namely, consider the net capacitance as the series combination of the capacitances of differential cylindrical shells. For this purpose it is, of course, better to work in terms of reciprocal capacitance or elastance. The other method is to determine the electric-field distribution by assuming a uniform distribution of charge on the conductor surfaces. This assumption is justified by the symmetry of the system.

The fundamental method of calculating the electric-field intensity $\mathcal{E}$ is to utilize Eq. 60 and integrate over the entire conductor surfaces. This is extremely tedious although probably very interesting and enlightening. Another way of getting $\mathcal{E}$ is to show first that the charge on the outer conductor produces no field intensity within the region completely surrounded by that conductor, on the authority of Faraday's ice-pail experiment, or by application of Gauss's law to an internal closed surface, considering only the surrounding charge. Hence only the charge on the surface of the inner conductor need be considered. Owing to the radial and concentric cylindrical symmetry, the field between the conductors is the same as though the outer conductor were removed and the charge on the inner conductor were concentrated along a filament coinciding with the axis of the inner conductor.

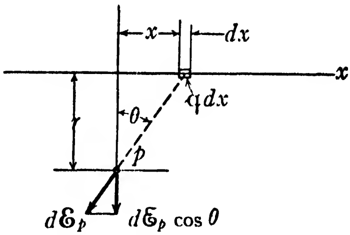


FIG. 15. Analysis of electric field due to uniformly charged long filament.

In Fig. 15 a uniform filamental charge distribution of q coulombs per meter is along the x axis. The net field intensity at the point p evidently is given by the integral

$$\mathcal{E}_p = \int_{-\pi/2}^{\pi/2} d\mathcal{E}_p \cos\theta, \tag{85}$$

since the horizontal components cancel. For the evaluation of Eq. 85 it is necessary first to obtain

$$d\mathcal{E}_p = \frac{q\,dx}{\varepsilon(r^2 + x^2)} \tag{60c}$$

by using Eq. 60, and to note that

$$\cos\theta = \frac{r}{\sqrt{r^2 + x^2}}. \tag{86}$$

Then,

$$\mathcal{E}_p = \int_{-\infty}^{\infty} \frac{q r dx}{\varepsilon(r^2 + x^2)^{3/2}} = \frac{q}{\varepsilon r}\int_{-\pi/2}^{\pi/2} \cos\theta d\theta = \frac{2q}{\varepsilon r} \quad \text{v/m.} \tag{85a}$$

From inspection of the integrands in these expressions it should be *observed in passing that only the charge in the more immediate vicinity* of the point p contributes materially to the field intensity.

A third way of obtaining the field intensity is to apply Gauss's law to a cylindrical surface of radius r one meter long, recognizing from the symmetry that the field intensity is uniform circumferentially and that hence

$$\frac{4\pi q}{\varepsilon} = 2\pi r\mathcal{E}, \tag{87}$$

from which Eq. 85a follows.

The potential difference between the inner and outer conductors is readily found:

$$V = \frac{2q}{\varepsilon}\int_{r_1}^{r_2} \frac{dr}{r} = \frac{2q}{\varepsilon} \ln \frac{r_2}{r_1} \quad \text{v} \tag{88}$$

if ε is in mks units. Since the charge per unit length is q coulombs, the capacitance parameter per unit length is

$$C_u = \frac{q}{V} = \frac{\varepsilon}{2 \ln \dfrac{r_2}{r_1}} \quad \text{farads/m.} \tag{89}$$

By substitution of numbers for r_1 and r_2

$$C_u = \frac{\varepsilon}{2 \ln \dfrac{7.60}{0.556}} = 0.1910\varepsilon \quad \text{farad/m} \tag{90}$$

as for Eq. 54b.

Finally, the inductance parameter can be calculated analytically. The magnetic-field intensity can be obtained from the use of Ampère's rule, Eq. 61, in a manner similar to that in which $\mathcal{E}$ is derived in Fig. 15. In fact the integrals are entirely similar, and one sees clearly by this method that the current in the immediate vicinity of the point in question contributes mainly to the result. From the symmetry, it follows that foı

the determination of the magnetic field outside the inner conductor, a current I may be thought of as concentrated along the axis. The current in the outer conductor produces no field intensity within the annular space, as is evident from Ampère's circuital law. The magnetic-field intensity at any point between the conductors is

$$\mathcal{H} = \int_{-\infty}^{\infty} \frac{Irdx}{(r^2 + x^2)^{3/2}} = \frac{I}{r}\int_{-\pi/2}^{\pi/2} \cos\theta d\theta = \frac{2I}{r}. \qquad [91]$$

Alternatively, by direct application of Ampère's circuital law, the familiar Biot-Savart law, Eq. 91 can be obtained immediately from

$$2\pi r\mathcal{H} = 4\pi I \qquad [92]$$

because, owing to the symmetry, $\mathcal{H}$ is uniform circumferentially.

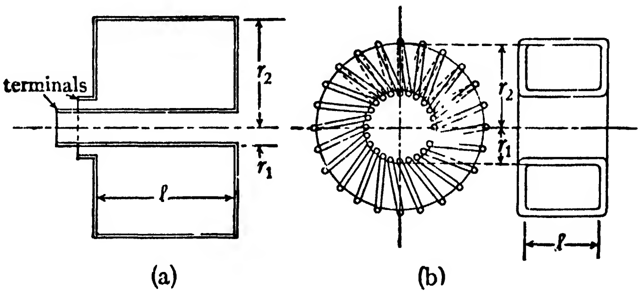


FIG. 16. Single-turn torus — Multiturn torus.

The flux per unit length between the two cylindrical conductor surfaces is

$$\phi_u = \int_{r_1}^{r_2} \mu\mathcal{H}dr = 2\mu I\int_{r_1}^{r_2}\frac{dr}{r} = 2\mu I \ln\frac{r_2}{r_1} \qquad \text{webers/m,} \qquad [93]$$

and hence the inductance parameter per unit length is

$$L_u = \frac{\phi}{I} = 2\mu \ln\frac{r_2}{r_1} \qquad \text{h/m} \qquad [94]$$

if mks units are used. By substitution of numbers for r_1 and r_2

$$L_u = 2\mu \ln\frac{7.60}{0.556} = 5.23\mu \qquad \text{h/m,} \qquad [95]$$

as for Eq. 56b.

Figure 16 shows two additional practical inductance configurations to which the present analysis applies. The derivation is made by assuming the conductors to be infinitely long, but the flow picture, so far as the magnetic field is concerned, is not altered if at any cross section a conduct-

ing wall is inserted which conducts the current radially from the inner to the outer conductor, as in the configuration shown in part (a) of the figure. This can be seen from the fact that the field is still uniform along any concentric cylindrical path, and the current in the wall in no way affects the value of the line integral taken around such a path. Hence the inductance formula for the single-turn torus shown in Fig. 16a is simply Eq. 94 multiplied by the length ℓ of the torus, or

$$L = 2\ell\mu \ln \frac{r_2}{r_1} \qquad \text{h} \tag{94a}$$

if mks units are used. Such a single-turn torus is found to have relatively a small resistance loss at high frequencies[7] as compared with ordinary coils, and is commonly used at such frequencies where only small values of inductance are required.

For the multiturn torus shown in Fig. 16b the detailed configuration of the field is quite complicated on account of the many possible flux paths encircling one or more of the single wires. Nevertheless, a good approximation to the inductance of such a coil is obtained by multiplying Eq. 94a by the square of the number of turns, according to the discussion preceding Eq. 59. Thus if the torus has N turns, the inductance is

$$L = 2\ell N^2\mu \ln \frac{r_2}{r_1} \qquad \text{h} \tag{94b}$$

if mks units are used.

The multiturn torus may be wound with a flat strip conductor instead of with wire, or may be obtained by cutting a helical slit in the single-turn torus in such a way as to finish at the starting point. For such a multiturn torus Eq. 94b is more nearly accurate because the field configuration departs very little from that for the single turn, especially if the slit is narrow. In the construction of such coils care must of course be taken to preserve the circular symmetry and longitudinal uniformity. This at the same time materially reduces the resistance losses, which at high frequencies are lowest in configurations whose conductor surfaces are everywhere parallel to the net flow direction.

The second case considered in this article is that of two identical parallel circular cylindrical conductors which are uniform in the longitudinal dimension. Figure 17 shows the appearance of the flow map in the cross-sectional plane. This can, of course, be obtained purely by graphical means, by noting that the circular cylindrical conductor surfaces must belong to one of the two families of orthogonal loci and that these loci must intersect so as to form similar rectangles (preferably squares)

[7] F. E. Terman, "Some Possibilities for Low Loss Coils," *I.R.E. Proc.* XXIII (1935), 1069–1075.

throughout. This method, however, is long and tedious, and in the end does not yield a basis from which to draw any more generally useful conclusions.

This example is interesting in that it illustrates how analytic resourcefulness and ingenuity may serve to provide the solution to a problem when the so-called straightforward methods become too complex. As pointed out earlier, the straightforward mathematical method is to find a solution to Laplace's equation which fits the given boundary conditions. Although this is not too difficult in this instance, it requires first that the

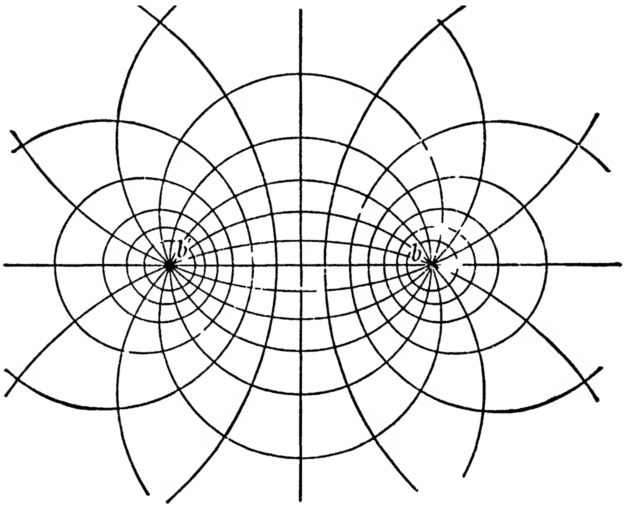


FIG. 17. Map of field between two parallel cylinders external to each other.

proper system of co-ordinates be found for which the variables in the equation separate. This system in itself must be found wholly through mathematical resourcefulness, so that the so-called straightforward process is not altogether straightforward.

The present problem and other similar ones are solved in a much simpler fashion. It is recalled from analytic geometry that a system of bipolar circles forms a pair of mutually orthogonal families of curves having the appearance of the plot in Fig. 17. The points b and b' are the poles. One family of circles has its centers on the vertical axis, and all these circles pass through the poles. The orthogonal family is an eccentric set with centers on the horizontal axis, all encircling either the pole b or b'. As the radius of a circle in this latter set is decreased, the center moves toward, and finally coincides with, the corresponding pole, but for finite circles the centers are either to the right of b or to the left of b'.

The general appearance of this orthogonal set of circles immediately suggests that this may be the solution to the problem. Since the curves *are all circles, any pair surrounding the poles b and b' may be identified* with the circular conductor cross sections. This fact and the orthogonal property of these curves are not by themselves, however, sufficient evidence that this map actually constitutes the solution to the given problem. That is, it is known from the discussion in Art. 7 that, for the longitudinally uniform system with transverse flow, the families of curves must intersect to form similar curvilinear rectangles *throughout.* Until it is proved that the bipolar circles have this property, they cannot be accepted as yielding the desired solution. Although this property of the bipolar circles may be shown purely on a geometrical basis, an alternative method which utilizes the electromagnetic features of the problem is simpler as well as more instructive. This method consists in assuming

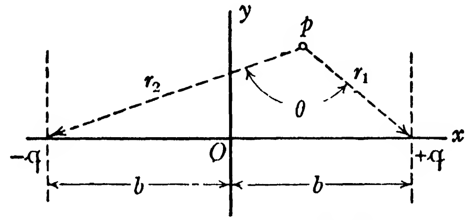


FIG. 18. For analysis of field mapped in Fig. 17.

that the bipolar plot does constitute the solution, and investigating whether the electrical properties of the problem are satisfied. If no contradictions are met, the validity of the assumed solution is proved.

In the physical problem for which the conductors carry equal and opposite net charges uniformly distributed in the longitudinal direction, the charges are *not* uniformly distributed circumferentially. Since the circumferential charge distribution is not known, it is not possible to map the field by means of Eq. 60. However, since in Fig. 17 the circles with centers on the vertical axis represent electric flux lines for this problem, and since these circles *all* pass through the poles b and b', the inference is drawn that the conductor charges may be considered concentrated along filaments located at the poles so far as the field external to the conductors is concerned. For such filamental charge distributions the field is readily calculated from Eq. 85a. If the corresponding equipotential loci and their orthogonal trajectories then turn out to be the set of bipolar circles, the sufficiency of the assumed solution is proved.

In Fig. 18 the problem is to find the potential at the point p located at the perpendicular distances r_1 and r_2, respectively, from the uniform fila-

mental charge distributions $+q$ and $-q$ coulombs per meter of length. From Eq. 85a, and since the field intensity is the negative gradient of the potential,

$$\mathcal{E} = \frac{2q}{\varepsilon r} = -\frac{d\mathcal{V}}{dr}, \tag{96}$$

the potential function has the form

$$\mathcal{V} = \frac{2q}{\varepsilon} \ln \frac{1}{r}. \tag{97}$$

Thus for the point p in Fig. 18,

$$\mathcal{V} = \frac{2q}{\varepsilon} \ln \frac{1}{r_1} - \frac{2q}{\varepsilon} \ln \frac{1}{r_2} = \frac{2q}{\varepsilon} \ln \frac{r_2}{r_1}. \tag{98}$$

Equipotential loci are evidently such curves for which

$$\frac{r_2}{r_1} = \text{constant} = k. \tag{99}$$

In terms of the variables x and y in Fig. 18 this reads

$$\frac{(b + x)^2 + y^2}{(b - x)^2 + y^2} = k^2. \tag{99a}$$

After multiplying out and collecting terms, one finds that this equation may be put into the form

$$(x - a)^2 + y^2 = r^2, \tag{99b}$$

in which

$$r = + \frac{2bk}{\sqrt{(k^2 - 1)^2}} \tag{100}$$

and

$$a = \pm\sqrt{b^2 + r^2}. \tag{101}$$

Equation 99b represents a circle with radius r, and center located on the x axis at distance a from the origin. Since the values of a occur in positive and negative pairs, the circles come in such pairs. These are the bipolar circles of Fig. 17 of which one surrounding each pole may be identified with a conductor surface. The orthogonal set of circles is not derived here, but it is useful to know that these are circles for which the angle between r_1 and r_2 at p in Fig. 18 remains constant. One circle in each family and a dotted circle passing through the poles b and b' are shown in Fig. 19. A number of simple geometrical relationships are indicated which facilitate the drawing of a family of circles.

As a result of this reasoning and analysis it is evident that for two parallel circular cylindrical conductors of radius r and axial separation $2a$, carrying longitudinally uniform charges of equal magnitude but opposite sign, the external field is determined as though these charges were concentrated along axes in the same plane but *separated by $2b$, as in Fig. 19, where*

$$b = \sqrt{a^2 - r^2}. \tag{101a}$$

The latter axes are eccentric and closer together than the conductor axes because of the mutual attraction between the surface charges, which is wholly responsible for the nonuniform circumferential distribution. As the separation between conductors increases, for a given conductor size, b merges with a. The charges may then be considered concentrated along

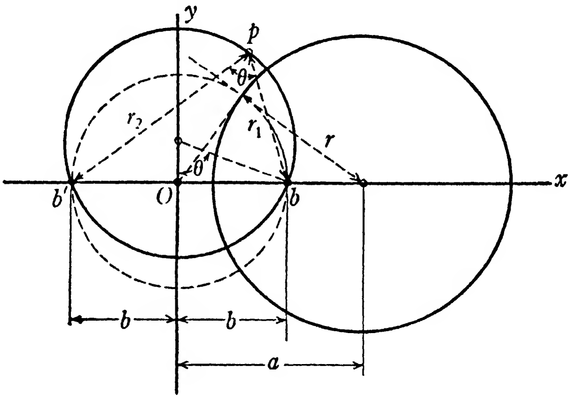


FIG 19. For analysis of field mapped in Fig 17.

the conductor axes, and the circumferential charge distribution becomes uniform, as it is for a single conductor.

The resistance per unit length between the two conductors, when the surrounding medium is homogeneous with conductivity and dielectric permittivity γ and ε respectively, can be computed as follows. The voltage between the adjacent conductor surfaces is

$$V = \frac{2q}{\varepsilon}\ln\frac{b+a-r}{b-a+r} - \frac{2q}{\varepsilon}\ln\frac{b-a+r}{b+a-r} = \frac{4q}{\varepsilon}\ln\frac{a+\sqrt{a^2-r^2}}{r}, \tag{102}$$

and the total current per unit length leaving the surface of one conductor is

$$i = \oint_s \mathcal{J}\cdot ds = \oint_s \gamma\mathcal{E}\cdot ds = \frac{4\pi\gamma q}{\varepsilon} \tag{103}$$

where the integral is taken over any unit of length of a shell completely surrounding the conductor. The resistance parameter for a unit length, therefore, is

$$R_u = \frac{V}{I} = \frac{1}{\pi\gamma} \ln \frac{a + \sqrt{a^2 - r^2}}{r} \quad \text{ohms for one meter,} \tag{104}$$

or

$$G_u = \frac{\pi\gamma}{\ln \dfrac{a + \sqrt{a^2 - r^2}}{r}} \quad \text{mhos/m} \tag{104a}$$

if mks units are used.

The capacitance parameter per unit length is determined from

$$C_u = \frac{q}{V} = \frac{\varepsilon}{4 \ln \dfrac{a + \sqrt{a^2 - r^2}}{r}} \quad \text{farads/m} \tag{105}$$

if mks units are used. If either G_u or C_u is known, the other can be obtained from Eq. 79.

When the separation between conductors becomes large compared with their radii, the following approximate relations may be used:

$$\frac{1}{R_u} = G_u \approx \frac{\pi\gamma}{\ln \dfrac{2a}{r}}, \tag{104b}$$

and

$$C_u \approx \frac{\varepsilon}{4 \ln \dfrac{2a}{r}}. \tag{105a}$$

In the calculation of the inductance parameter for this conductor arrangement, the necessity of considering internal, as well as external, flux linkages leads to complexity greater than that encountered in resistance and capacitance calculations. As previously pointed out, the relative importance of the internal linkages depends, for one thing, upon the conductor size as compared to the distance between adjacent surfaces; the importance of internal linkages depends also upon the frequency at which the value of the inductance parameter is desired. At zero frequency — that is, for steady currents — the distribution of current density over the cross section of either conductor is uniform. If conductor size and spacing are comparable, the internal linkages are then important. On the other hand, at frequencies for which the wavelength within the conductor material is comparable with the conductor size or is smaller, the current

density is much greater near the conductor surface and to a good approximation the total current may be considered to reside within a relatively thin surface layer. This phenomenon of current crowding toward the surface of the conductor is called *skin effect.* When skin effect is pronounced, the importance of internal linkages is then far less, even though the conductor size and spacing may be comparable.

Another important feature enters into these considerations in connection with the present problem, which is not in evidence in the previous problem involving circular symmetry. Not only is the current for larger frequencies crowded toward the conductor peripheries but it is crowded also toward the *adjacent* conductor surfaces. The latter phenomenon, quantitative establishment of which requires a somewhat lengthy analysis, is known as *proximity effect.* When it is absent, as with steady currents, the external magnetic field evidently can be calculated by assuming the net current to be concentrated along the conductor axes. When the proximity effect is complete,* however, a quantitative analysis shows that the distribution of the external field is correctly given by considering the net conductor currents to be concentrated on the same displaced axes along which the charges were considered concentrated in the corresponding electric-field problem just discussed. Since the skin and proximity effects are phenomena of essentially the same nature, the frequencies for which the one is found to be essentially complete will render the other essentially complete also. Under these circumstances *there are no internal linkages,* and the flow map for the magnetic field coincides with that for the electric field. In Fig. 17, the loci encircling the conductors become the lines of magnetic-field intensity, and the orthogonal family the equipotential loci. In other words, field strength and equipotential loci are interchanged as compared to the same map when representing the electric-field problem.

If $\pm q$ in Fig. 18 is replaced by $\pm I$, the magnetic-field intensity at a point on the x axis is given by the Biot-Savart law as

$$\mathcal{H} = \frac{2I}{b - x} + \frac{2I}{b + x}. \qquad [106]$$

The flux per unit length between adjacent conductor surfaces is

$$\begin{aligned} \phi_u &= 2\mu I \int_{-(a-r)}^{(a-r)} \left(\frac{1}{b - x} + \frac{1}{b + x}\right) dx \\ &= 2\mu I \Big\{ -\ln (b - x) + \ln (b + x) \Big\}_{-(a-r)}^{(a-r)} \end{aligned} \qquad [107]$$

* By skin and proximity effects being *complete* is meant the conditions which are approached as the frequency approaches infinity.

Substituting limits and making use of Eq. 101a give

$$\phi_u = 4\mu I \ln \frac{a + \sqrt{a^2 - r^2}}{r}. \qquad [107a]$$

The inductance parameter per unit length, therefore, is

$$L_u = \frac{\phi}{I} = 4\mu \ln \frac{a + \sqrt{a^2 - r^2}}{r} \quad \text{h/m} \qquad [108]$$

if mks units are used.

Comparing this with Eq. 105 for the capacitance parameter, one sees again that

$$L_u C_u = \varepsilon\mu. \qquad [81]$$

This and Eq. 108 assume, of course, that the frequency is so high that skin and proximity effects are complete. Theoretically this assumption requires an infinite frequency, but for many practical circumstances these relations are very nearly correct. With this restriction on L_u, if either G_u, C_u, or L_u is known, both the other quantities can be calculated by use of Eq. 79, 80, or 81.

It should be recognized, however, that for steady currents the external field is calculated by assuming that the net currents are concentrated along the conductor axes. The inductance parameter, considering the external field alone, is then given by replacing b by a in Eq. 107 and obtaining after insertion of limits and dividing by I:

$$L_{u0} = 4\mu \ln \frac{2a - r}{r}. \qquad [108a]$$

This must be supplemented by components due to the flux which passes through the conductor cross sections. The determination of these components, although straightforward analytically, is quite laborious even in this rather simple example.[8] In this article the result is derived only in part. The internal flux in each conductor is considered in two portions: that due to the current in the other conductor and that due to the current in the conductor under consideration.

For the first portion the statement is made, without proof, that integration of flux from the surface of one conductor to the axis of the other (instead of merely between surfaces) gives the correct result for total external flux linkages, plus the internal flux linkages of flux caused by

[8] A rigorous derivation is in the reference volume, Ch. XVIII, also in L. F. Woodruff, *Principles of Electric Power Transmission* (2d ed.; New York: John Wiley & Sons, 1938), Ch. iii, using the method of geometric mean distances. This method can be applied also to multiconductor systems. Other references on this method are in the bibliography.

current in one conductor, with current of the other. That is, Eq. 108a is *modified to read,*

$$L_{u1} = 4\mu \ln \frac{2a}{r}. \qquad [108b]$$

Inspection of Fig. 20 indicates that this internal flux links on the average about half the current. Hence it seems at least plausible that to integrate to the axis of the conductor and to consider that the flux between the inner side of the conductor and the axis links all of the current gives about the same result.

For the second portion, the calculation is not difficult. The distribution of magnetic-field intensity within each conductor due to the current

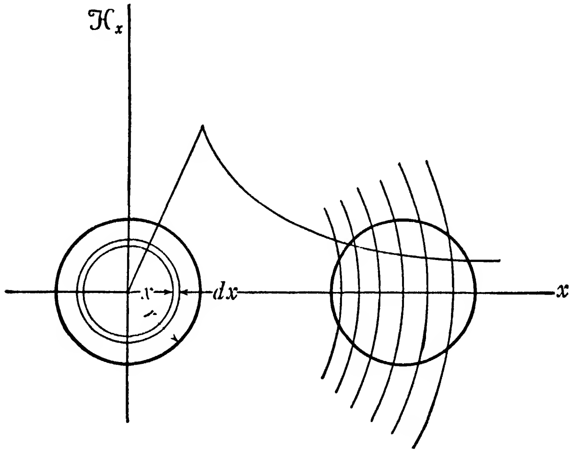


FIG. 20. Internal flux of parallel conducting cylinders.

in the conductor under consideration is symmetrical; that is, it is the same as for a single isolated conductor. If in Fig. 20 the radius is r and I is the total current, application of Ampère's circuital law to the circular path with radius x gives

$$2\pi x \mathcal{H}_x = \frac{4\pi I x^2}{r^2}; \qquad [109]$$

because the current inclosed by this path is x^2/r^2 times the total. Hence, within the conductor $\mathcal{H}_x$ is given by the following function of x

$$\mathcal{H}_x = \frac{2Ix}{r^2}. \qquad [110]$$

In Fig. 20, $\mathcal{H}_x$ is shown plotted versus x over the internal and over a portion of the external region.

In order to calculate the internal-inductance component by the method of flux linkages, a differential cylindrical element of thickness dx is considered. The flux per unit length of conductor within this element is $\mu\mathcal{H}_x dx$, and this flux links with the fraction x^2/r^2 of the total current. The flux linkage contributed by this element of cross section is therefore

$$d\lambda_u = \frac{x^2}{r^2}\mu\mathcal{H}_x dx = \frac{x^2}{r^2}\mathcal{B}_x dx = \frac{x^2}{r^2}d\phi = \frac{2\mu I x^3 dx}{r^4}. \qquad [111]$$

The internal linkages per unit length of one conductor are found by integrating this expression over the cross section:

$$\lambda_u = \frac{2\mu I}{r^4}\int_0^r x^3 dx = \frac{\mu I}{2}. \qquad [112]$$

The internal flux linkages for the two conductors are twice this value. The internal contribution to the inductance is

$$L_{u2} = \mu \qquad \text{h/m} \qquad [113]$$

if μ is in mks units.

The same result may alternatively be obtained from energy considerations. The energy density for an internal point is

$$w_m = \frac{\mu\mathcal{H}_x^2}{8\pi} = \frac{\mu I^2 x^2}{2\pi r^4}. \qquad [114]$$

The volume element is

$$dv = 2\pi x dx/\text{unit length}. \qquad [115]$$

Hence the differential contribution to the internal field energy is

$$dW_{u2} = w_m dv = \frac{\mu I^2 x^3}{r^4}dx/\text{unit length}. \qquad [116]$$

The total internal field energy for both conductors per unit length is

$$W_{u2} = 2\int dW_{u2} = \frac{2\mu I^2}{r^4}\int_0^r x^3 dx = \frac{\mu I^2}{2}. \qquad [117]$$

But

$$W_{u2} = \tfrac{1}{2}L_{u2}I^2; \qquad [118]$$

so again it is found that

$$L_{u2} = \mu. \qquad [119]$$

The total inductance per unit length at zero frequency for the pair of parallel conductors is given by the sum of Eqs. 108b and 113:

$$L_u = L_{u1} + L_{u2} = \mu \left(1 + 4 \ln \frac{2a}{r}\right) \qquad [120]$$

For the frequencies or geometrical relationships usually encountered in communication practice, either the internal increment is negligible and then Eqs. 108 and 120 amount to essentially the same thing, or the skin *and proximity effects cause Eq. 108 to be more nearly correct than Eq. 120.* In calculations for power-transmission lines, the contribution due to the internal linkages is usually considered, although even here it adds a correction of only a few per cent.

The solution to the parallel circular cylindrical conductor problem illustrated in Fig. 17 immediately yields the solution to a number of

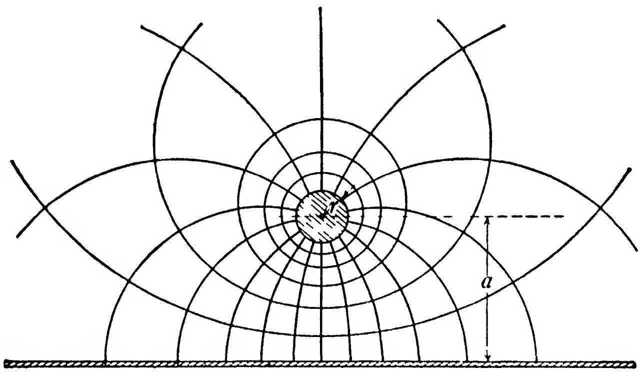


FIG. 21. Map of field between a plane and a cylinder parallel to it.

related problems. For example, if the two conductors have different radii, the field map is still given by the loci of Fig. 17, where now two unequal circles about b and b' are chosen to represent the conductor surfaces. The reader may, as an exercise, determine the parameter formulas for this more general case. A special case of particular simplicity and interest occurs if one of the conductor radii is allowed to become infinite for a constant separation between adjacent surfaces. The result is pictured in Fig. 21, which represents a circular cylindrical conductor opposite an infinite conducting plate. Through comparison with the previous example, it may be stated by inspection that the resistance and inductance parameters are now equal to one-half their previous values, while the capacitance parameter is doubled.

Another valuable point of view is obtained by considering Fig. 21 as the upper half of Fig. 17 turned on its side. The lower half is the *image* of

the upper half. It may be said, therefore, that the field distribution for the problem of Fig. 21 is obtained by considering the conductor of radius r together with its image with respect to the conducting plane. This artifice, called the *method of images*, was generally introduced in connection with problems of this sort by Lord Kelvin* (William Thomson). Further generalization of this principle leads to a valuable aid in the attack on more complex problems of this nature.

Practically, the situation illustrated in Fig. 21 may occur in cases where a single conductor is used for the transmission of electromagnetic waves with the earth serving as the return conductor. The exact location of the surface of the theoretical conducting plane is, in such a case, rather difficult to determine. At first thought it might be supposed that the earth's surface is to be identified with this plane, but since the current actually returns throughout a layer of appreciable depth, this is obviously not an altogether correct view. Furthermore, certain layers of the earth's crust at various depths may have widely varying conductivities, and the current favors those layers having the higher conductivities. Sometimes the topsoil is dry and sandy (a poor conductor), whereas a layer at greater depth is moist and compact. Evidently some knowledge of the geological strata is essential in dealing with problems of this sort. It is also significant that throughout the length of transmission such conditions and hence the parameters may vary considerably, with corresponding variations in the transmission properties of such circuits. These matters are sometimes of considerable importance.

It has been pointed out that the only problems practically susceptible to graphical solution are those in which longitudinal symmetry exists. For those illustrations involving longitudinal flow, introduced as an aid in establishing the theory and procedure, it was pointed out that actual graphical work is superfluous for the numerical evaluation of a particular parameter, since the computation can be made directly in terms of length and cross-section area of the specimen. Likewise in the examples of transverse flow given thus far, analytical computation is simpler for obtaining actual numerical results because the geometric configurations chosen lend themselves readily to the use of simple mathematics. Therefore as final examples in this classification it is instructive to consider some shapes which lend themselves to simple graphical solutions but require very complicated mathematical analysis.†

For this purpose a conductor having a right-angle bend is first selected,

* Although Kelvin is generally credited with having introduced the method of images as an aid in the solution of problems of electrostatic induction, the method was apparently known earlier in the field of mathematics.

† Field mapping can be carried out experimentally by locating equipotential lines in an electrolytic tank or on a sheet-metal model with the aid of a potentiometer circuit.

Fig. 22. In the solution of this problem graphically, the sole criterion is that the flow lines and equipotential lines must intersect orthogonally on a cross section so as to form curvilinear squares. Hence the procedure is to sketch flow lines and equipotential lines by trial so as to form such squares. The test of any trial map is in the subdivision of the squares. If by continued subdivision the curvilinear squares appear to approach true squares more and more closely, the trial is a good guess; if they do not, a poor guess has been made and a new trial must be started based upon the apparent mistakes of the previous one.

In Fig. 22, ℓ_u is a unit of length in the longitudinal direction, and the current is directed from t_1 to t_2. Since the nonuniformity of the current

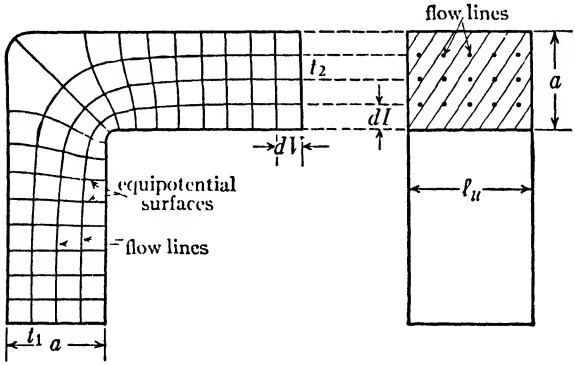


FIG. 22. Map of current in conductor with right-angle bend.

density evidently is most pronounced in the vicinity of the bend, this region is the most difficult to map. A good way to begin is to draw first the diagonal equipotential line in the corner, which, by symmetry and by uniformity of the dimension a, is evidently correct. That is, the map is symmetrical about this diagonal. Next the central current line is sketched, by guess at first, recognizing that the current is parallel to the conductor sides at large distances from the bend and that it tends to hug the inside corner because of the lower resistance of this shorter path. Next a number of curvilinear squares may be sketched by drawing additional equipotential lines to each side of the diagonal one. If the central current line has been correctly placed, this initial attempt at producing squares is successful, although it is at this stage rather difficult to recognize whether or not the meshes are true curvilinear squares. If this first attempt looks promising it may be checked more carefully by further subdivision, that is, by adding more current and more potential lines so as to divide each square into four sections. This process is continued until results of sufficient precision are obtained. Then if there are m equipotential lines (including

one of the terminal lines) and n flow lines (including one of the border lines), Eq. 52 gives

$$G_u = \frac{1}{R_u} = \frac{n\gamma}{m} \qquad \text{mhos/m} \tag{52c}$$

if γ is in mks units. As previously discussed, the resistance for one unit of length in the ζ direction can be obtained in ohms per centimeter, abohms per centimeter, microhms per centimeter, etc., merely by expressing γ in the proper units. It is interesting that the actual conductance per unit length is independent of the actual dimensions: All geometrically similar figures having the same conductivity have the same conductance per unit length.

A second example of graphical solution is illustrated by Fig. 23, in which the field of a rectangular parallel-plate condenser is mapped, taking into account edge effect.[9] As mentioned in part (b) of this article, this map represents a unit of length not near an end of a long, rectangular parallel-plate condenser having small spacing between plates compared with the plate dimensions. If there are m equipotential lines (including one of the plates) and n flow lines (for the entire figure), the capacitance per unit length is

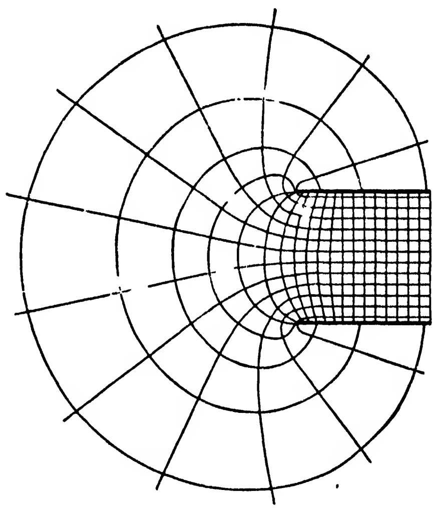


FIG. 23. Map of field at edge of parallel plate condenser.

$$C_u = \frac{n\varepsilon}{4\pi m} \qquad \text{farads/m} \tag{54}$$

if ε is in mks units.

Figure 24 shows a third, and more commonly encountered, example of the application of flux plotting to the determination of a parameter where analytic treatment is difficult if not impossible. This figure represents a section through the rotating pole and stationary armature of an alternator, the section plane being perpendicular to the axis of rotation. The problem is to determine the reluctance of the air gap per unit axial length of the machine.

[9] This illustration is taken from James Clerk Maxwell, *A Treatise on Electricity and Magnetism* (3d ed.; Oxford: at the Clarendon Press, 1892), I, Fig. XII.

In practice the stationary armature, represented by the arc *ABC*, contains a large number of radial slots into which the armature conductors are inserted. It is found that, provided the air gap is increased, a smooth arc representation of the armature, as in the figure, conveys the effect of these slots with sufficient accuracy for the graphical analysis. The amount of increase in the air gap necessary may be determined from the depth, width, and number of slots. For the present problem, the gap may be assumed to have been so increased.

A second assumption has to do with the permeability of the iron. *Under ordinary conditions this permeability exceeds that of the gap by a ratio of the order of a thousand. Therefore the assumption of infinite*

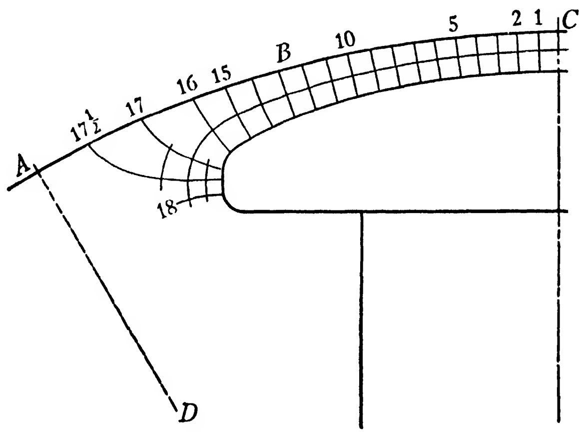


FIG. 24. Map of magnetic flux in air gap of alternator.

permeability, or zero reluctivity, for the iron compared to the gap is justified within the accuracy obtainable from a graphical solution of this type.

The face of the rotating pole is ordinarily shaped to give such a flux distribution that the voltage induced in the armature conductors is sinusoidal in form. The flux distribution can be obtained from the flux plot also, but this is outside the immediate problem of determining the reluctance.

The plot may best be drawn by laying off radial lines, beginning at the center of the pole in such a way that squares are produced by these lines and the boundaries of the armature and pole faces. The central equipotential line may then be drawn, and further subdivision carried out where it is necessary to check on the accuracy of the map. The radial line *AD* divides the armature at a point equidistant between pole centers. Hence any flux which does not reach the armature between *C* and *A* does

not reach it at all and therefore is not useful for the purpose of the problem.

If there are m equipotential lines, and the corresponding number of flux lines is n, the reluctance of the half pole for one unit of axial length is

$$\mathcal{R}_{1/2} = \frac{U}{\phi_{1/2}} = \frac{m}{n}\nu_0 \qquad [121]$$

where ν_0 is the reluctivity of free space (the reciprocal of the permeability), and the permeance per unit of axial length is

$$\mathcal{P}_{1/2} = \frac{n\mu_0}{m}. \qquad [121a]$$

For a full pole the reluctance is half as much, and the permeance twice as much, as for a half pole.

In the mks system ν_0 is 10^7, and in the machine represented m is 2 and n is about 17.6. Therefore, for a full pole

$$\mathcal{R} = \frac{2 \times 10^7}{2 \times 17.6} = 5.7 \times 10^5 \text{ pragilberts/weber for 1 m axial length of pole,} \qquad [121b]$$

$$\mathcal{P} = 0.176 \times 10^{-5} \text{ weber/pragilbert/m of axial length of pole.} \qquad [121c]$$

10d. Systems having spherical symmetry

A system exhibiting spherical symmetry, occurring commonly, is illustrated in cross section in Fig. 25, which consists of two concentric spherical conducting shells. The intervening space is filled with a homogeneous material having a conductivity γ and a permittivity ε.

As pointed out in Art. 7, the utility of the graphical method is confined practically to two-dimensional problems and is then used primarily where analytic methods are hopelessly complicated, or for the purpose of lending physical visualization to a possible analytic attack. The field plot of Fig. 25 can be drawn by spacing the flow lines and equipotential surfaces so as to satisfy Eq. 41c. However, in this example an analytic method is not only simpler and more direct but more generally useful as well. On account of the symmetry and the consequent fact that the flow lines diverge from the geometrical center of the configuration, it is clear that the field in the space between the conductors may be calculated as though all the surface charge on the inner sphere were concentrated as a point charge at the center of the sphere. The equal and opposite charge on the inner surface of the outer conductor has no influence upon the field.

The expressions for electric-field strength and potential, therefore, are

$$\mathcal{E} = \frac{Q}{\varepsilon r^2} \qquad [122]$$

and

$$\mathcal{V} = \frac{Q}{\varepsilon r}, \qquad [123]$$

in which Q is the assumed total charge on the inner sphere and r the radial distance from the center. If the inner radius of the outer sphere is r_2

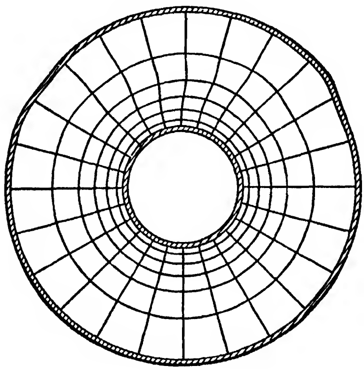


FIG. 25. Map of field between concentric spheres.

and the outer radius of the inner sphere is r_1, then the potential difference between the spheres is given by

$$V = \frac{Q}{\varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad [124]$$

In order to calculate the resistance, it is necessary to determine the net current diverging from the inner sphere. This is given by the integral

$$I = \oint_s \gamma \mathcal{E} \cdot ds \qquad [125]$$

taken over the surface of the sphere. But by Gauss's law

$$\oint_s \mathcal{E} \cdot ds = \frac{4\pi Q}{\varepsilon}, \qquad [126]$$

and hence

$$I = \frac{4\pi\gamma Q}{\varepsilon} \quad \text{amp.} \qquad [127]$$

The resistance parameter is

$$R = \frac{V}{I} = \frac{1}{4\pi\gamma}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \text{ohms,} \qquad [128]$$

and the capacitance parameter is

$$C = \frac{Q}{V} = \frac{\varepsilon}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \quad \text{farads} \qquad [129]$$

if mks units are used.

This three-dimensional flow problem exhibits an interesting characteristic not present in the one- or two-dimensional examples. Whereas in the latter types of flow problems the capacitance becomes zero and the resistance infinite for an infinite separation between the terminal conductors, these parameters approach finite nonzero values for infinite separation in the three-dimensional flow problems. When r_2 is allowed to become infinitely large, Eqs. 128 and 129 become

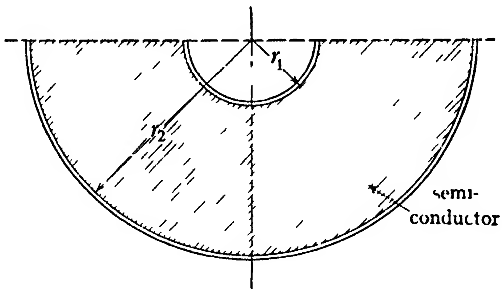


FIG. 26. Hemispherical conducting shells.

$$R = \frac{1}{4\pi\gamma r_1} \quad \text{ohms,} \qquad [128a]$$

$$C = \varepsilon r_1 \quad \text{farads.} \qquad [129a]$$

Their magnitudes depend solely upon the radius of the remaining sphere of finite size.

On the basis of the symmetry involved in this problem, it is possible to write down by inspection the solution to a closely related configuration, illustrated in Fig. 26. It consists of two concentric hemispherical conducting shells with a semiconducting intervening medium. Evidently the flow map is just half the picture in Fig. 25, and hence the resistance is twice

and the capacitance half the respective values for the spheres. Thus

$$R_{hemispheres} = \frac{1}{2\pi\gamma}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \text{ohms,} \qquad [128b]$$

$$C_{hemispheres} = \frac{\varepsilon}{2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \quad \text{farads.} \qquad [129b]$$

Again the radius of the outer sphere may become infinite without having C become zero or R infinite. Thus, for the single hemisphere with flow extending to infinity,

$$R = \frac{1}{2\pi\gamma r_1}, \qquad [128c]$$

$$C = \frac{\varepsilon r_1}{2}. \qquad [129c]$$

There is a good physical reason why these parameter values remain finite in spite of the infinite spacing. The resistance might be considered as the series combination of the component resistances of successive spherical shells of uniform thickness and increasing radii. To each of these the principles of linear (although radial) flow may be applied approximately. While the length of flow path is the same for all shells, the area increases as the square of the radius and hence the component resistance decreases inversely as the square of the radius. Thus the component shells in the more immediate vicinity of the inner sphere contribute heavily to the total resistance, while the more remote shells contribute very little and finally nothing.* The very remote ones may, therefore, be ignored without noticeably affecting the result. In other words, if the outer conductor radius is large compared with the inner one, the latter alone essentially determines the resulting resistance.

This fact becomes useful in the following practical example. In certain applications involving the transmission of electric currents over long distances, a considerable saving in transmission-line conductors may be effected by utilizing the conduction of the earth as a return path for the current. In telegraphy, for example, this is an expedient commonly used, and in the early days of telephonic communication the device was used also, although the necessity of minimizing interference between cir-

* It may be emphasized here that the negligibility of the contributions coming from the larger spherical shells rests upon the fact that their areas are proportional to the square of their radii. For example, in the case of concentric cylinders, which was treated previously, a similar argument would lead to a fallacious conclusion because the cylindrical surface is proportional only to the first power of the radius and the relative contribution of cylindrical elements does not converge rapidly enough.

cuits led later to its abandonment. The resistance of ground connections of power-transmission lines, ground wires, and lightning arresters also is important. In order to estimate the net resistance which might be expected of an earth path the contact with the earth may be thought of as effected through the surfaces of hemispherical embedded conductors as illustrated in Fig. 27. The net resistance R between the two hemispherical ground conductors is due principally to the resistance of concentric hemispherical shells of conducting earth in the more immediate vicinity of each terminal and hence is given by twice the value of Eq. 128c, or for this circuit by

$$R \approx \frac{1}{\pi \gamma r} \qquad [128d]$$

This assumes, of course, that the structure of the earth is more or less homogeneous throughout, particularly in the vicinities of the embedded terminals.

Experiments made chiefly in connection with electromagnetic wave propagation over the earth's surface[10] indicate that for damp soil the

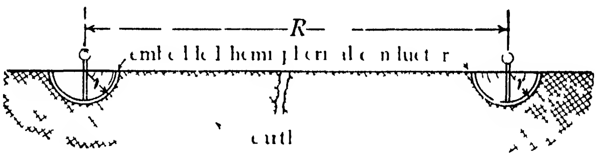


Fig. 27. Grounding hemispheres.

earth has a resistivity of 100 to 1,000 ohm meters, whereas for dry soil an average figure is in the vicinity of 10,000.* Thus the value of Eq. 128d for damp soil is somewhere in the vicinity of

$$R \approx \frac{30}{r} \text{ to } \frac{300}{r} \text{ ohms,} \qquad [128e]$$

in which r is expressed in meters.

For ground hemispheres with radii of one meter embedded in a damp soil one might therefore expect such a ground return circuit to have a net resistance somewhere between 30 ohms and 300 ohms, which value is essentially independent of the distance between terminals. However,

[10] J. Zenneck, "Über die Fortpflanzung ebener elektromagnetischer Wellen längs einer ebenen Leiterfläche und ihre Beziehung zur drahtlosen Telegraphie," *Ann. d. Phys.*, XXIII (1907), 846–866; J. A. Fleming, "The Function of the Earth in Radiotelegraphy," *Engineering* (London), LXXXVII (1909), 766–767, 776–777; G. W. Pierce, *Principles of Wireless Telegraphy* (New York: McGraw-Hill Book Company, Inc., 1910), Ch. xiv, xv.

* For sea water the average figure is 1, for fresh water 10, and for dry rock about 100,000.

ground connections are not ordinarily made by means of hemispherical shells. The most common methods are to make fast to water pipes, to drive rods into the ground, or to bury wires, grids, or plates. Sometimes very complicated interconnected configurations are used. The ground is sometimes specially treated by salting in order to reduce materially the resistivity in the neighborhood of the rods or other earthing conductors. It is obvious that precise computation of the resistance of a ground path is not feasible, even when simple grounding rods are used. However, Eq. 128d can be used as a basis of comparison; that is, it is possible to express the resistances for various configurations in terms of the radius of an equivalent sphere. For example,[11] the hemisphere equivalent in resistance to a driven cylindrical rod of length ℓ and radius r_c has a radius

$$r_s = \frac{\ell}{\ln \dfrac{4\ell}{r_c} - 1}. \qquad [130]$$

10e. Some additional remarks concerning problems with three-dimensional flow

The three-dimensional problem with spherical symmetry leads to such a simple analytic treatment principally because the question of charge or current-density distribution can be answered by inspection. Three-dimensional problems exhibiting other types of symmetry or possessing no symmetry are, however, in general very difficult to treat. Even such problems which at first sight appear to be manageable lead to unexpected difficulties when a rigorous treatment is attempted.[12]

As an illustration of this situation, the problem of two spheres with radii r_1 and r_2, and separation d, as illustrated in Fig. 28, is mentioned. The flow map, assuming that the spheres carry equal and opposite charges, is exceedingly difficult to derive. Offhand, it might be supposed that the map is obtained simply by rotating the one in Fig. 17 for the cylindrical conductors about the horizontal axis, and then identifying spherical surfaces about b and b' with the surfaces of the given spheres of Fig. 28. That the flow map thus obtained cannot be correct follows from the discussion in Art. 7 regarding systems with rotational symmetry. There it was shown that in the correct flow map for such systems the curvilinear rectangles in any cross section containing the axis of rotation must have

[11] H. B. Dwight, "Calculation of Resistances to Ground," *A.I.E.E. Trans.*, LV (1936), 1319–1328.

[12] The electrolytic tank can be adapted to three-dimensional field mapping. Other experimental methods have been developed by: D. Gabor, "Mechanical Tracer for Electron Trajectories," *Nature*, CXXXIX (1937), 373; and D. B. Langmuir, "Automatic Plotting of Electron Trajectories," *Nature*, CXXXIX (1937), 1066–1067.

ratios of length to width proportional to their distance from the axis of rotation. (" Length " refers to the flow direction and " width " to the orthogonal direction.) For the map in the cross section of a longitudinally uniform system with transverse flow, like that in Fig. 17, the ratio of length to width is the same for all rectangles. The system of bipolar circles which fulfills this latter condition cannot fulfill the former also, and therefore cannot yield the solution to the three-dimensional problem of Fig. 28.

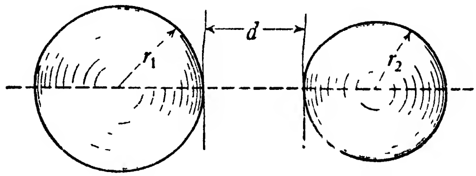


FIG. 28. Spheres external to each other.

These conditions on the ratios between the sides of curvilinear rectangles, or of parallelepipeds when three dimensions must be considered, are essentially the result of stipulating that each flow line must represent a constant amount of flux throughout its length, which is the same as stipulating that in any closed volume in the flow space as much flux must leave the surface as enters it. This is the essence of Laplace's equation. By fulfilling certain ratios when drawing the rectangles of a flow map, one in reality is satisfying Laplace's equation. In drawing the map so as to include the conductor boundaries in the family of loci, one is satisfying the boundary conditions of the problem. Since both must be satisfied, it is clear that not just any system of mutually orthogonal loci which fit the conductor surfaces yields the desired solution, for such a system might disregard Laplace's equation entirely.

The determination of the field of the two charged spheres of Fig. 28 can be carried through by freehand sketching, but this is hopelessly tedious and also very approximate. A successful analytic method[13] makes repeated use of the method of images and finally arrives at a solution in the form of an infinite series. For small spacings d the series converges very slowly and must be converted by means of suitable changes of variable to a more rapidly convergent form. It is certainly not evident at the outset that this problem should lead to such length in the calculation of a physically simple parameter. Perhaps the available mathematical methods of analysis are inadequate.

In some practical three-dimensional problems it is possible by ingenious combination of maps for simpler field distributions to obtain approximate solutions which are quite sufficient for the purpose. One typical case of this kind is discussed below.

It is pointed out in part (b) of this article that Eq. 59a for the induct-

[13] Alexander Russell, *A Treatise on the Theory of Alternating Currents* (2d ed.; Cambridge: at the University Press, 1914), I, 191.

ance of a length ℓ of an infinitely long solenoidal coil is sometimes used for coils of finite length but with rather poor accuracy if the ratio of length to diameter is not large. Lack of accuracy results from the fact that when the solenoid is finite, there *is* a magnetic field outside, and this part of the flow map should be taken into account if a precise result is expected.

The exact mapping of the whole field of the finite solenoid is a three-dimensional problem of considerable complexity, even though rotational symmetry is involved. Figure 29 shows, however, how the field map may be pieced together approximately out of simpler maps which have been considered in the present article. Here the map within the finite solenoid

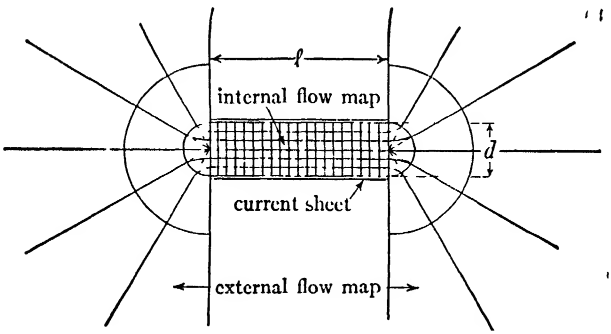


FIG. 29. Approximation to magnetic-field map for a finite solenoid.

is considered to be the regular square grid as for the infinitely long solenoid. At the ends the field is assumed to diverge uniformly and have the same general character as for the concentric hemispheres with the larger one at infinity.

The resultant inductance parameter may be calculated from the fundamental Eq. 56, which is repeated here for convenience,

$$L = \frac{4\pi\phi}{F}. \tag{56}$$

In calculating the inductance of a length ℓ of the infinite solenoid, as is done in part (b) of this article, magnetomotive force F is the difference in magnetic potential between the two ends of this length. For the determination of the net inductance of the finite solenoid of length ℓ, F must be the magnetic potential difference taken completely around a closed path linking the solenoid.

According to the approximate map of Fig. 29, this net potential difference is conveniently calculated as the sum of the potential difference throughout the region of linear flow, and the equal potential differences

between each end and the hemispheres at infinity. The first of these is equal to ℓ pragilberts, as shown in the consideration of the infinite solenoid. The second contributions are calculated in a fashion analogous to that used in the electric-field determination for the flow problem involving spherical symmetry. The magnetic potential at a radial distance r from a fictitious magnetic point pole with pole strength m is

$$\mathcal{M} = \frac{m}{\mu r} = \frac{\phi}{4\pi\mu r} \qquad [131]$$

where ϕ is the total magnetic flux emanating from this pole. No such source of magnetic flux exists, of course, but the mathematical analogy is nevertheless complete. If the magnetic flux from the pole is over one *hemisphere* instead of spherical, ϕ is $4\pi m$ instead of $2\pi m$, and

$$\mathcal{M} = \frac{\phi}{2\pi\mu r}. \qquad [131a]$$

This is the situation which exists at the ends of the finite solenoid when the flow map is as pictured in Fig. 29 and hence is approximately realized physically.

If the radius of the solenoid is r, this is also the radius of the hemisphere which just caps the end of the solenoid. The potential of this hemispherical equipotential surface is then given by Eq. 131a; and, since the potential of the hemisphere at infinity is zero (the value of Eq. 131a when r approaches infinity), it follows that the value of Eq. 131a is the desired difference of magnetic potential between the end of the solenoid and infinity. Hence the net potential difference to be used in Eq. 56 for the calculation of L is

$$F' = \ell + \frac{\phi}{\pi\mu r}. \qquad [132]$$

But from the previous discussion of the flow map for the infinite solenoid

$$\phi = \mu s = \mu\pi r^2; \qquad [133]$$

so Eq. 132 becomes

$$F' = \ell + r. \qquad [132a]$$

The inductance of the finite solenoid on the basis of the approximate flow map of Fig. 29, therefore, is

$$L = \frac{4\pi^2\mu r^2}{\ell + r} \quad \text{h}, \qquad [134]$$

or expressed in terms of the diameter of the solenoid

$$L = \frac{\mu\pi^2 d^2}{\ell}\left\{\frac{1}{1 + \dfrac{d}{2\ell}}\right\} \quad \text{h}, \tag{134a}$$

or when the current sheet is divided into N turns

$$L = \frac{\mu N^2 \pi^2 d^2}{\ell}\left\{\frac{1}{1 + \dfrac{d}{2\ell}}\right\} \quad \text{h} \tag{134b}$$

if mks units are used.

This is the same as the inductance for a length ℓ of the infinitely long solenoid, multiplied by the factor

$$k = \frac{1}{1 + \dfrac{d}{2\ell}}, \tag{135}$$

which may be termed a correction factor for the end effects. The correct values for this correction factor, taken from Table 10, p. 283, Bureau of Standards Circular No. 74, are plotted in Fig. 30, in which the values of

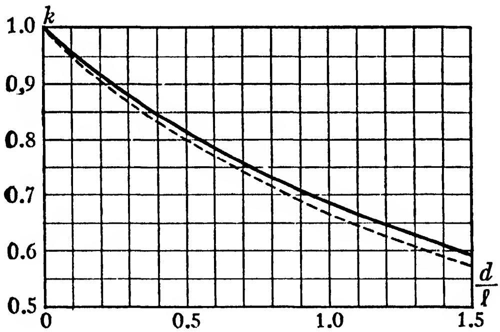


FIG. 30. Plot of inductance correction factor versus ratio of diameter to length for a long solenoid.

Eq. 135 are shown by the dotted curve. The agreement is remarkably good considering the approximate nature of the flow map used in the determination of Eq. 135.

The inaccuracies in this flow map are particularly marked in the vicinity of each end of the solenoid. A more careful analysis shows that at the ends of a long solenoid the axial component of the field strength is approximately half the value at the center, whereas in Fig. 28 the intensity of the

field at the ends is the same as at the center or the same as for the infinite solenoid. The exact calculation of inductance for a finite solenoid is mathematically quite involved. Such calculations are out of place here, but it may nevertheless be of interest to present some of the more exact formulas, of which there is a large variety[14] for any one geometrical configuration.

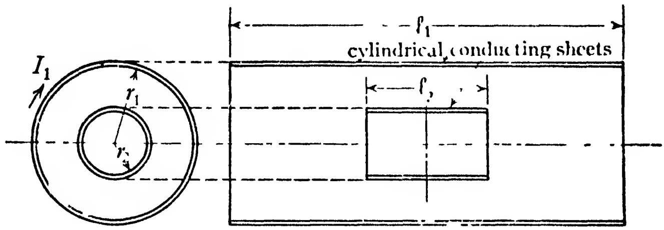


FIG. 31. Solenoids magnetically coupled.

The Rayleigh and Niven[15] formula is particularly adapted to short solenoids:

$$L = 4\pi\mu r N^2 \left\{ \ln \frac{8r}{\ell} - \frac{1}{2} + \frac{\ell^2}{32r^2}\left(\ln \frac{8r}{\ell} + \frac{1}{4}\right) \right\}. \qquad [136]$$

A formula by H. B. Dwight[16] converges rapidly for relatively longer coils:

$$L = \frac{4\pi^2\mu r^2 N^2}{\ell} \times \frac{r}{\ell}\left\{\frac{1}{m} - \frac{8}{3\pi} - \frac{m^3}{8} + \frac{m^5}{16} - \cdots\right\}, \qquad [137]$$

where

$$m^2 = \frac{r^2}{r^2 + \ell^2}. \qquad [137a]$$

These formulas are for thin current sheets.

10f. CALCULATION OF THE MUTUAL-INDUCTANCE PARAMETER

The method of calculating mutual inductance is essentially the same as that for self-inductance. The difference is that two current paths are involved instead of one, and the flux linking one of the paths is considered with respect to the magnetomotive force of the other current path, which is assumed to cause the flux. Since the mutual inductance of path 1 with respect to path 2 is the same as that of path 2 with respect to path 1, it is immaterial which one of the paths is assumed to be the cause of the flux.

[14] E. B. Rosa and F. W. Grover, "Formulas and Tables for the Calculation of Mutual and Self Inductance" *Sci. Paper Nat. Bur. Stand.* No. 169 (3d ed. [revised]; Washington: Government Printing Office, 1916).

[15] *Id.*, p. 116.

[16] H. B. Dwight, "Some New Formulas for Reactance Coils," *A. I. E. E. Trans.* XXXVIII Part 2 (1919), 1688.

In a given example, the choice as to which path causes the flux is made according to what seems to lead to the simpler analysis of the resulting flow problem.

As an example, the pair of cylindrical current sheets shown in Fig. 31 is considered. Here it is simpler to consider the larger and longer cylinder to be carrying a current I_1 and producing the flux, because the field is then essentially uniform over the region occupied by the second current sheet. *In accordance with the flow picture for the infinite solenoid discussed in* part (b) of this article the flux linking the inner cylinder is

$$\phi_{21} = \mu s_2, \tag{138}$$

where

$$s_2 = \pi r_2^2 \tag{139}$$

is the area of the inner cylinder. The mutual-inductance parameter is

$$M_{12} = M_{21} = \frac{\phi_{21}}{I_1} = \frac{4\pi\phi_{21}}{F_1}, \tag{140}$$

in which F_1 is the magnetic potential difference corresponding to the flux of Eq. 138 taken around a path completely linking both cylinders. From the discussion in part (e) of this article this is

$$F_1' \approx \ell_1 + r_1 \tag{141}$$

on account of the finite length of the outer cylinder. Hence the mutual-inductance parameter is

$$M_{12} = M_{21} \approx \frac{4\pi\mu s_2}{\ell_1 + r_1} = \frac{4\mu\pi^2 r_2^2}{\ell_1 + r_1}; \quad (\ell_1 > \ell_2). \tag{140a}$$

The mutual inductance is then independent of ℓ_2 and varies almost inversely as ℓ_1.

Incidentally it is easy to see that if ℓ_2 is greater than ℓ_1 and the inner cylinder is assumed to be the cause of the flux, then the mutual flux is still given by Eq. 138, and the magnetic potential difference is

$$F_2' \approx \ell_2 + r_2. \tag{141a}$$

Hence for this case

$$M_{12} = M_{21} \approx \frac{4\mu\pi^2 r_2^2}{\ell_2 + r_2}; \quad (\ell_2 > \ell_1). \tag{142}$$

For the boundary case of ℓ_1 and ℓ_2 equal, neither Eq. 140a nor Eq. 142 applies unless the length is large compared to either radius or unless the radii are almost equal. When they are unequal it may improve the accuracy to use an average radius in the denominator.

The formulas derived here are, of course, entirely inadequate for work requiring precise calculations. For certain geometrical configurations the Eqs. 140a and 142 give results which are accurate to within a few per cent. In any case, they serve to yield a rough estimate; but on the whole they are less reliable than the approximate formulas derived for the calculation of self-inductance because there are more geometrical variables involved and hence there is a greater chance of meeting situations in which the approximations made here are inappropriate. For example, the approximate character of the flow map of Fig. 29 is not justified in the calculation of mutual inductance when one of the solenoids is much longer than the other. As pointed out in part (e) of this article, the axial flux density diminishes toward the ends of the solenoid; that is, the flow lines spread apart and so do the equipotential surface. Actually then, the magnetic potential difference given by Eq. 132a is too large, but the value ϕ for the flux linkage in the case of self-inductance calculation is too large also. In calculating the mutual inductance of the configuration shown in Fig. 31, the spreading of the flow lines does not appreciably affect the value of ϕ_{21} when the inner solenoid is relatively short. In that case the term r_1 in the denominator of Eq. 140a should be dropped. Similarly r_2 should be dropped in the denominator of Eq. 142 when ℓ_2 is much greater than ℓ_1.

The Bureau of Standard's paper referred to in part (e) of this article, p. 71, also contains numerous formulas for the precise calculation of mutual inductance for various configurations.

If the current sheet I_1 is divided into N_1 turns to form a uniform helical winding, then

$$I_1 = N_1 I_{t1} \tag{143}$$

where I_{t1} is the current per turn. Instead of Eq. 140, the relation for mutual inductance then is

$$M_{12} = M_{21} = \frac{N_2\phi_{21}}{I_{t1}} = \frac{N_1 N_2 \phi_{21}}{I_1} = \frac{4\pi N_1 N_2 \phi_{21}}{F_1}, \tag{140b}$$

in which N_2 is the number of turns into which the inner sheet is similarly subdivided. The magnetic potential difference causing ϕ_{21} is the same as before. Hence the mutual-inductance parameter becomes

$$M_{12} = M_{21} \approx \frac{4\mu N_1 N_2 \pi^2 r_2^2}{\ell_1 + r_1}; \quad (\ell_1 > \ell_2), \tag{140c}$$

or

$$M_{12} = M_{21} \approx \frac{4\mu N_1 N_2 \pi^2 r_2^2}{\ell_2 + r_2}; \quad (\ell_2 > \ell_1). \tag{142a}$$

An interesting method of obtaining a variable mutual inductance is shown in Fig. 32. Here the inner coil is mounted on a shaft so that the plane of the coil may be rotated through an arbitrary angle θ as indicated in the figure. If the inner coil is short, the flux ϕ is nearly proportional to $\cos\theta$, and hence for this configuration

$$M_{12} = M_{21} \approx \frac{4\mu N_1 N_2 \pi^2 r_2^2}{l_1} \cos\theta. \qquad [144]$$

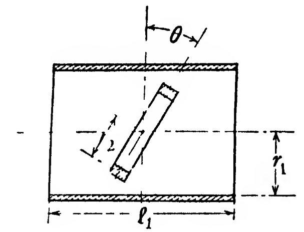


FIG. 32. Arrangement of coils in variometer.

Another commonly occurring practical example of mutual-inductance parameter calculation is illustrated in Fig. 33a. This represents a cross-sectional view of a longitudinally uniform system of circular cylindrical conductor pairs. The conductors numbered 1 and 2 form a pair labeled a which constitutes the down and return path for the current I_1, while the conductors 3 and 4 form a pair labeled b which represents a similar current path for the current I_2. The problem is to determine the mutual inductance between the conductor pairs a and b per unit length.

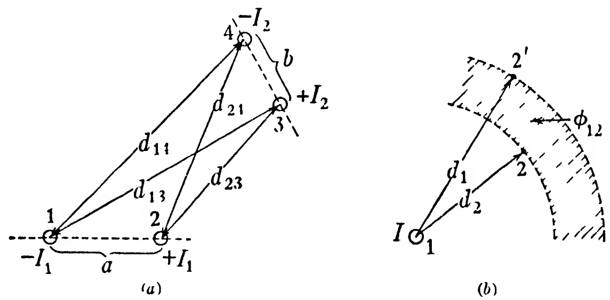


FIG. 33. For calculation of mutual inductance between parallel cylindrical conductor pairs.

Preliminary to determining the flux linkage produced through one conductor pair by the other pair, it is advantageous to consider the situation illustrated in Fig. 33b. Conductor 1 carries a current I. At the points 2 and 2′ are situated other conductors whose radii are small compared with the interaxial distances d_1 and d_2 from the conductor 1. The flux linkage per unit length through the pair 2–2′ produced by the current I in conductor 1 is denoted by ϕ_{12} and occupies the space shown by the shaded ring area.

In order to calculate this flux, use is made of the Biot-Savart relation

$$\mathcal{B} = \frac{2\mu I}{r}, \qquad [91a]$$

which gives the flux density at a point r units from the axis of conductor 1. Hence,

$$\phi_{12} = \int_{d_1}^{d_2} \mathfrak{B} dr = 2\mu I \ln \frac{d_2}{d_1} . \qquad [145]$$

In the situation illustrated in Fig. 33a, if it is assumed that the conductor radii are negligibly small compared to all the interaxial distances (as is usually allowable in practice), the flux per unit length linking conductors 3 and 4 due to the currents $\mp I_1$ in conductors 1 and 2 is found by using Eq. 145 together with the principle of superposition, thus:

$$\phi_{34} = 2\mu I_1 \left\{ \ln \frac{d_{24}}{d_{23}} - \ln \frac{d_{14}}{d_{13}} \right\} . \qquad [146]$$

The desired mutual-inductance parameter per unit length is

$$M_{ab} = \frac{\phi_{34}}{I_1} = 2\mu \ln \frac{d_{13}d_{24}}{d_{23}d_{14}} . \qquad [147]$$

11. Variation of resistance with temperature

It is found experimentally that, for most metallic conductors, the change of resistance is nearly proportional to the change of temperature over a range of a few tens of degrees centigrade. This relation is frequently written, for the resistance of any conductor, as

$$R_2 = R_1 [1 + \alpha_1(t_2 - t_1)], \qquad [148]$$

in which R_1 is the resistance at temperature t_1, R_2 is the resistance at temperature t_2, and α_1 is the temperature coefficient of resistance at temperature t_1 degrees. Hence α is the fractional change in resistance due to unit change in temperature. For pure metals α is of the order of 0.004 per degree centigrade at 20 degrees centigrade. Another form that is convenient for calculation is

$$\frac{R_2}{R_1} = \frac{K + t_2}{K + t_1}, \qquad [149]$$

in which

$$K = \frac{1}{\alpha_0} \qquad [150]$$

depends upon the material of the conductor. For standard annealed copper,

$$K = 234.5 \text{ C} \qquad [151]$$

and,

$$\alpha_0 = 0.00427/C \qquad [152]$$

at zero degrees centigrade.*

A conductor carrying current dissipates energy in the form of heat, and its temperature therefore rises. In metallic conductors (with the exception of some alloys) this rise causes an increase† in the resistivity ρ. Since the temperature rise and the accompanying change in resistivity are functions of the energy input, they become functions of the current and its variation with time. In conductors used as standard resistances in measurements, this change in resistance with current, due to self-heating, is important and must be taken into account in careful work. This same change in resistance is useful in another application. For example, when suitably mounted in a nonoxidizing atmosphere, small wires of certain materials can be made to change resistance rapidly with change in current, due to self-heating, in such a way that the current remains nearly constant in spite of changes in the voltage. These devices, known as *regulator* or *ballast tubes*, are useful in obtaining a nearly constant current from a variable-voltage source.

Courtesy Western Electric Co.

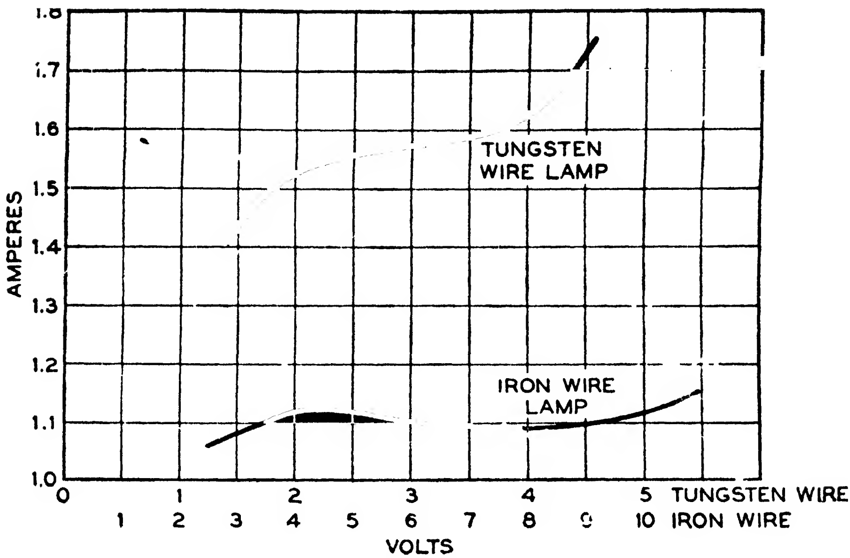
Iron-wire ballast lamp, characteristic shown on the plot on p. 77.

It is interesting that some metals become *superconducting;* that is, their electrical resistivity becomes zero at temperatures slightly above absolute zero. As yet no engineering application of this remarkable effect has become practicable.

The effects of other conditions, such as mechanical strain in various forms, heat treatment, age, very high and very low temperatures, on the conductivity of metals may be significant in unusual circumstances. The study and explanation of these phenomena are largely in the hands of the atomic physicist.

*A table giving the characteristics of various materials as conductors and another table giving the physical and electrical characteristics of copper wire are given in App. A. Extensive tables covering the physical and electrical properties of materials are in the Smithsonian Physical Tables and in the International Critical Tables. Detailed tables on copper and other electrical conductors are in electrical engineering handbooks. Manufacturers of materials used in the electrical art often can supply data on conductivity and its variation with temperature.

† Carbon and most materials commonly classed as insulators show a decrease in resistance with an increase in temperature. The same is true of most liquids.



Characteristics of Ballast Lamps. *Courtesy Western Electric Co.*

12. Practical electrical conductors

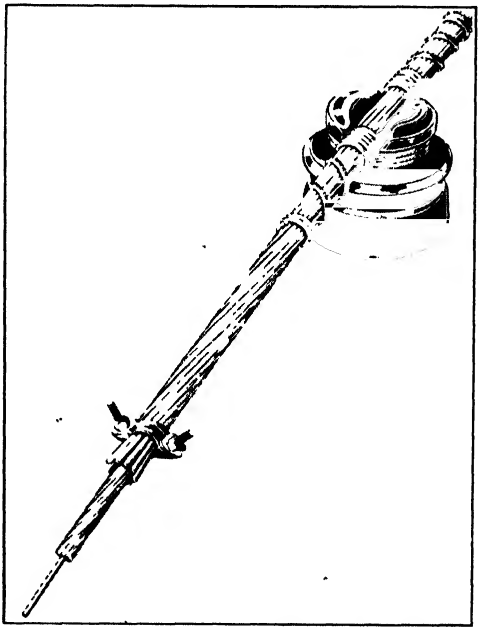
Practical electrical conductors may be divided into two classes, according to the purpose they fulfill: Those used primarily to conduct electrical energy from one point to another with a minimum of loss form one class; those used primarily to obstruct or resist the flow of charge may be grouped into a second class. Electrical lines of all sorts, including those used both in power transmission and in communications, fall into the first class. Resistors of all varieties, as well as heating devices, make up the second class.

Power-transmission and distribution lines offer examples of the highest development of electrical conductors for the purpose of carrying energy from one point to another with small loss. Where power is to be transferred in large quantities over long distances, conductors of either copper or aluminum are used almost exclusively. These two metals possess high conductivity and at the same time are available in sufficient quantities at a reasonable cost. The principal factors which enter into a choice between them for a large power-transmission line are four, namely, electrical conductivity, density, tensile strength, and cost. The electrical conductivity of aluminum is only 60 per cent of that of copper. Offsetting this disadvantage is the fact that aluminum weighs only one-third as much as copper per unit volume. The tensile strength of aluminum is less than half that of copper, but aluminum conductors can be reinforced easily by the use of a steel core, about which the aluminum is laid in

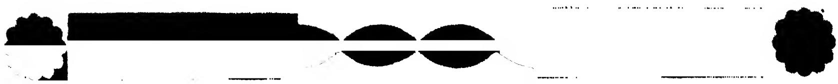
strands. From the point of view of cost, the choice between aluminum and copper for a line of given conductivity is dictated largely by the relative prices of the two metals at the time of construction. For certain applications, however, especially in transmission lines and rural distribution lines, steel-reinforced aluminum conductors often show an advantage in cost over copper because their use permits installation of lighter-weight supporting towers or poles or of a smaller number of supports per mile, thereby reducing the total installed cost of the line. For applications where many branch connections are made, copper is preferred because it is more flexible.

Power-distribution systems in urban areas are now almost entirely underground, in ducts carrying cables - the name given to assemblies of conductors surrounded with layers of insulation and inclosed in semi-flexible lead sheaths. The insulation in modern power cables commonly consists of paper tapes wound spirally about the inner conductor and then impregnated with oil. Because the cost of cable insulation and of ducts in which to lay cables represents a substantial part of the total cost of any cable installation and also because a cable must be flexible, its volume must be kept at a minimum. Small volume is secured by the use of a copper conductor, which offers the greatest practicable conducting power per unit volume.

Aluminum cable, steel reinforced, and protected by armor rods at the points of support. The aluminum wires are stranded around the steel core. The armor rods prevent mechanical failure of the conductors due to fatigue resulting from vibration caused by wind. The armor rods also permit a stronger tie to the insulator, and protect the conductor against wear from the tie wire, chafing in the tie groove and burning caused by flash-over.



Courtesy Aluminum Company of America.



Courtesy Anaconda Wire and Cable Co.

Bare conductor with concentric stranding on twisted I-beam core.

Bare and insulated conducting wires and cables are obtainable in sizes from several circular mils to several million circular mils in cross-sectional area. Conductors are made also in the form of tubes, flat straps, channels and miscellaneous forms. The current-carrying capacity goes over a range from millionths of amperes to thousands of amperes and while directly related to the cross-sectional area depends also on so many other circumstances that cross-sectional area can be taken only as a very crude guide. There are many forms of insulation suitable under various circumstances for nominal voltage ratings up to 230,000 volts (effective alternating voltage at 60 cycles per second; 325,000 volts peak). Wire and rectangular strap for winding coils is ordinarily insulated with one or more layers of asbestos, cotton, silk, paper, enamel, or enamel covered with cotton or silk and can be use[1] up to several hundred volts. The wires of telephone cables for indoor use are ordinarily insulated with silk, cotton, enamel, or enamel covered with silk or cotton; for outdoor use the wires ordinarily have dry paper ribbon or extruded wood pulp insulation. The insulation of the wires of telephone cables is tested at voltages up to 1,400 volts (maximum instantaneous value at 60 cycles per second). The various general types of insulation used for power cables are ordinarily recommended within the following limits of effective alternating voltages:

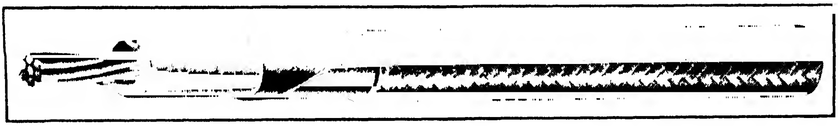
Asbestos	up to 13,800
Rubber	up to 15,000
Varnished cambric	7,500 to 26,000
Oil-impregnated paper (solid type) .	10,000 to 75,000
Oil-filled paper (with oil ducts) . .	35,000 to 230,000

The actual selection of a proper conductor for an installation depends upon the current and voltage requirements, the temperature, whether indoor, outdoor, aërial underground or submarine, the possibility of corona formation, corrosion or other special difficulties, the installation facilities at hand, and the overall economic considerations. For detailed information the student is referred to manufacturers' literature such as:

General Electric Co., *How to Select Insulated Cable,* GEA 1837, May 1934.
Industrial Cable, GEA 1838, September 1934.
Tips on Cable Uses, GEA 2532, January 1937.
Western Electric Co., *Telephone Apparatus and Cable,* Catalog No. 10, 1939.
General Cable Corporation, *Bare and Tinned Wire and Cables,* Catalog BT-37.
Paper Insulated Power Cables, Catalog PIC-37.
Alectral, 1930.
Aluminum Company of America, *Electrical Characteristics of A.C.S.R.,* 1934.
A.C.S.R. Rural Lines, 1939.
Channeluminum Electrical Conductors of Alcoa Aluminum, 1931.

Courtesy General Cable Corp

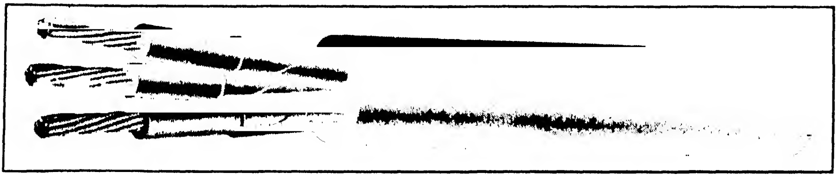
Bare hollow conductor composed of interengaged tongue and groove segments



Courtesy General Electric Co

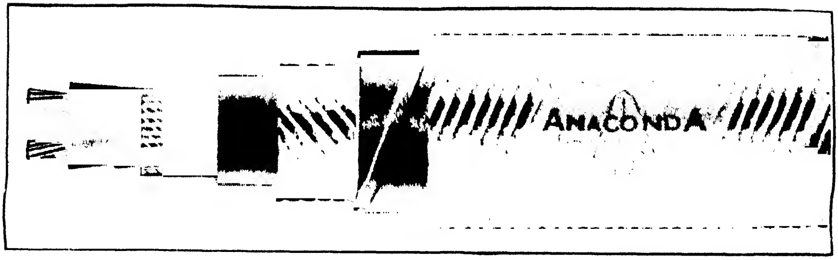
Weatherproof braided sheath rubber insulated single conductor cable used for all ordinary outdoor and indoor wiring or in conduit or reasonably dry ducts

Armored Bushed Cable (ABC) Commonly used in house wiring Rubber and cotton braid insulated wires wrapped in paper tape and armored with galvanized iron interlocked tape Earlier cable known as BX was similar except that braided cotton fabric was used instead of paper tape (Manufactured by various companies)



Courtesy General Electric Co

Nonhygroscopic rubber-sheath rubber insulated three conductor parkway cable, for laying in earth without use of ducts

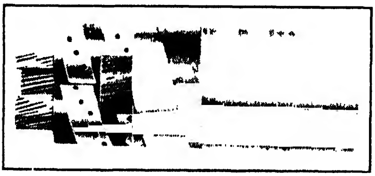


Courtesy Anaconda Wire and Cable Co

Armored three conductor parkway cable for laying in the earth without use of ducts

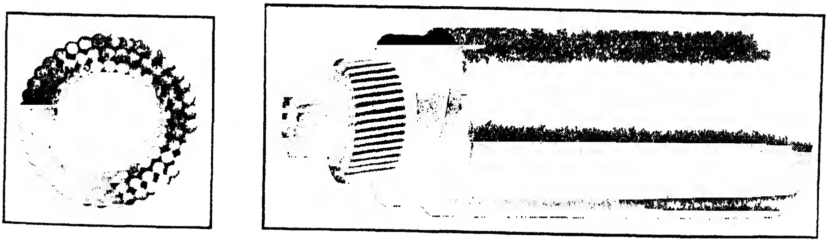
Courtesy General Electric Co

Asphalt compound covered steel wire armored asphalt jute covered rubber insulated three-conductor submarine cable



Courtesy Anaconda Wire and Cable Co

Three conductor solid type shielded cable The term *solid type* means that the insulation is composed of layers of paper tapes applied helically over the conductor and impregnated with oil The inner spirally wound ribbon shields are to obtain approximately radial flux distribution about each conductor The outer spirally wound ribbon shield is to avoid potential between the outer insulation and the sheath

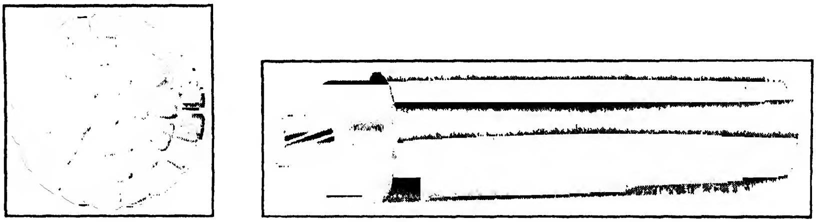


Courtesy General Cable Corp

Solid type single conductor cable having annular concentric stranding over a rope core

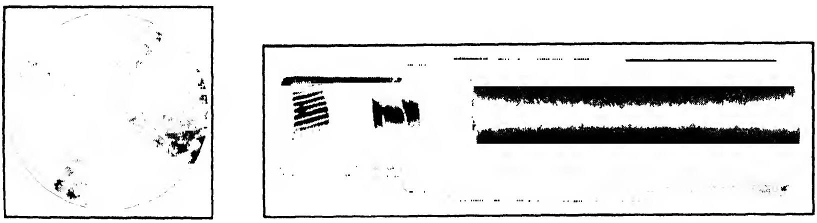
Courtesy General Cable Corp

Oil filled single-conductor cable having annular concentric stranding over a hollow core of helically wound steel or copper strip An oil filled cable differs from a solid type cable in that a channel is provided for oil flow, in this case the hollow core



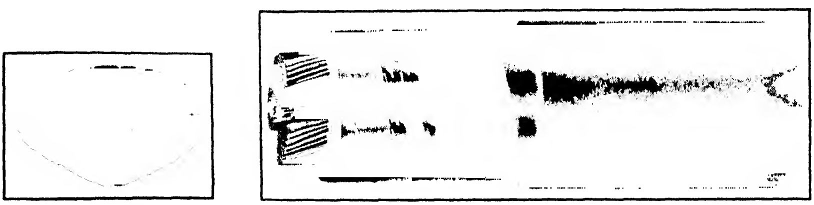
Courtesy General Cable Corp

Solid-type single conductor cable with concentric stranding rolled to eliminate space between strands



Courtesy General Cable Corp

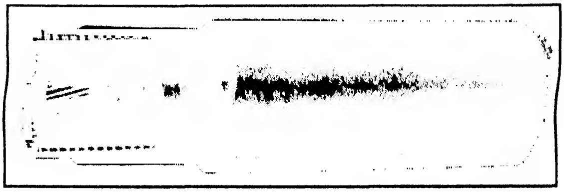
Solid type single conductor segmental cable The insulation of the four segments from each other makes large conductors practical for alternating current without the increase in conductor diameter inherent in the annular construction



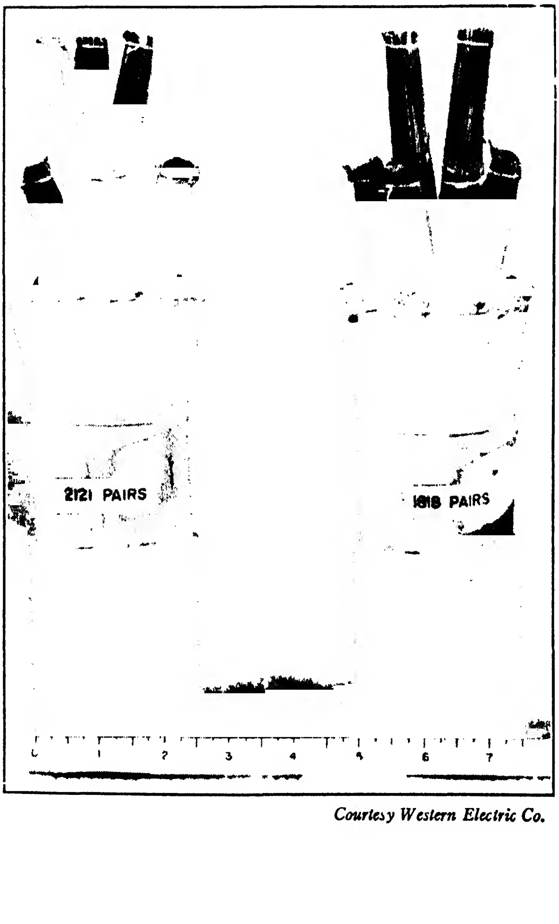
Courtesy General Cable Corp

Solid-type, three-conductor shielded sector cable

Oil filled, three-conductor shielded sector cable.

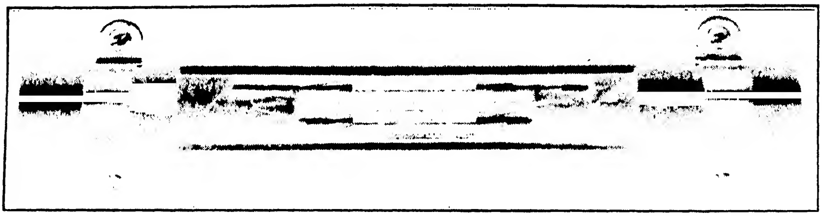


Courtesy General Cable Corp.



Courtesy Western Electric Co.

Multipair telephone cable, 26 AWG wire, pulp insulation, lead sheath.



Courtesy Bell Telephone Laboratories

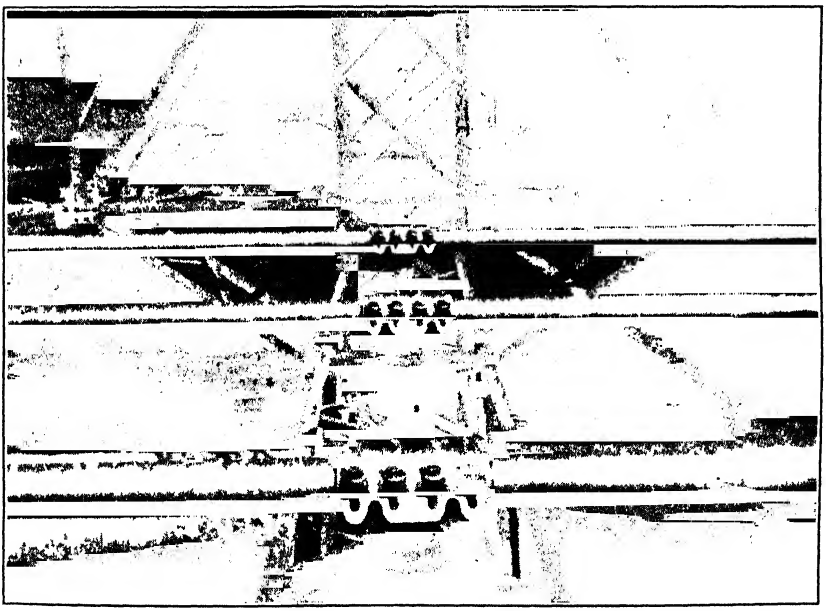
Two coaxial cable units and two 19 AWG paper insulated quads. Each coaxial unit consists of a central wire supported by hard rubber disks spaced $\frac{3}{4}$ inch apart, around which is wrapped the outer conductor, composed of interlocked copper tapes.

Courtesy Aluminum Company of America

Aluminum channel bus mounted on indoor porcelain insulators.

Courtesy Aluminum Company of America

Aluminum channel and flat bus conductors at the plant of the Aluminum Company of America at Alcoa, Tennessee. The main bus run is over 600 feet long.



Courtesy Aluminum Company of America

Aluminum tubular bus installation in the Columbia Primary Substation of the Tennessee Valley Authority. The spans are 40 feet long.

Many applications of electrical energy require its conversion directly into heat. The object then becomes not to conduct the current as perfectly as possible but to oppose its passage in a particular section of the circuit. Copper or aluminum conductors of sufficiently small cross section would accomplish this result, were it not for their rapid disintegration at comparatively low temperatures and their rapid rise of resistance with increase in temperature. Alloys of nickel, chromium, copper, and other metals have been developed which can withstand temperatures of 1,000 degrees centigrade for long periods of time and which, at the same time, have low temperature coefficients of resistivity. Fortunately for this purpose, these alloys, in common with most others, have resistivities many times greater than the metals of which they are composed. One such alloy in common use, nichrome, has a resistivity more than fifty times that of copper.

One of the standard electrical quantities on which all electrical measurements are based is the unit of resistance, the ohm. For purposes of

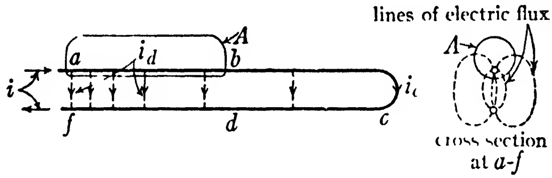


FIG. 34. Conductor bent back on itself.

measurement it is essential that resistance standards have values which remain fixed under normal working conditions. To this end alloys have been developed whose temperature coefficients of resistance are not only small but completely negligible over a small range in the region of room temperature. The conductors of nearly all standard or precision resistors are made of these materials and, in addition, frequently are cooled in oil baths and very conservatively rated, to guard against overheating. In precision resistors great pains are taken to avoid displacement currents, skin effect, and inductive effects. Skin effect can be reduced by the use of sufficiently fine wires. Both capacitive effects and inductive effects can be greatly reduced by the use of one or more of the many winding methods which have been developed for this purpose.[17]

Capacitance effects are reduced by making the connections in such a way that adjacent turns have small differences of potential. If the conductor is doubled back on itself and wound as a pair, the inductance can be reduced to a low value, though the capacitance is increased thereby.

[17] F. E. Terman, *Measurements in Radio Engineering* (New York: McGraw-Hill Book Company, Inc., 1935), 92–94.

This result may be visualized more clearly by reference to Fig. 34. If the conductor resistance is R and it carries a current i, the potential difference between points a and f is Ri volts. Therefore an electric field and hence an electric displacement exist between the conductors, as indicated in the cross-sectional sketch. If i is changing with time, this electric flux changes, constituting a displacement current between the conductors. It can be seen by inspection, or, more rigorously, by applying the principle of conservation of charge to a surface whose traces with the plane of the paper are shown as lines A, that the conduction current is less at b than at a and is least at c.

As a result of the conditions just described, the potential difference between a and f as calculated by the relation

$$v = Ri \tag{153}$$

using

$$R = \rho \frac{\ell}{A} \tag{52c}$$

is in error in two ways: First, i is a function of the distance of a point on the conductor from a; second, it is a function of dv/dt because the displacement current is. The result of the second is that the conductor acts as if a condenser were connected in parallel with it, as in effect there is. The conductor therefore cannot be characterized electrically by the single constant parameter R as given by Eq. 52c but is more accurately approximated by the combination of equal lumped resistances and equal lumped capacitances as shown in Fig. 35. This representation becomes more nearly correct the larger the number of equal components into which the total resistance and capacitance are subdivided. For the actual configuration, the resistance and capacitance effects are uniformly distributed; this characteristic is often described by referring to the parameters as distributed parameters.

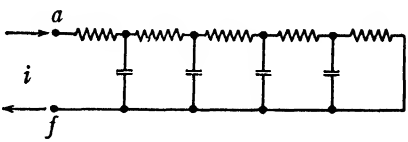


FIG. 35. Electric circuit representation approximation equivalent to the conductor of Fig. 34.

The development of the radio industry has made easily available low-power resistors of a wide range of values. Many of these are wound with wire of low-temperature coefficient, considerable precaution being taken to avoid inductive and capacitive effects. Others are made of carbon or some other semiconductor mixed with ceramic materials. Although these have very small inductance or capacitance effects, their temperature coefficients are usually high. Still others are made by the evaporation and subsequent condensation of metal in a very thin layer on the surface of an insulator. These possess the characteristics of a metallic resistor

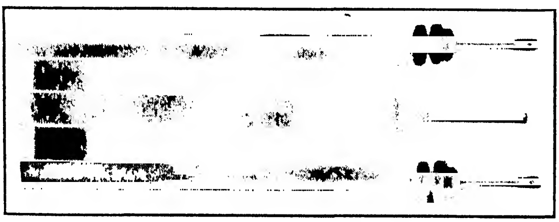
and are frequently more stable and less affected by temperature than those made from semiconductors

Resistors of these three types are available in the ranges of resistance and power rating shown in the following table. The first and last types lend themselves to fabrication in the form of rheostats.

REPRESENTATIVE RESISTOR RATINGS FOR LOW POWER RESISTORS

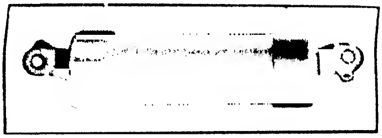
Material	*Approximate Range of Resistance (ohms)*		*Approximate Range of Power Dissipation (watts)*
	Fixed	*Adjustable*	
Wire	$0\ 10^{7}$	$0\ 10^{5}$	1–10
Metal in thin layers	$10^{2}\ 10^{8}$	—	0.1–2
Semiconductors	$10^{2}\ 10^{12}$	10^{3}-10^{6}	0.1–2

Most of the resistors used in practice depend on convection by air currents for the dissipation of the heat developed in them. Therefore, their ability to absorb electrical power without damage to themselves depends on their surface areas and their safe operating temperatures. The heat lost by a surface in still air varies from about 0.005 to 0.010 watt per square inch per degree centigrade above air temperature. For dissipating large amounts of energy, as in railway motor control; iron grids supported by procelain and mica are used. This construction permits large temperature rises and free air circulation. In machinery insulation, where organic materials are employed, temperature rises of 50 degrees to 70 degrees centigrade are allowed, with forced air circulation aiding in the dissipation of heat. In measuring devices, where constancy of resistance is essential, the allowable energy dissipation per unit area is usually about one-tenth that permitted in machinery.



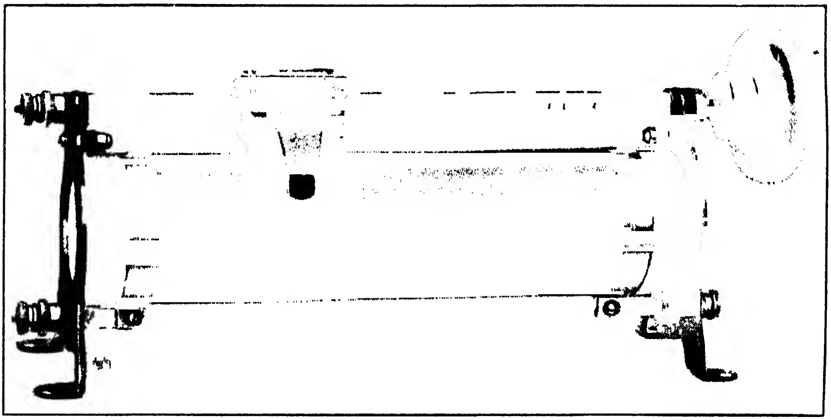
Courtesy Western Electric Co.

Wire-wound center-tap resistor for rack mounting used in telephone circuits, wound for a fraction of an ohm up to several thousand ohms.



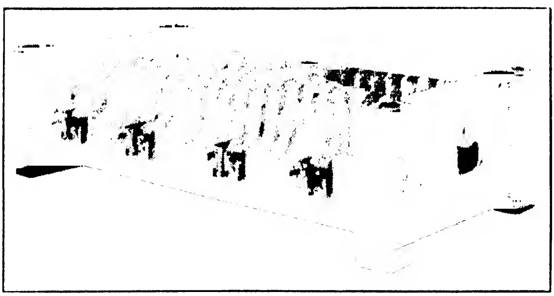
Courtesy Western Electric Co

Carbon filament resistor wound for 4,000 to 100,000 ohms



Courtesy James G. Biddle Co

Resistance wire wound on porcelain tube, for a fraction of an ohm up to 30 000 ohms, arranged for use as series rheostat or dropwire

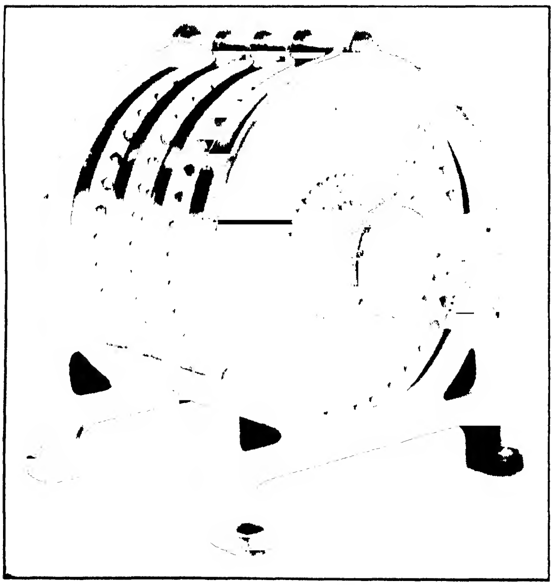


Courtesy General Electric Co

Cast grid type resistor for motor starting or control rheostat

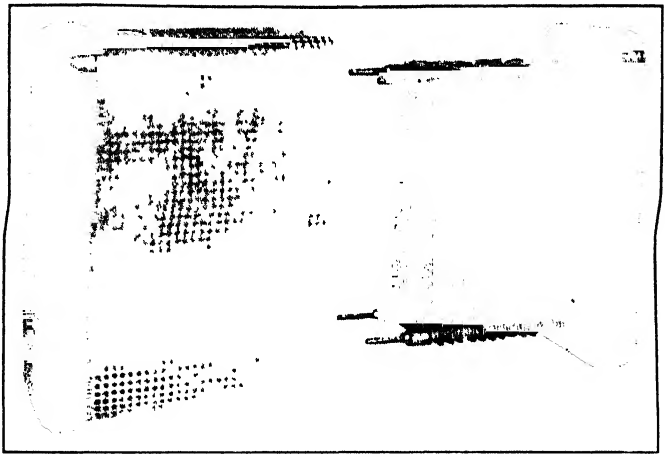
Courtesy General Electric Co.

Enclosed motor-starting rheostat for direct-current motors.



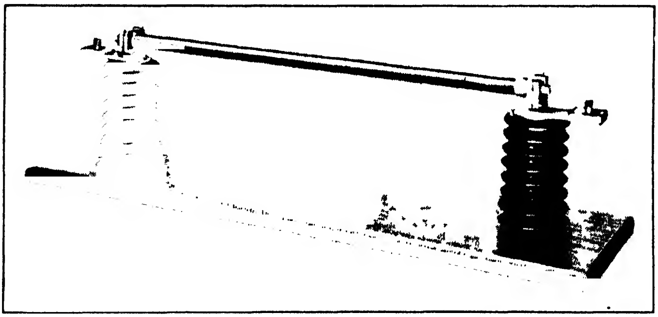
Sprocket driven field rheostat assembly, several sections in parallel for adequate current-carrying capacity.

Courtesy General Electric Co.



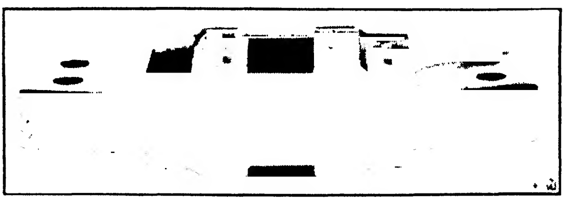
Courtesy Weston Electrical Instrument Co.

Voltmeter series multiplier resistor, zero temperature coefficient wire wound noninductively on cards; resistance up to 5 megohms for use up to 5,000 volts with instruments having resistance of 1,000 ohms per volt full scale.



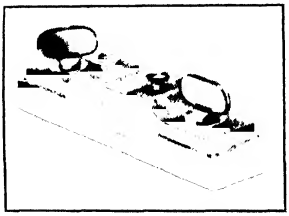
Courtesy Weston Electrical Instrument Co.

High-voltage tubular series multiplier resistor, 5 to 30 megohms for 5,000 volts and up.



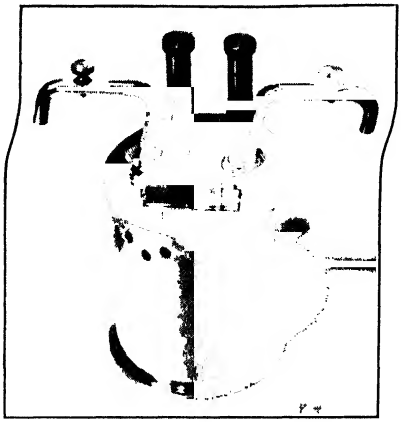
Ammeter shunt for insertion in bus structure; for use with instruments requiring 50 millivolts drop for full-scale deflection; ratings up to 15,000 amperes; zero temperature coefficient

Courtesy Weston Electrical Instrument Co.



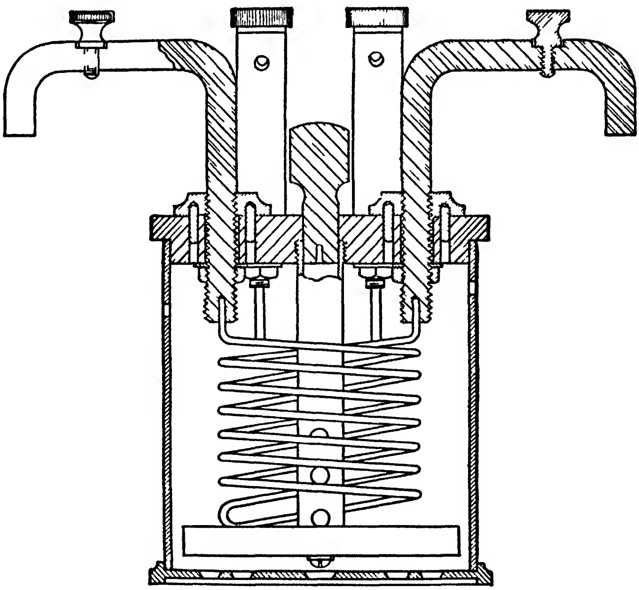
Courtesy Weston Electrical Instrument Co

Portable Ammeter Shunt for use with instruments requiring 50 millivolts drop for full scale deflection ratings up to 500 amperes zero temperature coefficient



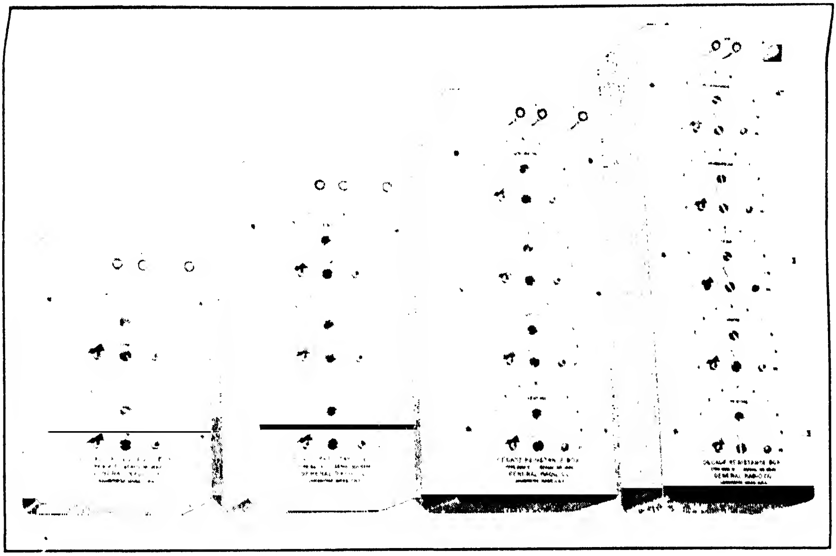
Courtesy Leeds and Northrup Co

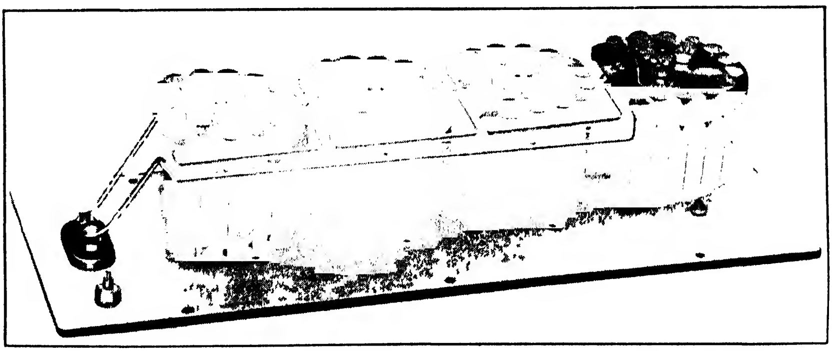
Standard resistance 0 1 ohm unit of style made in ratings from 0 001 to 0 1 ohm, zero temperature coefficient



Courtesy Leeds and Northrup Co

Section through standard resistance





Courtesy General Radio Co.

Standard decade resistance, zero temperature coefficient, wire wound on cards in manner which minimizes inductance and capacitance.

13. Series and parallel combinations of resistance

If a number of resistances are connected in series to form a circuit, such as $R_1, R_2 \ldots$, it follows from the definition of the resistance parameter (and has been often checked experimentally) that the resistance of the series combination is equal to the sum of the individual resistances. That is, the potential drops across the separate resistances are additive while the current is the same in each and

$$V = R_1I + R_2I + \cdots = (R_1 + R_2 + \cdots)I = R_0I \qquad [154]$$

if

$$R_0 \equiv R_1 + R_2 + \cdots, \qquad [155]$$

where R_0 is the total resistance of the combination.

When resistances appear in parallel the potential difference across each resistance is the same and the currents are additive. For each resistance

$$I = \frac{V}{R} = GV. \qquad [156]$$

If the several resistances of a parallel combination have individual conductances $G_1, G_2 \ldots$, the total current I is

$$I = G_1V + G_2V + \cdots = (G_1 + G_2 + \cdots)V = G_0V \qquad [157]$$

if

$$G_0 \equiv G_1 + G_2 + \cdots, \qquad [158]$$

where G_0 is called the total conductance of the combination.

Series and parallel combinations of resistances may be treated, on the above bases, by considering the series and parallel groups as separate elements and then combining these to form the resultant resistance or conductance, whichever may be desired.

14. Illustrative examples of resistance calculations

The first example illustrates the calculation of resistance from dimensions, the total resistance of series-parallel combinations, and the use of a temperature coefficient. One part, or phase, of the armature winding of an alternating-current generator contains 1,200 inductors, each extending from the front of the machine to the back, a distance of 60 inches. Each inductor consists of six copper straps 0.114 inch by 0.258 inch, connected in parallel. The completed armature winding of this phase is divided into three groups of coils, these groups being connected in parallel. The problem is to find the resistance of this phase at 25 degrees centigrade,

taking the resistivity as that for annealed copper. No allowance is made for rounding of the corners of the copper straps.

Solution: From Table IV of App. A the resistivity ρ for annealed copper is 1.724×10^{-6} ohm-cm and the temperature coefficient α_{20} is 0.00393/C. The resistance at 20 C of one strap is, from Eq. 52c,

$$\frac{1.724 \times 10^{-6} \times 60 \times 2.54}{0.114 \times 0.258 \times (2.54)^2} = 1.384 \times 10^{-3} \text{ ohm}. \qquad [159]$$

The resistance of one inductor is, from Eqs. 157 and 158,

$$\frac{1.384 \times 10^{-3}}{6} = 2.306 \times 10^{-4} \text{ ohm}. \qquad [160]$$

There are 1,200/3, or 400, inductors in series in one group; hence using Eqs. 154 and 155 gives for the resistance of one group

$$400 \times 2.306 \times 10^{-4} = 9.224 \times 10^{-2} \text{ ohm}. \qquad [161]$$

Since there are three such groups in parallel, the resistance of the armature at 20 C is

$$R_{20} = \frac{9.224}{3} \times 10^{-2} = 0.0307 \text{ ohm}. \qquad [162]$$

By use of Eq. 148 the desired resistance at 25 C is

$$R_{25} = 0.0307 [1 + 0.00393 (25 - 20)] = 0.0313 \text{ ohm}. \qquad [163]$$

The second example illustrates the use of the change of resistance to measure temperature. The cold resistance of the high-voltage winding of a transformer is measured with direct current and is found to be 5.7 ohms at 21.0 degrees centigrade. After a heat run, the transformer is taken off the alternating-current power line and the resistance of the same winding is found to be 6.6 ohms. The temperature of the air at that time is 18 degrees centigrade. What is the average temperature rise of the winding due to its load current?

Solution: Equation 149 gives

$$\frac{6.6}{5.7} = \frac{234.5 + t_2}{234.5 + 21.0}, \qquad [164]$$

from which,

$$t_2 = 62 \text{ C}. \qquad [165]$$

The average temperature rise is then

$$62 - 18 = 44 \text{ C}. \qquad [166]$$

15. Practical condensers

From the point of view of nearness to the ideal, condensers may be divided into two groups, namely, those using air or other gases as dielectric, and those using solid or liquid materials. Air condensers approach

the ideal of pure capacitance so closely that elaborate methods are required to detect their deviation from perfection. Even when such methods are applied the deviations measured can usually be traced to effects of the solid insulators which support the condenser plates. By the use of gases at high pressure the energy storage per unit volume can be made to approach that obtainable with solid dielectrics.

Condensers using solid or liquid dielectrics are not so near the ideal as those using air as dielectric, but nevertheless their efficiency is sufficient so that in many engineering uses the deviation may be neglected.

In early condensers glass was used almost exclusively as dielectric. Because of its fragility and its unadaptability to economic use at low voltage, it has been almost entirely superseded. The most common dielectric at present is paper impregnated with oil, paraffin, or certain noninflammable synthetic organic liquids. Since paper is available in thin sheets it can be applied economically to low-voltage uses and can be built up to any desired thickness for high-voltage uses. During the past few years important advances have been made in the production, on a commercial scale, of paper condensers impregnated with noninflammable synthetic organic compounds. Condensers of this type for use at low frequencies have become so compact and inexpensive that they are being used extensively for power-factor correction on power systems. They are used also with induction furnaces and in wave filters for rectifier outputs.

In many ways mica is a highly desirable dielectric for condensers, but its use is limited because of the small quantity available and the high cost. Standards of capacitance are frequently built of mica because of its desirable mechanical properties and its freedom from aging effects. In radio-frequency applications, where air condensers of sufficient size are unwieldy and other condensers using solid dielectric have excessive losses, mica is frequently used.

The deviation from perfection in impregnated paper condensers appears in two forms, namely, as a slight variation in the apparent value of the permittivity of the dielectric, and as a leakage current through the dielectric. The variation in the apparent permittivity follows no simple law but varies from condenser to condenser and from time to time. It depends on temperature, applied voltage, voltages previously applied, and frequency. As a result, its apparent magnitude depends on the method by which, and the conditions under which it is measured. The leakage resistance likewise follows no simple law, and is dependent on the same factors as the apparent permittivity.

The variations in effective capacitance of an impregnated paper condenser rarely exceed one per cent. While the variations in resistance may amount to several hundred per cent, even at its worst the leakage is so small that it is usually unimportant. A good modern condenser has a

leakage resistance of at least 1,000 ohms per daraf and frequently ten or a hundred times that much.

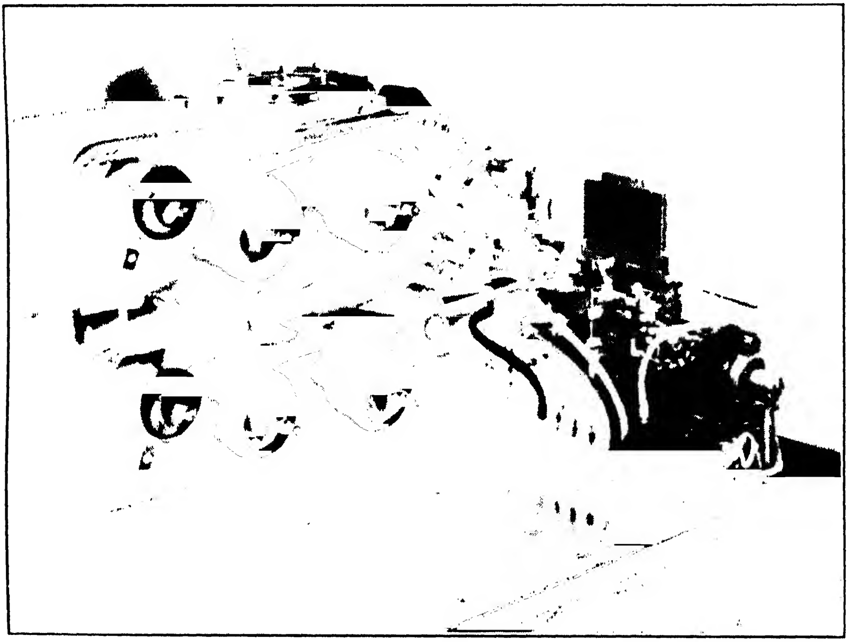
Electrolytic condensers are used where a large value of capacitance is required for use at low voltage. Their condenser action is centered in an insulating layer a few molecules thick which is formed on one electrode of an electrolytic cell. Because of the thinness of the layer a great concentration of capacitance is possible in a small space, but for the same reason the electrical breakdown strength is small. The insulating properties of electrolytic layers are so far inferior to those of impregnated paper that the leakage resistance of condensers employing them may be more than a thousand times smaller than in equivalent paper condensers. The resistance of the insulating layer is much greater for one direction of current across it than for the other direction; therefore, unless the internal construction is such as to render the condenser nonpolarized, it is necessary to connect the condenser in accordance with its polarity marks. Polarized condensers are not suited for use in alternating-current circuits. Electrolytic condensers nevertheless find a wide variety of applications where their comparatively large losses can be tolerated.

The choice among the air, mica, and impregnated-paper types of condenser for a given use depends to a considerable extent on the value of capacitance required. Air-insulated condensers are ordinarily made for the range of capacitance between a few micromicrofarads and a few thousand micromicrofarads, with voltage ratings ranging from a few hundreds to a few ten thousands of volts. In condensers of this type continuous adjustment of capacitance over a wide range is accomplished easily by change of the relative positions of the plates. For this reason air condensers are often made continuously adjustable. For the range of capacitance between a few hundred micromicrofarads and a few tenths of a microfarad, mica condensers are commonly used. They are not susceptible to continuous adjustment as are air condensers. In the ranges above a few tenths of a microfarad, paper condensers are used extensively. They are available in a wide variety of sizes, ranging up to several tens of microfarads in a single unit and in voltage ratings up to several tens of thousands of volts. Electrolytic condensers are seldom used where voltages exceed a few hundred volts but for lower voltages they are available in units having capacitances up to several thousands of microfarads.

One important criterion for judging the effectiveness of a particular type of condenser is its ability to store energy. The storage takes place in the dielectric and therefore the nature of the dielectric, as exhibited in its permittivity and breakdown strength, is the determining factor. The following table gives approximate magnitudes of energy storage per unit volume for condensers made of the materials commonly used. These values hold only at the frequencies to which the materials are suited.

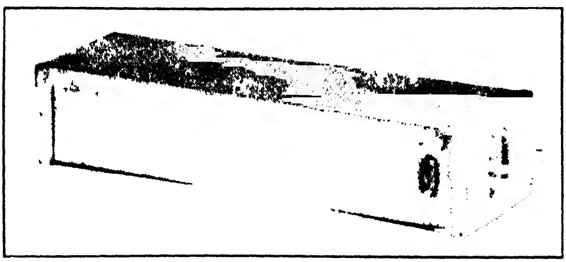
ENERGY STORAGE IN CONDENSERS

Material	Energy in joules per cubic meter
Air (at atmospheric pressure)	5– 10
Air (10 atmospheres pressure)	500– 1,000
Paper impregnated with oil	1,000– 2,000
Paper impregnated with synthetic compound	3,000–15,000
Mica	1,000– 2,000
Electrolytic	5,000–40,000



Courtesy Western Electric Co.

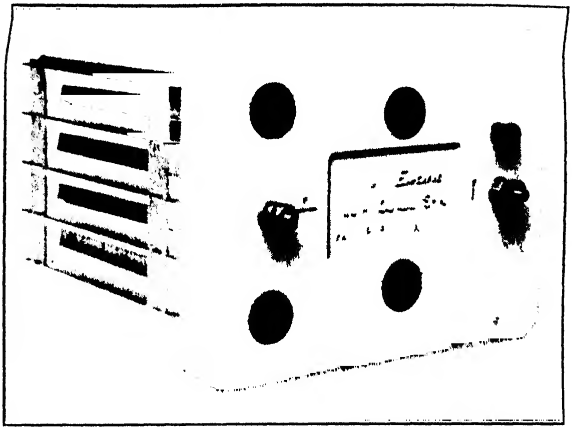
Machine for winding paper condensers.



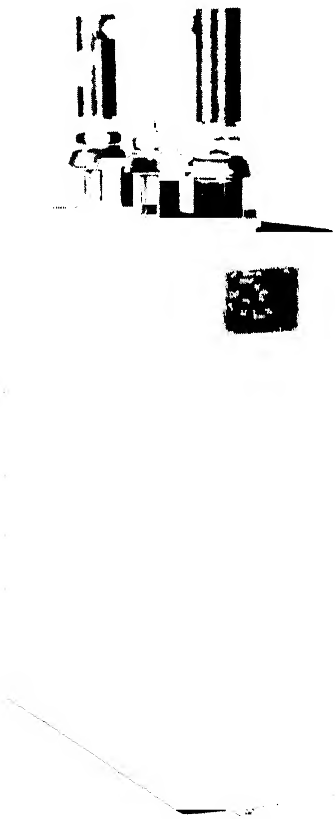
Two-unit (0 5 and 2 0 microfarad) paper condenser used in telephone circuits

Courtesy Western Electric Co.

Electrolytic Condenser 0 33 direct voltage bias, 1550 microfarads 0 25 ohm resistance at 60 cycles per second, 800 microfarads, 0 15 ohm resistance at 2 000 cycles per second

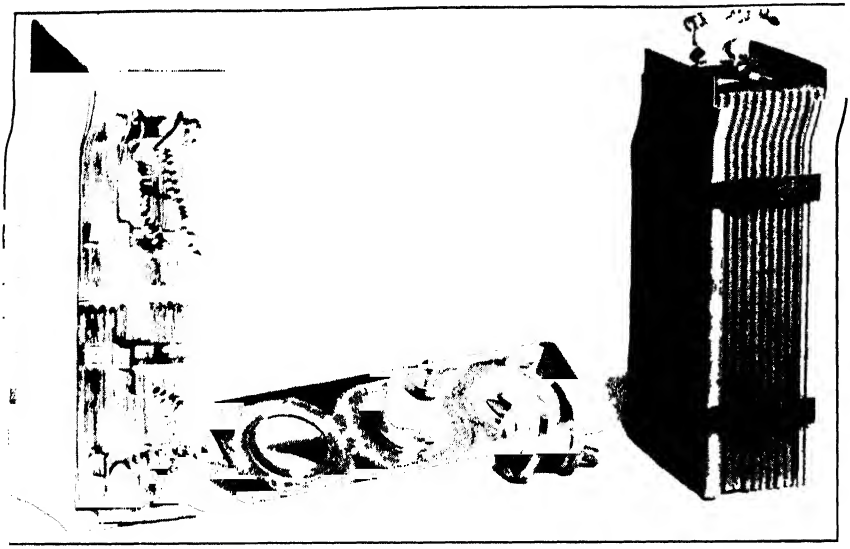


Courtesy Western Electric Co



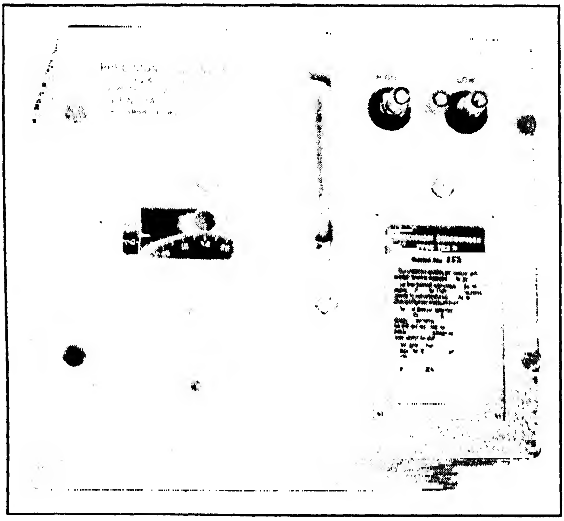
Three phase power condenser having Inerteen a noninflammable liquid dielectric, soaked into paper wound on aluminum foil made in ratings from 0 5 to 15 kilovolt amperes 230 to 4 600 volts, 60 cycles per second ratings

Courtesy Westinghouse Electric and Manufacturing Co.



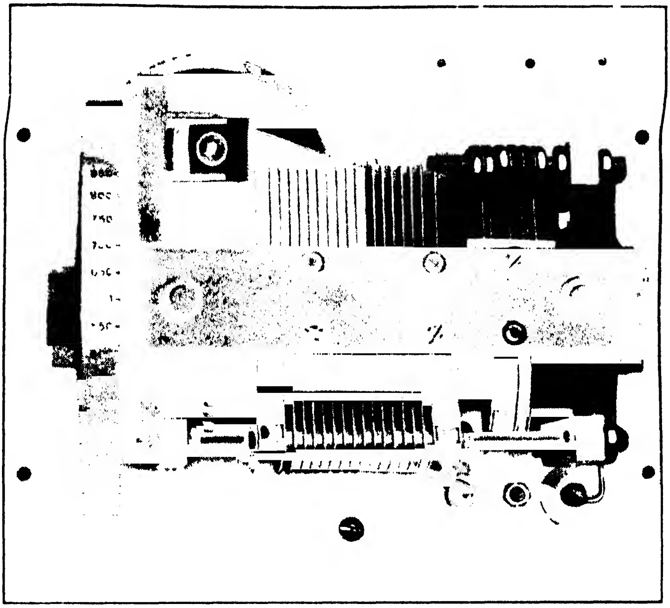
Courtesy Westinghouse Electric and Manufacturing Co

Parts of three phase Inerteen condenser



Courtesy General Radio Co

Precision condenser having plates movable in air dielectric.



Courtesy General Radio Co.

Inside view of precision condenser.

16. Series and parallel combinations of capacitances

In Fig. 36 are shown three elastances S_1, S_2, and S_3 charged with $q_1(t)$, $q_2(t)$, and $q_3(t)$,* respectively, by displacement of positive charge in the arrow direction (electrons in opposite direction). Since the potential differences are additive, the potential difference $v(t)$ of the polarity indicated is

$$v(t) = S_1q_1(t) + S_2q_2(t) + S_3q_3(t). \tag{167}$$

This relation can be simplified because $q_1(t)$, $q_2(t)$, and $q_3(t)$ are not independent, in general. If conduction in the dielectrics is negligible, the charge on each elastance is the sum of two components. The first is the charge on the individual elastance at some arbitrary instant from which time is measured. The second is the charge $q(t)$ that is circulated from

* In this article it is convenient to use functional notation for the voltage and charge, which are in general functions of time In this way, for example, the value of a function $q(t_1)$ corresponding to a given value t_1 of the argument t can be readily indicated, as well as the general function $q(t)$ of the variable t Further use is made of functional notation in Chs III, V, VII, and VIII; it is discussed principally in Ch. III.

a to f around the group as a whole thereafter. Thus if the initial charges are $q_1(0)$, $q_2(0)$, and $q_3(0)$, respectively,

$$q_1(t) = q_1(0) + q(t), \tag{168}$$

$$q_2(t) = q_2(0) + q(t), \tag{169}$$

$$q_3(t) = q_3(0) + q(t). \tag{170}$$

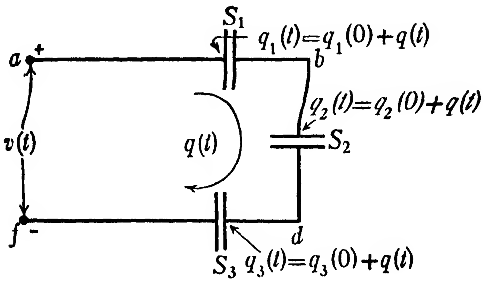


FIG. 36 Capacitances in series.

By use of these relations, Eq. 167 becomes

$$\left.\begin{aligned} v(t) &= S_1[q_1(0) + q(t)] + S_2[q_2(0) + q(t)] + S_3[q_3(0) + q(t)] \\ &= S_1q_1(0) + S_2q_2(0) + S_3q_3(0) + [S_1 + S_2 + S_3]q(t) \\ &= V(0) + S_0q(t), \end{aligned}\right\} \tag{171}$$

in which $V(0)$ is the potential difference across the combination at the first instant, or

$$V(0) \equiv S_1q_1(0) + S_2q_2(0) + S_3q_3(0) \tag{172}$$

and

$$S_0 \equiv S_1 + S_2 + S_3. \tag{173}$$

Equation 171 suggests that the circuit of Fig. 36 with initial charges on the elastances can be represented between its terminals a and f by a simple equivalent circuit. Two circuits are said to be equivalent if they are indistinguishable by any electrical tests made at specified corresponding terminals. Figure 37 shows this equivalent circuit, which consists of a battery, or source of constant potential difference $V(0)$ of the polarity shown, in series with an elastance S_0. The physical behavior of this equivalent circuit is more easily visualized than that of the original, and for this reason it may be useful. If there is no initial charge on the elastances or, more generally, if $V(0)$ is zero, the equivalent circuit reduces to the elastance S_0 as defined by Eqs. 172 and 173.

The foregoing discussion applies to three elastances in series but its

extension to the more general case of n elastances in series involves nothing more than adding more terms to the equations as given.

For the consideration of condensers in parallel, the use of capacitance C rather than the elastance S leads to more convenient forms. For example, if three condensers having capacitances C_1, C_2, and C_3, respectively, are connected in parallel across a potential difference $v(t)$, then for each capacitance

$$q(t) = Cv(t), \qquad [174]$$

and evidently the total charge on the three capacitances is

$$\left.\begin{aligned} q_0(t) &= (C_1 + C_2 + C_3)v(t) \\ &= C_0 v(t) \end{aligned}\right\} \qquad [175]$$

if

$$C_0 = C_1 + C_2 + C_3. \qquad [176]$$

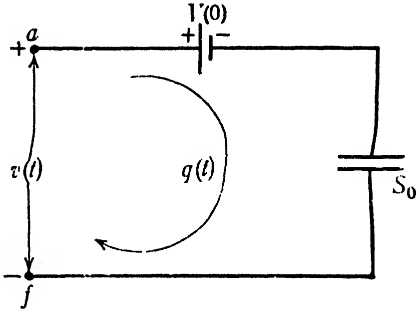


FIG. 37. Circuit equivalent to the circuit of Fig. 36.

Thus it is seen that capacitances in parallel behave like a single capacitance equal to the sum of the individual capacitances.

It is mentioned that any of the foregoing relations between charge $q(t)$ and voltage $v(t)$ can be converted into relations between current $i(t)$ and the time derivative of voltage by differentiation.

17. ILLUSTRATIVE EXAMPLE OF CAPACITANCE CALCULATION

Insulated cable designed for use on high-voltage circuits frequently has a thickness of insulation greater than the diameter of the conductor. In such situations serious variations of potential gradient may occur within the dielectric. Some improvement of conditions may be obtained by using two or more insulating materials having different values of *relative permittivity* or *dielectric constant* K. Thus by use of a dielectric with higher permittivity nearest the inner conductor, the electric-field strength in this vicinity is kept more nearly like that in the outer dielectric.

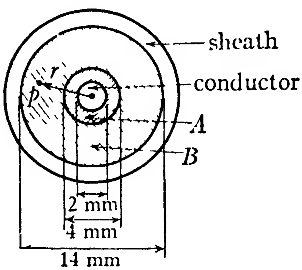


FIG. 38. Cross section of cable having graded insulation.

Such a cable with graded insulation is shown in Fig. 38. If the value of K_A for insulating medium A is 3.0 and the value of K_B for the insulating medium B is 2.0 what is capacitance between conductor and sheath in microfarads per foot?

Solution: The solution is readily obtained by recognizing that the resultant capacitance is the series combination of two component capacitances each of which is calculated from Eq. 89:

$$C_u = \frac{\varepsilon}{2 \ln \frac{r_2}{r_1}}. \tag{89}$$

The resultant capacitance is

$$C_u = \frac{1}{\frac{2 \ln 2}{\varepsilon_A} + \frac{2 \ln 3.5}{\varepsilon_B}}, \tag{177}$$

in which

$$\varepsilon_A = K_A \varepsilon_0; \tag{178}$$

$$\varepsilon_B = K_B \varepsilon_0; \tag{179}$$

$$\varepsilon_0 = 1.113 \times 10^{-10} \text{ in mks unrationalized units.} \tag{180}$$

Hence

$$\left. \begin{aligned} C = \frac{1.113 \times 10^{-10}}{\frac{2}{3} \ln 2 + \ln 3.5} &= 0.649 \times 10^{-10} \text{ farad/m} \\ &= 0.198 \times 10^{-4} \ \mu\text{f/ft.} \end{aligned} \right\} \tag{181}$$

18. Practical inductors

As is true of condensers, inductors are used in a variety of applications in the power, wire-communication, and radio branches of the electrical industry. Some of the more important of their characteristics and applications merit brief consideration at this point.

Although the characteristics of practical condensers are such that in many applications they may be treated as pure capacitances without accompanying resistance and inductance effects, a correspondingly close approach to the ideal inductor has not been attained in practice. In cases in which power loss is significant, the product RC may be taken as a measure of the perfection of a condenser, R being the equivalent parallel resistance and C the capacitance. The analogous quantity in the case of an inductor is the ratio L/R, L being the inductance and R the equivalent series resistance. In practical condensers it is not uncommon for the product RC to be greater than a thousand, whereas in practical inductors the ratio L/R rarely attains a value as great as ten. In other words, the resistive losses in practical inductors are relatively far more important than they are in practical condensers. In addition to resistive losses in practical inductors, the effects of distributed capacitance are often important, whereas the inductive effects in condensers can be detected only by refined measurements. The resistive losses and the effect of dis-

tributed capacitance seriously limit the variety of circuit conditions under which a practical inductor may be treated as a pure inductance.

The amount of inductance which can be obtained in a given volume depends on the number of linkages of magnetic flux with conductor turns. Increasing the number of turns increases the inductance but at the same time increases the resistance. However, the flux can be increased several thousandfold by providing the coil with a core of high-permeability material of which many are now available. Changes in the flux in magnetic materials are accompanied by the absorption of power, but the net gain in L/R over that which can be obtained by any reasonable increase in the amount of conducting material alone is considerable.

The use of high-permeability materials in the magnetic fields of inductors introduces complications arising from variations in the permeability of these materials with field strength. The inductance of the circuit element is no longer substantially independent of current but may vary with it over a wide range. To reduce this variation it is common practice to insert a small air gap in the magnetic material at such a place that all the flux lines must cross it. The maximum value of inductance is thereby greatly reduced, but the variation in inductance with current is also materially reduced.

Inductors with cores of high-permeability materials are in very common use for a wide variety of purposes. An important present application is their use to suppress undesirable high-frequency components in the output currents from mercury-arc rectifiers supplying railways. Likewise radio receivers operated by alternating current employ iron-core inductors to aid in suppressing fluctuations which arise in the process of rectifying the plate-supply current. Power sources for radio transmitters use similar inductors for the same purpose.

In communication lines, capacitance effects are usually too large relative to those of resistance and inductance. By the use of properly designed inductors at intervals along the line, the predominance of capacitance can, in effect, be neutralized, and speech transmission can thereby be extended to much greater distances than would otherwise be possible. The inductors, known as loading coils, use magnetic materials in forms chosen to reduce losses to a minimum. The principle of loading has been extended to ocean cables, in which the inductance is increased uniformly along the cable by winding the central conductor with a tape of material which has very great permeability at low field strengths.

Many different types of *space-wound* coils have been developed for use in the radio-frequency circuits of radio receivers. Windings of this type are used to reduce capacitance effects by securing the largest separation convenient size will allow between turns whose conductors differ greatly in potential. The natural resonance frequency above which coils of this

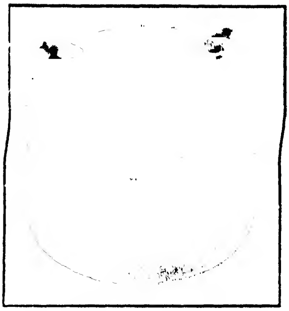
type cease to act as inductances may be as high as a hundred million cycles per second. In radio-frequency power circuits, in which voltages are large and currents are mostly in the surface layers of conductors, coils are made of hollow copper tubing wound in a comparatively open spiral.

The property of mutual inductance between coils can be used to advantage in the construction of continuously adjustable inductors. A common form of inductor for a small fraction of a henry consists of a movable coil mounted within a fixed coil of slightly greater diameter, with the inner coil capable of rotation about an axis as shown in Fig. 32. If the coils are connected in series, the total inductance can be varied over a range of nearly ten to one.

Inductance standards have fallen into disuse for precise measurements because of the difficulty in shielding them adequately to eliminate the mutual inductance with other circuits. Where possible, methods of measurement which determine the unknown in terms of capacitance and resistance are employed.

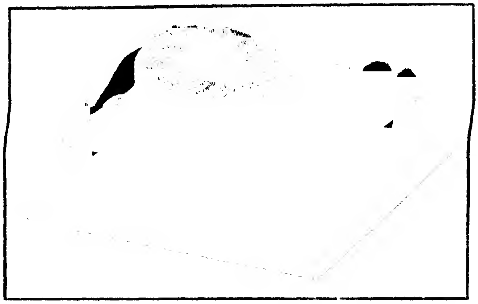
The following estimate is intended to give a practical concept of the magnitude of the inductance to be expected from short, air-core coils. The inductance of a single turn ten centimeters in diameter, made of wire whose diameter is small compared to ten centimeters, is approximately 0.3 microhenry. The inductance of such a configuration is approximately proportional to the diameter of the turn, and if more than one turn is involved, to the square of the number of turns. Thus if the diameter is doubled, and the number of turns increased to 100, the inductance is about $2 \times 100^2 \times 0.3$, or 6,000 microhenrys, or 6 millihenrys. To obtain an inductance of one henry in a coil 20 centimeters in diameter, the number of coincident turns required by this approximate method is 1,300. Since these turns cannot all occupy the same space, all the flux produced by each turn cannot link every other turn, and a considerably larger number of turns actually is required. Inductance formulas are available for the precise calculation of both self- and mutual inductance in coils of many forms.[18]

[18] "Radio Instruments and Measurements," *Circ. Nat. Bur. Stand.* No. 74 (2d ed.; Washington: Government Printing Office, 1924), 242 286; E. B. Rosa and F. W. Grover, "Formulas and Tables for the Calculation of Mutual and Self Inductance," *Sci. Paper Nat. Bur. Stand.* No. 169 (3d ed. [revised]; Washington: Government Printing Office, 1916).



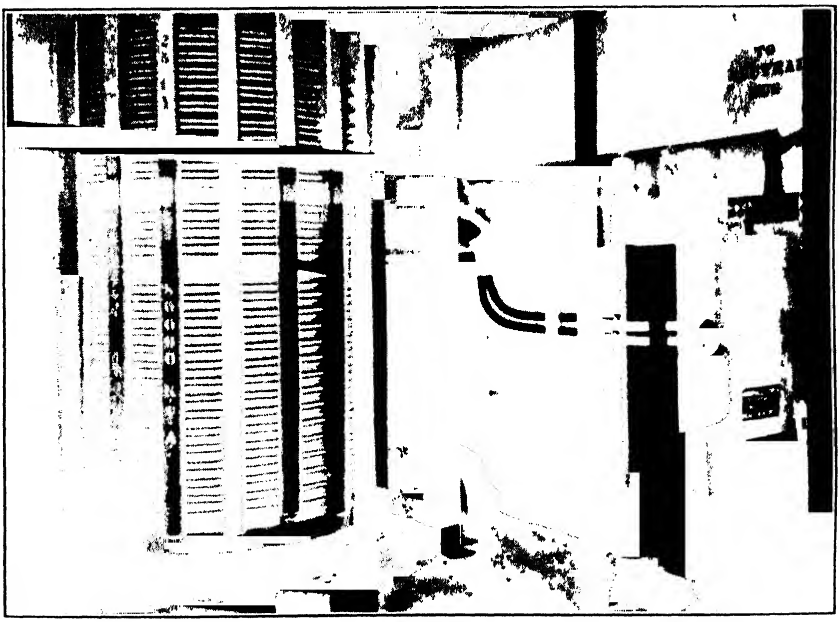
Courtesy Western Electric Co.

Air core inductor or *retardation coil* used in telephone systems; 63 ohms resistance, 0.05 henry direct current inductance



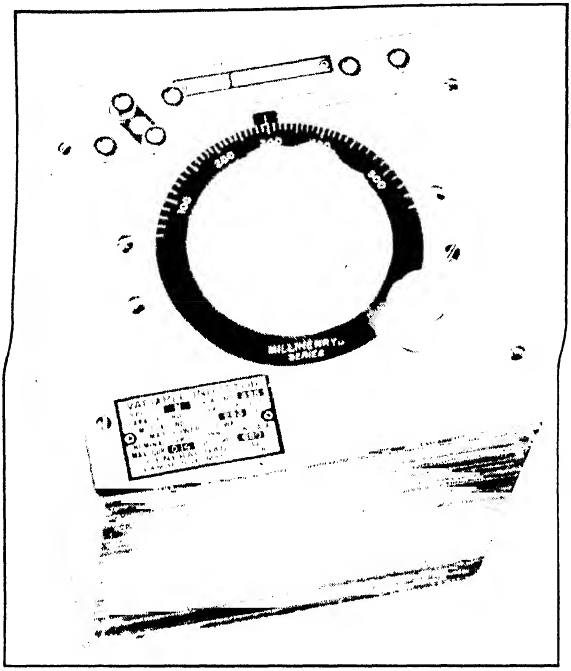
Courtesy General Radio Co.

Fixed inductance standard, 0.10 millihenries



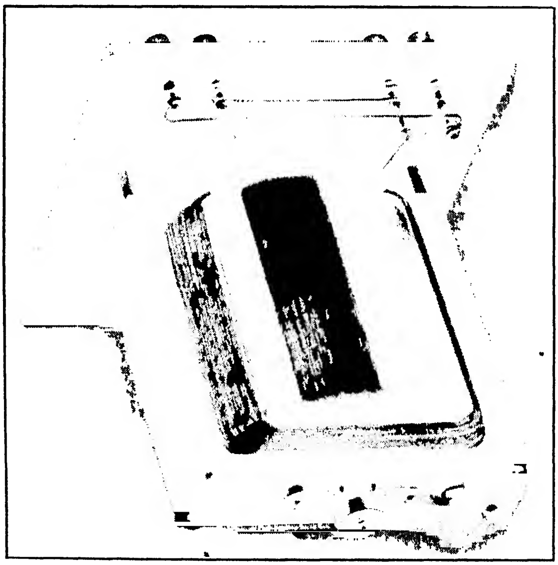
Courtesy General Electric Co.

Neutral grounding reactor, 80,000 kilovolt amperes, 8,000 volts, 10,000 amperes, in system of Consolidated Gas, Electric Light and Power Co., Baltimore, Md.



Courtesy General Radio Co

Variable inductor or *variometer*



Courtesy General Radio Co

Inside view of variometer.

19. Series and parallel combinations of inductance

For a single inductance L the voltage-current relations are

$$v = L\frac{di}{dt}, \tag{182}$$

$$i = \Gamma\int vdt. \tag{183}$$

Here Γ is used to denote reciprocal inductance.

When several inductances are considered, either in series or parallel combinations, and the voltage-current relation for the resultant is wanted, the problem is similar to, although more complicated than that for resistance or capacitance parameters because of the fact that there are two kinds of inductance: *self* and *mutual*. Both may be involved simultaneously. In the following, this more general situation is assumed.

If there are two inductances L_1 and L_2 with a mutual inductance M between them, and the voltages and currents v_1, i_1 and v_2, i_2 refer respectively to these two inductances, a simple extension of Eq. 182 yields the following pairs of relations:

$$v_1 = L_1\frac{di_1}{dt} \pm M\frac{di_2}{dt}, \tag{182a}$$

$$v_2 = \pm M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \tag{182b}$$

or, inversely,

$$i_1 = \Gamma_1\int v_1dt \pm W\int v_2dt, \tag{183a}$$

$$i_2 = \pm W\int v_1dt + \Gamma_2\int v_2dt. \tag{183b}$$

The second pair is not an independent set of equations but the solution to (or inversion of) the first pair, and vice versa. Hence by the theory of linear simultaneous algebraic equations, the coefficients of the two pairs are related as follows:

$$\Gamma_1 = \frac{L_2}{L_1L_2 - M^2}, \tag{184a}$$

$$\Gamma_2 = \frac{L_1}{L_1L_2 - M^2}, \tag{184b}$$

$$W = \frac{-M}{L_1L_2 - M^2}, \tag{184c}$$

$$L_1 = \frac{\Gamma_2}{\Gamma_1\Gamma_2 \mp W^2}, \tag{185a}$$

$$L_2 = \frac{\Gamma_1}{\Gamma_1\Gamma_2 \mp W^2}, \tag{185b}$$

$$M = \frac{-W}{\Gamma_1\Gamma_2 \mp W^2}. \tag{185c}$$

Now if the two inductances are connected in series, their currents are equal and their voltages are additive. Adding Eqs. 182a and 182b gives

$$v_1 + v_2 = v = (L_1 + L_2 \pm 2M)\frac{di}{dt}. \tag{182c}$$

The quantity in the parentheses is the resultant inductance for the series combination.

On the other hand, if the two inductances are connected in parallel, their voltages are equal and their currents are additive. Adding Eqs. 183a and 183b gives

$$i_1 + i_2 = i = (\Gamma_1 + \Gamma_2 \pm 2W)\int v\,dt. \tag{183c}$$

The quantity in the parentheses is the resultant reciprocal inductance parameter for the parallel combination. Expressed as an inductance, and in terms of the original self- and mutual inductances,

$$\frac{1}{\Gamma_1 + \Gamma_2 \pm 2W} = \frac{L_1L_2 - M^2}{L_1 + L_2 \mp 2M}. \tag{186}$$

It is emphasized that Eqs. 183c and 186 *cannot* be applied to the inductances of *coils* connected in parallel unless the error caused by the presence of the associated resistance is tolerable. For this reason these formulas are not commonly of practical use.

If the two self-inductances are not coupled by mutual inductance the formulas derived in this article still apply with M equal to zero.

20. Summary of voltage-current relations for the circuit elements

In the following, the current in an element — that is, the rate of movement of charge in one terminal and out the other — is i, and the potential drop between terminals in the direction of the current is v. Both i and v are in general functions of time. In any one element, v or its time derivative or integral is a linear function of i or the time derivative or integral of i.

For the resistance or conductance element the following relations hold:

$$v = Ri, \tag{187a}$$

$$i = Gv. \tag{187b}$$

These two forms are alternative ways of stating the same relation. The choice between them is a matter of convenience only.

For the elastance or capacitance element the following relations hold:

$$v = Sq = S\int idt, \tag{188a}$$

$$i = C\frac{dv}{dt}. \tag{188b}$$

These two forms also are alternative forms of the same relation. The choice between them is a matter of convenience.

Finally, for the self-inductance element the following relations hold:

$$v = L\frac{di}{dt}, \tag{189a}$$

$$i = \Gamma\lambda = \Gamma\int vdt. \tag{189b}$$

Again, convenience alone determines which of these equivalent forms shall be used.

The mutual-inductance parameter is similar to self-inductance except that v is in one element and i in another.

21. Summary of energy relations

The electrical power input p to an element is

$$p = vi, \tag{190}$$

in which v and i are defined as in the preceding article.

This follows from the definition of potential difference in terms of work done on a unit charge, and from the fact that current is the charge moved per unit of time. Equation 190 is quite independent of the nature of the element. In fact, any two-terminal electrical device having v volts potential drop across it in the direction of a current i amperes in it absorbs p watts of electrical power, or joules of electrical energy per second. For a generator, p as thus defined is, of course, negative numerically.

Applying Eq. 190 to the resistance element and using Eqs. 187a and 187b give for the absorbed power p_R,

$$p_R = vi = Ri^2 = Gv^2. \tag{191}$$

The integral of this power — energy — is converted into heat and disappears irrecoverably from the electrical system.

Applying Eq. 190 to the elastance element and using Eq. 188b give for the absorbed power p_C,

$$p_C = vi = Cv\frac{dv}{dt}. \tag{192}$$

This power represents a rate of energy storage. In time dt, an amount of energy

$$dw_C = p_C dt = Cvdv \tag{193}$$

is put into the capacitance, while, in charging it from zero voltage to a voltage v, the energy input is

$$w_C = C\int_0^v vdv = \frac{Cv^2}{2} = \frac{Sq^2}{2}. \tag{194}$$

Lastly, the inductance element is considered, for which the power input p_L is derived by using Eqs. 189a and 190:

$$p_L = Li\frac{di}{dt}. \tag{195}$$

This power also represents a rate of storage of energy. In time dt an amount of energy

$$dw_L = p_L dt = Lidi \tag{196}$$

is put into the inductance, and during the time in which the current builds up from zero to i the energy w_L stored is

$$w_L = L\int_0^i idi = \frac{Li^2}{2} = \frac{\Gamma\lambda^2}{2}. \tag{197}$$

The energies w_L and w_C are independent of the time during which the current and charge, respectively, are built up to their final values.

22. Network equations

In the preceding articles of this chapter, the manner in which a complicated electric circuit can be resolved into an aggregation of individual elements or *network* is indicated, the equations for the electrical behavior of the individual elements are developed, and various methods for evaluating the parameters are illustrated. This concluding article gives the general mathematical relations which express the behavior of any network. These are known as *Kirchhoff's laws,* because they were first formulated by him in 1845.[19] Both are special expressions of relations implicit in the field equations, applicable to electric circuits.

[19] G. Kirchhoff, *Gesammelte Abhandlungen* (Leipzig: Johann Ambrosius Barth, 1882), 15.

The first law, known as the voltage law, follows from Faraday's induction law, Eq. 1, by a process similar to that carried out in Art. 2. Stated in words, it is:

▶If in an electric network a closed path is traversed, the algebraic sum of the voltage drops across the individual elements in the direction of traversal is zero ◀

For example, in Fig. 39 the closed path is *abcdfgha* and it is traversed in this direction. Each simple series combination of elements on the path, such as *abc* or *cdf*, has a current indicated by i with an identifying sub-

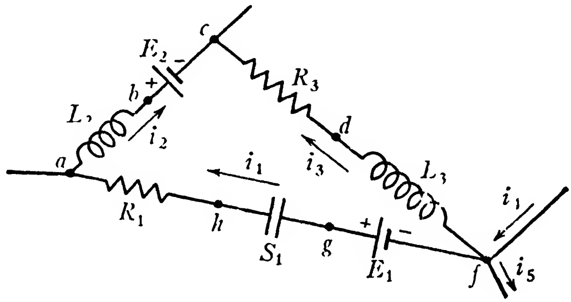


FIG. 39 Portion of an electric network.

script. As the actual direction of current is in general unknown and a function of time, a direction indicated by an arrow is arbitrarily selected as positive. If positive charge moves in the arrow direction, i has a positive value; if in the opposite direction, i has a negative value.

In order to conform to the statement given for the first law, Eq. 1 is written in the form*

$$\oint \mathcal{E} \cdot d\ell + \sum L \frac{di}{dt} = 0 \qquad [1c]$$

and applied to the path *abcdfgha*. Beginning at a, the contributions to $\oint \mathcal{E} \cdot d\ell$ and to $\sum L \frac{di}{dt}$ are:

$$\text{Potential drop from } a \text{ to } b = L_2 \frac{di_2}{dt}, \qquad [198a]$$

$$\text{Potential drop from } b \text{ to } c = E_2, \qquad [198b]$$

* It is understood that $\sum L \frac{di}{dt}$ includes terms of both self- and mutual inductance. The mutual terms need not be of concern for this example.

$$\text{Potential drop from } c \text{ to } d = -R_3 i_3, \tag{198c}$$

$$\text{Potential drop from } d \text{ to } f = -L_3 \frac{di_3}{dt}, \tag{198d}$$

$$\text{Potential drop from } f \text{ to } g = -E_1, \tag{198e}$$

$$\text{Potential drop from } g \text{ to } h = S_1 \int i_1 dt = S_1 q_1, \tag{198f}$$

$$\text{Potential drop from } h \text{ to } a = R_1 i_1. \tag{198g}$$

Equation 1c, with the results of Eqs. 198a to 198g, gives

$$L_2 \frac{di_2}{dt} + E_2 - R_3 i_3 - L_3 \frac{di_3}{dt} - E_1 + S_1 \int i_1 dt + R_1 i_1 = 0. \tag{1d}$$

Equation 1d, which is derived from Eq. 1 using the circuit-element concepts, is identical with the equation that is obtained by applying the Kirchhoff voltage law to this loop. Such a process as that used in obtaining Eq. 1d can be applied to any closed-circuit path. The result is identical with the equation which is obtained by an application of Kirchhoff's voltage law to the path. This fact establishes the general validity of the Kirchhoff voltage law.

The currents i_1, i_2, and i_3 in general are unknown in such a case as that shown in Fig. 39. They may be and usually are functions of time. Whatever their values, however, they must be such that Eq. 1d is satisfied. They can be determined only after additional equations such as Eq. 1d are written for the other paths of the network together with certain current equations discussed below.

Kirchhoff's second law, known as the current law, follows from Eq. 2. Stated in words, it is:

▶The algebraic sum of all currents directed toward a junction point is zero.◀

To establish this current law any junction of conductors may be inclosed in a surface, as is done on Fig. 2. Then the algebraic sum of the conduction currents entering the surface is equal to $1/4\pi$ times the rate of increase of electric flux directed outward through the surface. But by the assumptions used in establishing the circuit relations, the displacement current is assumed to be negligible in comparison with conduction current except as associated with the circuit element capacitance. Therefore, unless the conductors in the vicinity of the junction are in effect functioning as an electrode of a condenser — as they may do in the case of high-frequency circuits — the algebraic sum of the conduction currents entering the junction is zero. If, on the other hand, the current law is

understood to include displacement as well as conduction currents, it is universally true for conditions to which the field equations apply.

By means of these two circuit laws and the circuit-element relations it *is possible to determine the resultant behavior of any lumped-parameter* linear network. The formulation of the voltage and current equations is a necessary part of network analysis that is carried out in detail in subsequent chapters.

PROBLEMS

1. It is often of importance to know the ratio between the amplitudes of the displacement current and the conduction current in certain materials. To determine this, an electric field strength within the material is assumed to be of the form $\mathcal{E}_m \sin 2\pi ft$.

(a) What is the frequency at which this current ratio is unity?
(b) What is the frequency of (a) for the following materials?

Material	*Resistivity (ohm-cm)*	*Dielectric Constant*
Copper	1.72×10^{-6}	1 (assumed)
Sea water	0.5	81
Distilled water	0.5×10^{6}	81
Bakelite	*2×10^{11}	5

* Value depends considerably upon composition

(c) Which is the predominant term for frequencies below the frequency of (a)?
(d) What are the values of the ratio for copper when the frequency is 60 ~ (power); 30 megacycles/sec (short-wave radio)?

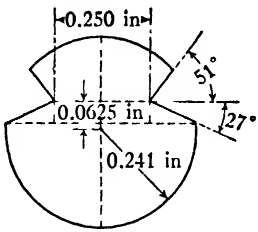


FIG. 40 Cross section of trolley wire, Prob. 3.

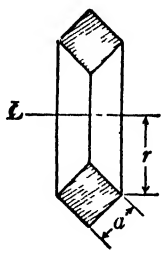


FIG. 41. Diagram of toroidal inductor having square cross section, Prob. 5.

2. In studies of the electrolysis of underground metallic structures caused by the action of stray direct currents, it is often necessary to measure the current in a pipe. One method of doing so is to measure the voltage drop across a measured length of pipe.

A " 6-in " class A cast-iron gas pipe has an inside diameter of approximately 6 in, and a wall thickness of 0.44 in. What current is indicated by a voltage drop of 60 mv over an 8-ft length?

3. What is the resistance per mile at 20 C of a hard-drawn copper trolley wire which has the cross section shown in Fig. 40?

4. *What is the capacitance of a parallel-plate condenser consisting of M equal* plates of one polarity interleaved uniformly with $M + 1$ similar plates of the opposite polarity?

5. An inductor is made by winding a single layer of wire on a toroidal form which has the square cross section indicated in Fig. 41.

(a) What is the inductance of the winding?
(b) What is the resistance of the winding?
(c) What is the number of turns in terms of wire diameter and other linear dimensions?

6. One of the condensers used at the Bureau of Standards consists of two concentric cylinders, whose radii are 6.26 cm and 7.24 cm, respectively. They are equipped with suitable arrangements at the ends to eliminate the fringing of electric flux. The effective length of the condenser is 20.1 cm.

(a) What is the capacitance of this condenser?
(b) What is the capacitance if the distance between the axes of the two condensers is 1 mm (cylinders not concentric)?
(c) Is the effect of this eccentricity on the capacitance of the condenser a second-order effect or an important effect?

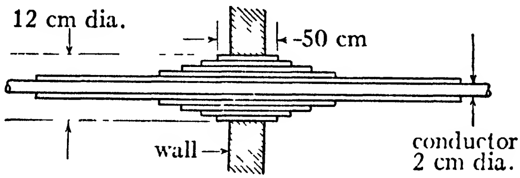


FIG. 42. Diagram of condenser-type bushing, Prob. 8.

7. Occasionally in the past, the mistake has been made of running unsheathed insulated cables in metallic conduits, with the result that sparking has taken place in the air space between the insulation surface and the conduit. This has resulted in the rapid deterioration and early failure of the insulation.

Generally the cable is kinked and follows an irregular path through the duct, touching it here and there for support. The analysis of what happens may be simplified by treating the simple case of an insulated cable concentric with the duct; for example, a conductor of $\frac{3}{4}$-in. diameter surrounded by a $\frac{1}{2}$-in. layer of varnished cambric in a duct of internal diameter 2 in.

(a) If the dielectric constant of the varnished cambric is 5, and the breakdown gradient of air is 3,000,000 v/m, what is the difference in potential between the conductor and the pipe at which sparking occurs?
(b) If the allowable working gradient in varnished cambric is 6,000,000 v/m, what is the difference in potential between the conductor and pipe at which the cambric breaks down if the pipe is shrunk to a tight fit on the cambric, excluding all air?

8. A high-voltage condenser-type bushing for carrying a conductor through a wall is constructed somewhat as is shown in Fig. 42. There are five layers of equal thickness of oil-filled paper insulation. Each layer is covered with metal foil for substantially its entire length. The surface in contact with the wall is at ground potential. The lengths of the respective layers are adjusted so that the potential gradients at the inner surfaces of each layer are equal. Leakage currents may be neglected.

(a) If the maximum allowable potential gradient is 8,000,000 v/m, what is the maximum allowable voltage from conductor to ground?
(b) If the dielectric constant of the insulation is 5, what is the capacitance of the bushing between conductor and ground?

9. A 0000 AWG hard-drawn copper trolley wire is 25 ft above the ground. What is the capacitance per mile to ground? The ground may be assumed to be an equipotential surface.

10. When a charged cloud appears over an open-wire electric-power transmission line or a telephone line, an abnormal difference in potential may be induced between the line and ground. If the presence of the cloud persists for some time, this difference in potential is gradually neutralized by the electric field due to charges which gradually leak on to the line over its insulation. If the cloud then suddenly discharges, the line may again be at an abnormal potential with respect to the earth owing to the fact that the charges which have leaked on to it cannot escape instantaneously. This situation is idealized in the arrangement which follows:

Two very extensive parallel conducting planes carry uniformly distributed electric charges of surface density $+\sigma$ and $-\sigma$, which give a resultant electric-field intensity $\mathcal{E}$ at any point between the planes, as indicated in Fig. 43. A very long wire of radius r now is inserted between the planes, parallel to the lower plane and connected to it by a very high resistance.

$$H \gg r, \qquad h \gg r.$$

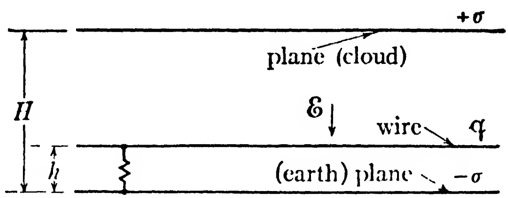


FIG. 43. Charged cloud over transmission wire, Prob. 10

(a) If the wire has been connected to the lower plane long enough for wire and plane to acquire the same potential, what is the charge per unit length q carried by the wire?

After the wire has acquired the same potential as the lower plane, the upper plane is suddenly discharged to the lower plane and completely removed. (This simulates the discharge of the cloud.)

(b) What is the initial difference in potential between the wire and the lower plane, that is, the potential difference existing before any appreciable charge can pass through the connecting resistance?
(c) What is the potential difference expressed by (b) for

$$\mathcal{E} = 1{,}000 \text{ v/ft} \qquad [199]$$
$$h = 40 \text{ ft} \qquad [200]$$
$$H = 1{,}000 \text{ ft} \qquad [201]$$
$$r = 0.230 \text{ in. (0000 AWG)?} \qquad [202]$$

11. A transmission line is composed of two 0000 AWG solid copper wires, their axes separated by 2 ft. What is the inductance per mile of line,

(a) neglecting the flux within the wires?
(b) considering the flux within the wires, and assuming uniform current distribution in the wires?

12. A double-circuit electric-power transmission line (Fig. 44) uses four 0000 AWG solid copper wires spaced at the corners of a 4.0-ft square. Current is in one direction in two of the wires (connected in parallel) and in the opposite direction in the other two wires (also connected in parallel).

(a) What is the resistance of the complete double-circuit line in ohms per mile of line?
(b) What is the self-inductance in h/mi of line, assuming that current is in one direction in the two wires in the upper plane and in the opposite direction in the two wires in the lower plane? The internal flux linkages in the wires may be neglected.
(c) What correction should be added to the result of (b) to take into account the internal flux linkages in the wires, assuming that the current density is uniform over the cross section of each wire?

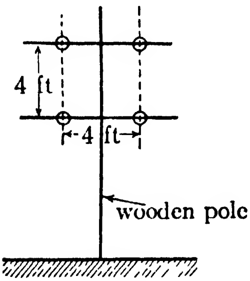


FIG. 44. Conductor arrangement for double-circuit transmission line, Prob. 12

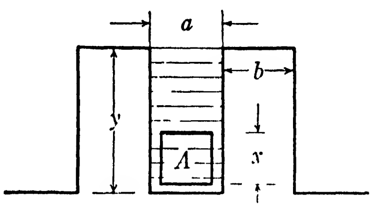


FIG. 45. Conductor at bottom of iron slot, Prob. 13.

13. One of the important considerations in the design of electrical machinery is the amount of magnetic flux which links the conductors, particularly where they are placed in slots in a steel structure. While in general no exact method of solution is available, certain methods of approximation are used which give results that are distinctly helpful. The following problem illustrates one such method.

In Fig. 45, A represents a copper conductor of rectangular cross section, at the bottom of a slot in a piece of annealed steel. The dimensions are as follows:

$$x = 2.5 \text{ cm} \quad [203]$$

$$y = 5.5 \text{ cm} \quad [204]$$

$$a = 1.8 \text{ cm} \quad [205]$$

$$b = 2.0 \text{ cm} \quad [206]$$

A current of 100 amp is in A. If the steel has an infinite permeability, the conductor and slot are of infinite length, and the conductor fills the bottom of the slot:

(a) How much magnetic flux per meter length of conductor crosses the slot above the conductor but below the top of the slot?

(b) How much magnetic flux passes through the conductor per meter of length?
(c) About how much error is introduced by assuming the permeability of the steel to be infinite?

14. A metal sphere of radius r is maintained at potential V with respect to a very extensive metal plate at a distance h from the sphere; $h \gg r$. What is the capacitance of the system?

15. A copper-coated steel rod $\frac{3}{4}$ in. in diameter is driven 7 ft into damp soil. What is the approximate range of values for the ground resistance of this rod?

16. A mutual inductor is constructed as follows: A fiber tube 1 m long and 10 cm outer diameter is wound over its entire length with a single layer of 16 S.C.E. (Single, Cotton, Enamel) copper wire. The enamel insulation is 1 mil thick and the cotton insulation is $2\frac{1}{2}$ mils thick. Around the middle of this coil, there is wound a 500-turn coil 10 cm long and 1 cm thick.

What is the mutual inductance of these two coils?

17. A power line consists of two 00 AWG solid copper wires 4 ft apart (center to center) in a horizontal plane 30 ft above the earth. A telephone line running parallel to the power line consists of two 12 AWG solid copper wires 12 in apart (center to center) in a horizontal plane 20 ft above the earth. The adjacent power and telephone wires are horizontally 30 ft apart.

(a) What is the voltage induced electromagnetically in the telephone line per mile by a current in the power line

$$i = 150 \cos 377t + 5 \cos 1{,}131t \qquad \text{amp}, \qquad [207]$$

where t is in seconds?

(b) How can this induced voltage be avoided practically?

18. What is the resistance at 30 C of 1,500 ft of 16 AWG annealed copper wire?

19. The field coils of a shunt generator have a measured resistance of 138 ohms after the machine has been standing for some time in a room whose temperature is 20 C. After the machine has been in operation for 3 hr, the resistance is again measured and found to be 152 ohms. The field coils are constructed of annealed copper.

What is the average temperature of the field coils at this time?

20. It is desired to make a resistor whose resistance at 20 C is 1,000 ohms and whose temperature coefficient is 0.0020 at 20 C. This is to be made by placing two windings in series, one wound with wire of one material and the other wound with wire of another material. The following wire is available.

Material	*Sizes AWG*
Copper	20, 24, 28, 32, 36, 40
Constantin	28, 32
Nichrome	22, 24, 26

For each usable combination of two materials:

(a) What is the resistance of each of the two windings?
(b) What wire size of each material should be used to get the smallest volume of each wire?
(c) What lengths of each material are required, using the sizes selected in (b)?

21. An immersion type of electric heating unit for heating liquids as made by one manufacturer consists of a helically wound coil of nichrome wire embedded in a jacket of compressed magnesium oxide. This combination is surrounded by a tubular metallic sheath. The heat is produced by passing electric current through the nichrome

resistance wire, whence the heat is conducted through the magnesium oxide and the sheath to the liquid being heated. In designing such a unit, the manufacturer wishes to keep costs at a minimum.

The following data apply to nichrome wire suitable for the windings:

AWG	*Diameter In*	*Weight lb/1,000 ft*	*Cost Dollars/lb*	*Resistance Ohms/ft*, 20 C
20	0.032	2.915	1.82	0.635
22	0 0253	1.807	2.06	1.017
24	0 0201	1.139	2.38	1.609
26	0.0159	0 719	2.73	2.571
28	0.0126	0.454	3.08	4.090
30	0.010	0.2845	3.44	6.500

Resistance factor $= R_t/R_{20}$, R_t = resistance at t C, R_{20} = resistance at 20 C

Temperature, C	20	100	200	300	400	500	600	700
Factor	1.00	1.017	1.035	1.052	1.060	1.068	1.066	1.063

In order to conform to a standard line of external dimensions, the resistor element must be wound so that the length of the helical coil shall be 13.5 in measured along the main axis of the helix and the outside diameter of the helix shall be 0.270 in. The unit is to be rated 1,000 w, at 230 v. Wire smaller than 30 has insufficient mechanical strength and so cannot be considered for the element.

(a) What are the specifications for the resistor element if its wire is intended to operate at 90 C for heating water:

(1) Size of wire, AWG,
(2) Length of wire in inches at 20 C,
(3) Number of turns per inch of coil,
(4) Cost of resistor wire per 1,000 units manufactured?

(b) What are the above specifications for a similar unit intended to operate at 450 C for melting lead?

(c) If the labor and overhead costs of winding the resistor helix are proportional to the number of turns per unit, should any different specification be made in (*a*) or (*b*)?

CHAPTER II

Elementary Network Theory

1. Network geometry

The conception of two-terminal electric-circuit elements, or lumped parameters, is developed in Ch. I. In practice, electric circuits commonly consist of many such elements connected to form a complicated system. For ease in defining certain terms used in describing these combinations, the nature of the individual elements is disregarded for the moment and each is represented in Fig. 1 as a line extending between two small dots. The entire aggregation of elements is termed an *electric network*, often abbreviated *network*. Each element such as *ga*, *fl*, or *b2c* is termed a *branch*. Each junction point of two or more branches is termed a *node*.* A closed path such as *ab2cdfga* or *kjfdhlk* is called a *loop*. A loop which cannot be subdivided into other loops, such as *fdhlf*, is called a *mesh*.† The circuit of Fig. 5a, for example, consists of seven branches, five being passive elements and two being voltage sources; it contains six nodes, three loops, and two meshes.

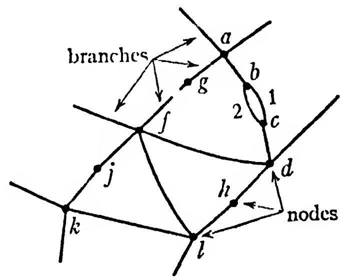


FIG. 1. For definition of network terminology.

A lumped-parameter circuit always can be broken down into a network of individual elements or branches, a process which is the preliminary to its analysis. More detailed treatment of network geometry is given in Ch. VIII; in this chapter discussion is limited to the simpler fixed-resistance networks in which fixed voltages are applied.

2. Directions of voltages and currents

In Arts. 2 and 3, Ch. I, the conventions of direction for voltage and current are established. These are repeated in this section and are discussed particularly from the point of view of electric-circuit relations.

There is said to be a rise in electric potential from point *b* to point *a* if

* It is sometimes convenient to enlarge the concept of a branch to include a combination of elements that has only two terminals, such as *ad*, thus dropping out nodes *b* and *c*.

† The actual elements which form a mesh depend upon the manner in which the network diagram is drawn. The definition becomes meaningless for networks which cannot be drawn in one surface.

energy must be expended to transfer positive charge from b to a. Point a then is said to be at a higher potential than point b, or to be positive with respect to point b. Conversely, there is said to be a drop in electric potential from point a to point b, and point b is said to be at a lower potential than point a, or to be negative with respect to point a. A positive charge transferred from a to b gives up energy. The conception of potential (or voltage) rise or drop, of course, can be defined equally well in terms of transfer of negative charge, by statements opposite to those made above, but it is obviously superfluous to make the statements both ways.

There is said to be an electric current from point b to point a if positive charges move from b to a, or there is said to be a positive current from b to a. Conversely, a negative current is said to be from point a to point b. It is now recognized, of course, that a positive current from b to a may consist largely or (in the case of metallic conduction) entirely of negative charges flowing from a to b.

When there is current in a resistance branch, energy is liberated in the form of heat in accordance with Joule's law. Hence there is a decrease of

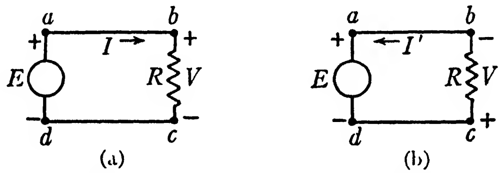


FIG. 2. Use of polarity marks and arrows.

potential in the direction of the current, that is, in the direction of flow for positive charge. Hence the resistance drop, or RI drop, so common in electrical-engineering parlance, *always is in the direction of the current.*

All the preceding, while very elementary, is very important. Though in simple resistance circuits having steady voltages and currents, little difficulty is encountered in keeping track of voltage and current directions, in more complicated circuits and more complex circuit conditions, involving transient- and alternating-current phenomena introduced in the following chapters, much more difficulty is met. A sound method of systematization is needed. It is important, therefore, to develop the system and practice its application on simple cases wherein likelihood of confusion is small.

In designating polarities, plus (+) and minus (−) signs may be placed at the terminals of branches, and a symbol for voltage may be associated with each branch, as in Fig. 2. In this way the voltage symbols are clearly defined by reference to the diagram. The direction of any current may be designated by means of an arrow associated with a symbol for current. However, in some complicated circuits, this system confuses the diagram

with too many marks and symbols. In such circumstances recourse may be had to the system of *double-subscript notation.* In this system as applied to electric circuits, the order of subscripts indicates the direction of traversal of a branch (or branches) from node to node. It is important to recall, however, that with voltages a symbol such as V_{bc} is meaningless unless it is known whether it stands for a voltage rise or for a voltage drop *in the direction $b - c$. In the treatment of circuits in this chapter, therefore*, the symbol E always stands for the voltage rise or electromotive force of a source, and the symbol V always stands for the voltage drop through a resistance, *in the direction of the appended subscripts.* For currents, the symbol I always stands for flow of positive charge *in the direction of the appended subscripts.* Double subscripts appended to resistance or conductance symbols have no directional sense. These serve merely to locate the parameters with respect to loops or nodes with which they are associated. From Fig. 2a, by Kirchhoff's voltage law,

$$\left.\begin{aligned} E &= V = RI, \\ E_{da} &= V_{bc} = R_{bc}I_{ab}. \end{aligned}\right\} \qquad [1]$$

Also,

$$E_{da} = -E_{ad}, \qquad [2]$$

$$V_{bc} = -V_{cb}, \qquad [3]$$

$$I_{ab} = -I_{ba}. \qquad [4]$$

In this book either the system of polarity marks and arrows, or the system of double-subscript notation, or a combination of them, may be used, as proves to be expedient.

In the solution of an electric network, it sometimes happens that the magnitudes of the voltages and currents are desired, but that the actual polarities and directions are unimportant. Even so, it is necessary to solve the circuit with due regard to a system which keeps track of voltage polarities and current directions *as a relative matter within the circuit,* though actual polarities and directions may never be known. That is, voltage polarities and current directions must be assigned arbitrarily and the circuit equations must be written in consistent conformity with the assumptions. The solution of such equations gives the correct magnitudes for all voltages and currents. Negative values appearing for some voltages and currents are the means mathematics has for saying that in those instances the assumed polarities or directions turned out to be inconsistent with the rest. This situation, in fact, may arise even for a circuit in which, for example, the actual source polarities are known, because even then it is likely to be impossible to guess correctly the voltage polarities and current directions everywhere. In Fig. 2b, as an illus-

tration, the polarity of the source may be correctly known, but the *polarity of the resistance branch and the direction of the current are (obviously in this case) incorrectly marked.* However,

$$E = -V' = -RI' \tag{1a}$$

is the correct equation corresponding to the marking. In order to make the example numerical,

$$E = 115 \text{ v}, \tag{2a}$$

$$R = 265 \text{ ohms}. \tag{5}$$

Hence,

$$115 = -V', \quad V' = -115, \tag{3a}$$

or the *voltage drop* through the resistance branch *in accordance with its polarity marking* is -115 volts. Also,

$$115 = -265I', \quad I' = -\frac{115}{265} = -0.434, \tag{4a}$$

or *the current in the arrow direction* is -0.434 ampere.

3. The application of Kirchhoff's laws

Kirchhoff's two laws which were discussed in Art. 22, Ch. I, are restated below:

(a) The sum of the voltage drops taken in a specified direction around any loop equals the sum of the voltage rises in that direction.
(b) The sum of the currents directed away from any node equals the sum of the currents directed toward that node.

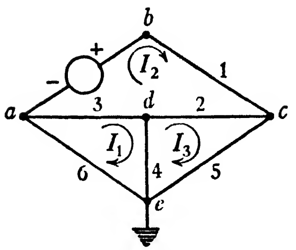


Fig. 3. For illustration of methods of solving networks.

These two laws form the bases of two methods of writing equations for the solution of network problems: the loop method and the node method. In the solution of simple networks the systematizing of the work by means of one or the other method is not always outstandingly advantageous; it is in fact sometimes advantageous to use a mixture of the two. However, as networks become more complicated, some method for systematization of work becomes essential to avoid confusion.

In the solution of any network problem it is necessary to write a number of independent equations equal to the number of unknown quantities, and to include all the network voltages, currents, and parameters in so doing. This is illustrated by means of Fig. 3, in which it is assumed that

the source voltage (or current) and the resistance (or conductance) of each branch are known. There are, therefore, 13 unknown quantities, the current and the voltage drop of each passive branch, and the current or voltage of the source. For each passive branch, by Ohm's law:

$$V_{bc} = R_{bc}I_{bc} \quad \text{or} \quad I_{bc} = G_1 V_{bc}, \tag{6}$$

$$V_{cd} = R_{cd}I_{cd} \quad \text{or} \quad I_{cd} = G_2 V_{cd}, \tag{7}$$

$$V_{da} = R_{da}I_{da} \quad \text{or} \quad I_{da} = G_3 V_{da}, \tag{8}$$

$$V_{de} = R_{de}I_{de} \quad \text{or} \quad I_{de} = G_4 V_{de}, \tag{9}$$

$$V_{ce} = R_{ce}I_{ce} \quad \text{or} \quad I_{ce} = G_5 V_{ce}, \tag{10}$$

$$V_{ea} = R_{ea}I_{ea} \quad \text{or} \quad I_{ea} = G_6 V_{ea}. \tag{11}$$

For loops 1, 2, and 3,

$$V_{ea} - V_{da} + V_{de} = 0, \tag{12}$$

$$V_{bc} + V_{cd} + V_{da} = E_{ab}, \tag{13}$$

$$V_{ce} - V_{de} - V_{cd} = 0. \tag{14}$$

For nodes a, b, c, and d,

$$-I_{ea} - I_{da} = -I_{ab}, \tag{15}$$

$$I_{bc} = I_{ab}, \tag{16}$$

$$I_{ce} + I_{cd} - I_{bc} = 0, \tag{17}$$

$$I_{de} - I_{cd} + I_{da} = 0. \tag{18}$$

These 13 equations all are independent; that is, no one can be obtained from any other or any combination of others. It is evident that all possible branch equations have been written; additional loop or node equations evidently can be written, but a little study shows that such are not independent of these already written; that is, the additional loop or node equations can be obtained by combining existing equations. The number of independent equations, therefore, is just equal to the number of unknowns; so the problem is determinate. It is, of course, unnecessary to use the loops or nodes chosen. Other combinations serve equally well. The use of meshes when feasible in writing loop equations, however, makes it more readily apparent that the equations are independent.

The number of independent node equations is one less than the number of nodes; that is, there are five nodes and but four independent nodes. There are seven branches There are three independent loop equations, or the number of independent loops is equal to the number of branches minus the number of independent nodes. This is generally true

for complicated networks as may be found by trial by building complicated networks from simpler components. That is,

$$\ell = b - n, \tag{19}$$

where ℓ is the number of independent loops, b the number of branches, and n the number of independent nodes. In general, therefore, there are an unknown voltage and an unknown current for each passive branch, and an unknown current or voltage for each source branch, or a total of $2b - s$ unknowns, where s is the number of source branches. It is possible to write $b - s$ independent branch equations, ℓ independent loop equations, and n independent node equations, or

$$b - s + \ell + n = b - s + b - n + n = 2b - s \tag{20}$$

independent equations or exactly the number required to make the solution unique. If certain parameters or source currents or voltages are unknown, a combined number of branch currents and voltages equal to the number of unknown parameters must be known instead.

As is evident from the network of Fig. 3, this general method of solution produces a rather formidable array of equations even for a relatively simple case. Considerable simplification is accomplished by assigning a circulating current to each independent loop and deriving the branch currents therefrom or by assigning a potential to each node with respect to some reference node and deriving the branch voltages therefrom. The former is known as the loop method and the latter as the node method of solution.

By the loop method,

Voltage drops *Voltage rises*

Loop 1:

$$I_1R_{ea} + (I_1 - I_2)R_{ad} + (I_1 - I_3)R_{de} = 0. \tag{21}$$

Loop 2:

$$I_2R_{bc} + (I_2 - I_3)R_{cd} + (I_2 - I_1)R_{da} = E_{ab}. \tag{22}$$

Loop 3:

$$I_3R_{ce} + (I_3 - I_1)R_{de} + (I_3 - I_2)R_{cd} = 0. \tag{23}$$

Assuming that E_{ab} and the various resistances are known, the three loop currents can be computed. The writing of these equations is equivalent to substituting Eqs. 6 to 11 into Eqs. 12 to 14 with the branch currents expressed in terms of loop currents;* Eqs. 15 to 18 are automatically

* For example, $I_{ea} = I_1$, $I_{da} = I_2 - I_1$, $I_{de} = I_1 - I_3$, etc. For the sake of systematization, all loop currents are assigned the same direction in this chapter, ordinarily clockwise.

satisfied because each loop current passes through the various nodes of the loop.

By the node method, using node e as the reference node,

	Currents leaving	*Currents approaching*	
Node a:	$V_aG_6 + (V_a - V_d)G_3$	$= -I_{ab}.$	[24]
Node b:	$(V_b - V_c)G_1$	$= I_{ab}.$	[25]
Node c:	$V_cG_5 + (V_c - V_d)G_2 + (V_c - V_b)G_1$	$= 0.$	[26]
Node d:	$V_dG_4 + (V_d - V_c)G_2 + (V_d - V_a)G_3$	$= 0.$	[27]

If I_{ab} and the various conductances are known, the four node potentials can be computed. As used above, the various V's represent the potentials of the respective nodes above the reference node. The writing of these equations is equivalent to substituting Eqs. 6 to 11 into Eqs. 15 to 18 with the branch voltages expressed in terms of node voltages;* Eqs. 12 to 14 are automatically satisfied because the potential of any node is independent of the path by which it is approached.

If Eq. 21 is rewritten

$$I_1(R_{ea} + R_{ad} + R_{de}) - I_2R_{ad} - I_3R_{de} = 0, \qquad [21a]$$

the various resistances can be defined as self- and mutual resistances,

$$R_{11} = R_{ea} + R_{ad} + R_{de}, \qquad [28]$$

the *self-resistance* of loop 1, or the total resistance around its contour, and

$$R_{12} = -R_{ad} \qquad [29]$$

$$R_{13} = -R_{de}, \qquad [30]$$

the ***mutual resistances*** between loops 1 and 2, and loops 1 and 3, respectively. Similar quantities may be defined for the other loops. Equations 21 to 23 then may be written

$$I_1R_{11} + I_2R_{12} + I_3R_{13} = 0, \qquad ▶[21b]$$

$$I_1R_{21} + I_2R_{22} + I_3R_{23} = E_{ab}, \qquad ▶[22a]$$

$$I_1R_{31} + I_2R_{32} + I_3R_{33} = 0. \qquad ▶[23a]$$

* For example, $V_a = V_{ae}$, $V_{ad} = V_a - V_d$, etc. In this chapter the symbol for node potential always represents the potential of the particular node above the reference node, that is, the voltage drop from the particular node to the reference node. It is therefore unnecessary to append the reference-node subscript if the reference node is clearly indicated on the diagram or by a statement in the text.

The reason for defining the mutual resistance as a negative quantity is purely the desire to systematize the writing of equations so as to have all terms preceded by + signs in the symbolic expressions. If directions of loop currents are assigned arbitrarily, the sign of the mutual resistance is *negative for branches in which the loop currents are in opposite directions and is positive for branches in which the loop currents are in the same direction.* Except in a few simple instances, it is desirable to assign the current directions systematically. Obviously,

$$R_{12} = R_{21}, \text{ etc.} \tag{31}$$

If Eq. 26 is rewritten

$$V_c(G_5 + G_2 + G_1) - V_dG_2 - V_bG_1 = 0, \tag{26a}$$

the various conductances can be defined as self- and mutual conductances,

$$G_{cc} = G_5 + G_2 + G_1, \tag{32}$$

the *self-conductance* of node c, or the total conductance of the branches terminating upon it, and

$$G_{cd} = -G_2, \tag{33}$$

$$G_{cb} = -G_1, \tag{34}$$

the *mutual conductances* between node c and other nodes except the reference node. Similar quantities can be defined for other nodes. Equations 24 to 27 then may be written,

$$V_aG_{aa} \qquad\qquad + V_dG_{ad} = -I_{ab}, \qquad ▶[24a]$$

$$V_bG_{bb} + V_cG_{bc} \qquad = I_{ab}, \qquad ▶[25a]$$

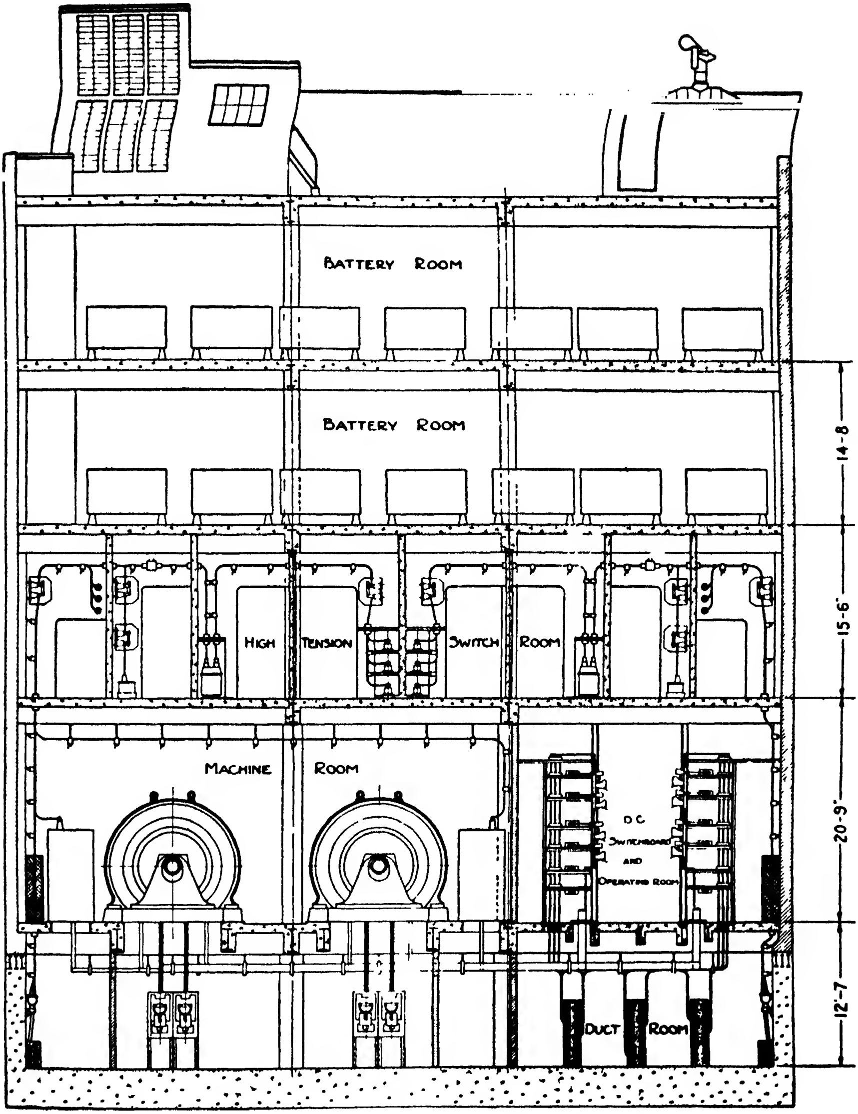
$$V_bG_{cb} + V_cG_{cc} + V_dG_{cd} = 0, \qquad ▶[26b]$$

$$V_aG_{da} \qquad + V_cG_{dc} + V_dG_{dd} = 0. \qquad ▶[27a]$$

The mutual conductance is defined as a negative quantity purely to systematize the writing of equations so as to have all terms preceded by + signs in the symbolic expressions. If the node potentials are assigned arbitrarily above or below the reference node, the sign of the mutual conductance is negative for nodes designated the same polarity with respect to the reference node and is positive for nodes designated opposite in polarity. Except in very simple instances, it is desirable to assign the node potentials systematically. Obviously,

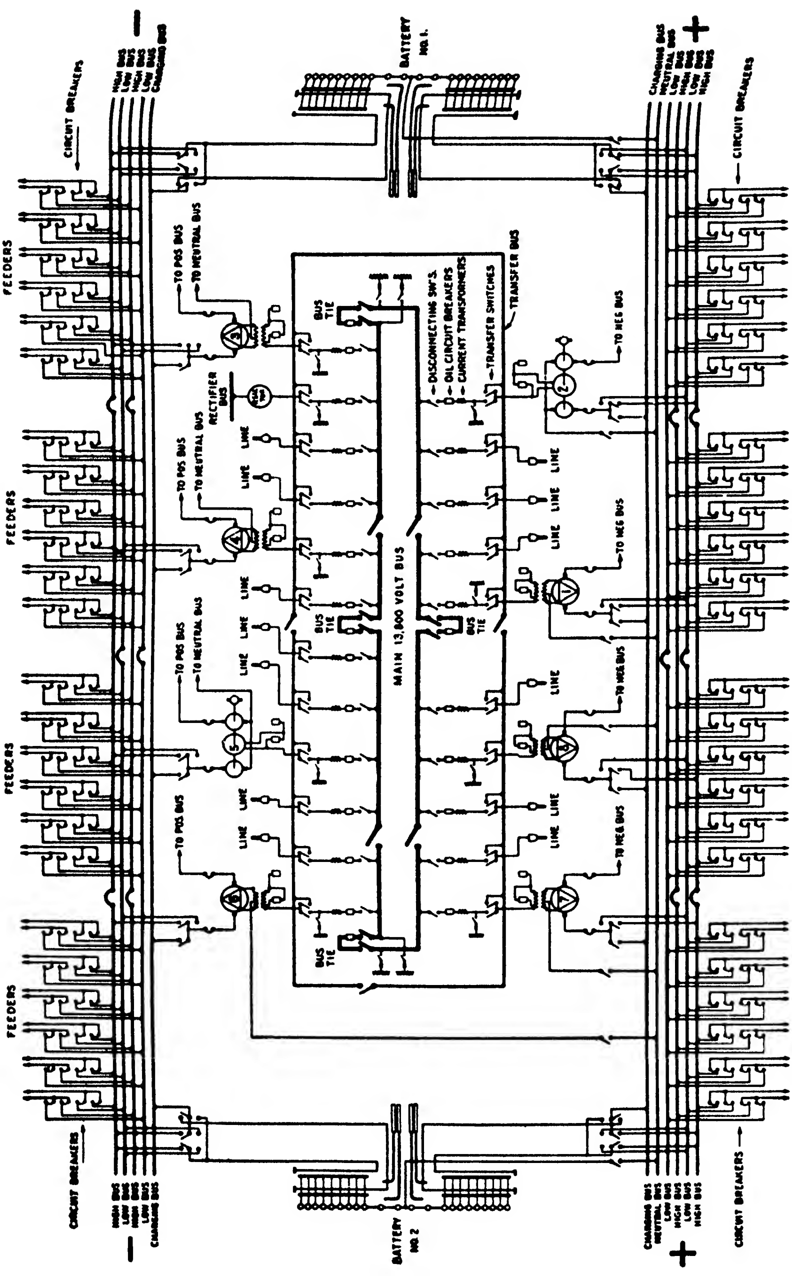
$$G_{bc} = G_{cb}, \text{ etc.} \tag{35}$$

The foregoing indicates that the number of equations required for the loop or the node method is equal respectively to the number of inde-

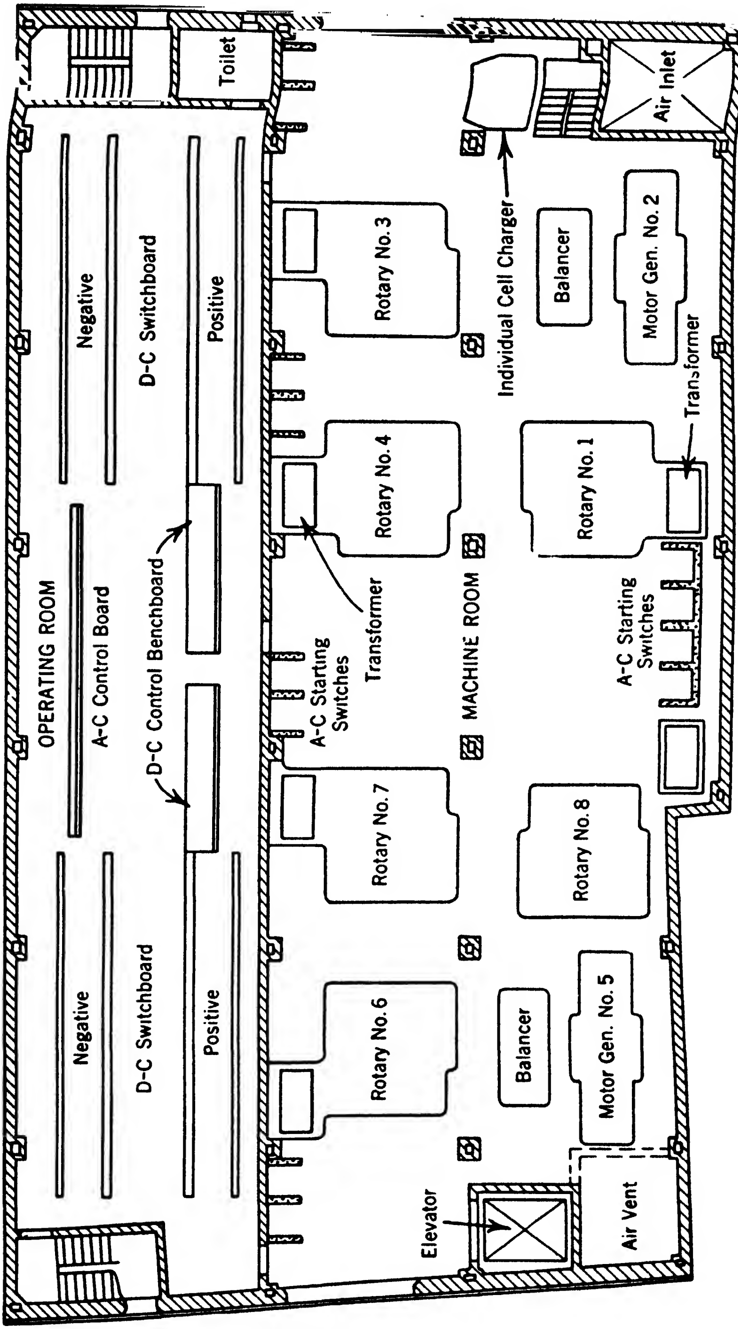


Cross section through building direct current substation.

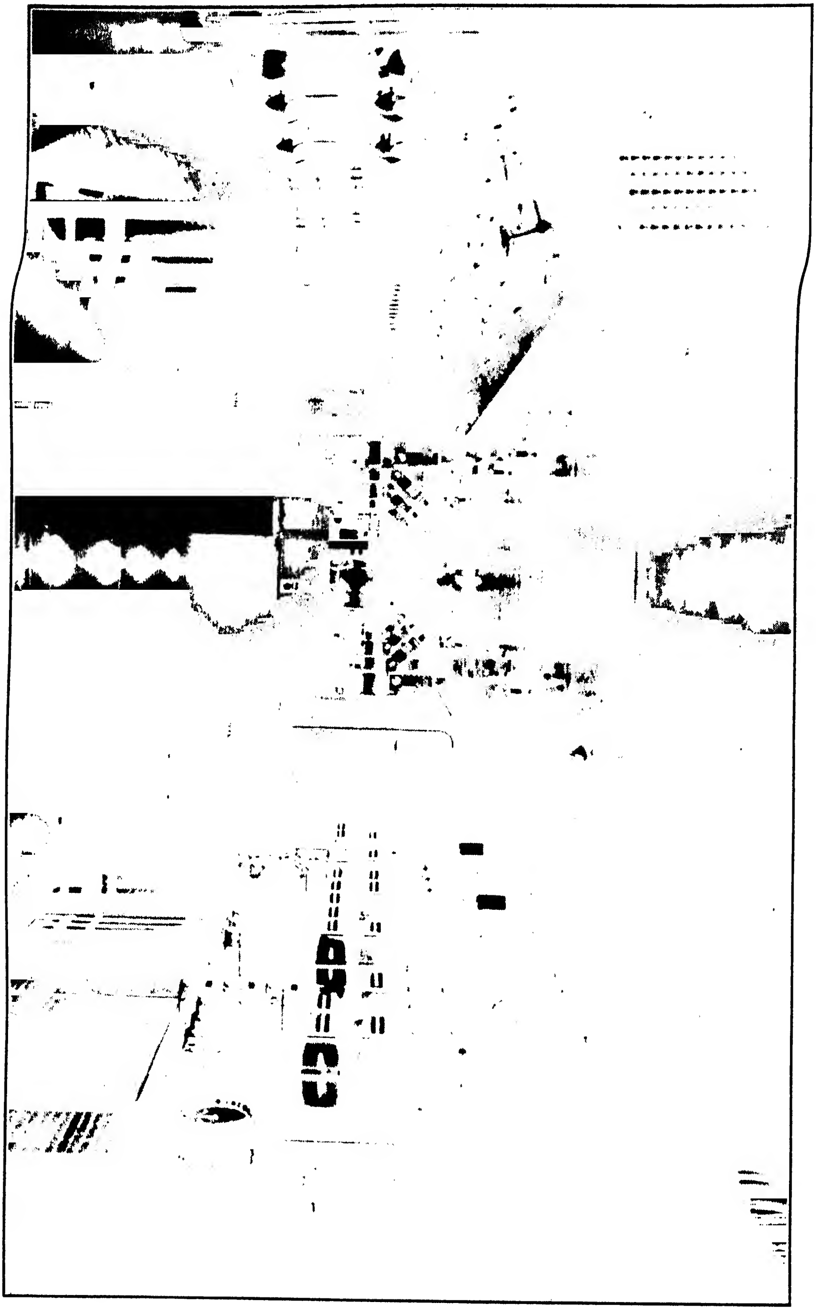
The most important direct-current networks are the street railway networks and the power and light networks of the high-load density areas of metropolitan centers. Very little direct current is generated, it is obtained principally by conversion from alternating current by means of motor-generator sets, rotary converters or mercury-arc rectifiers in substations near the load centers.



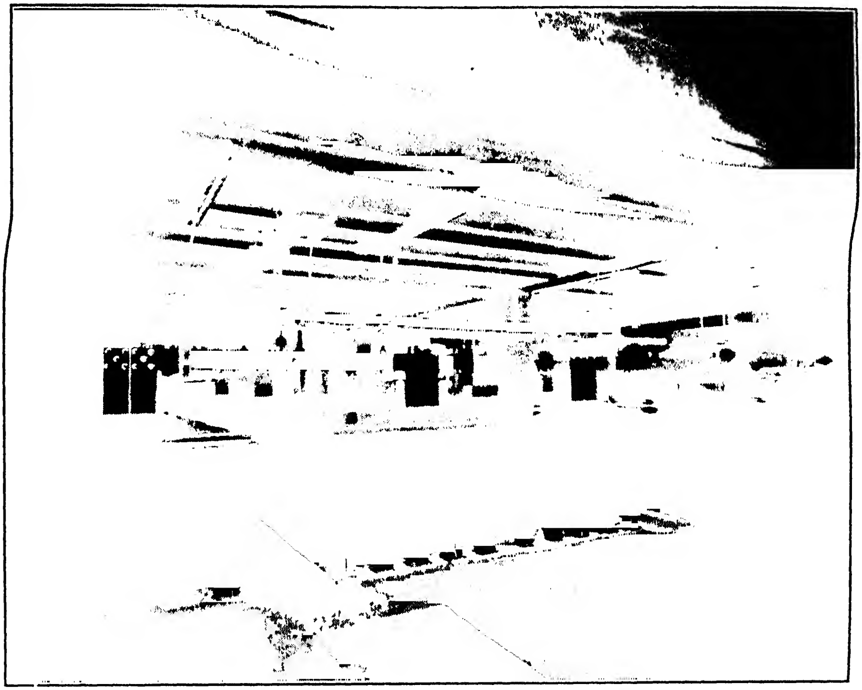
Single line wiring diagram of substation.



MACHINE AND OPERATING ROOMS 1st FLOOR



Control room.



Battery room — the battery can supply 20,000 amperes for 20 minutes, larger current for shorter period.

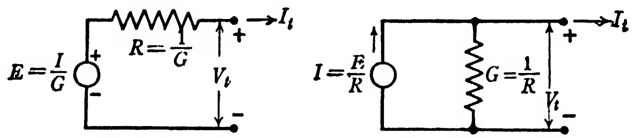
pendent loops or nodes. Hence the number of equations required can be ascertained by counting nodes and branches, and applying Eq. 19; ordinarily the method requiring the fewer equations then is selected. As the number of equations increases, it becomes increasingly important to have a systematic way for solving them as well as for writing them.*

4. Voltage and Current Sources

The development of the loop and node methods indicates also that it is convenient to know the voltage of the source in applying the loop method, and to know the current of the source in applying the node method. Since a physical source of electrical energy — that is, one as found in actual physical apparatus — always contains resistance (also inductance and

* Methods of simultaneous solution of linear algebraic equations are outlined in App. B. An illustrative application of the method of determinants is in Art. 7 of this chapter. For very complex networks, solution is made by construction of models in which the currents and voltages can be measured.

capacitance), the potential difference between its terminals is a function of the current which it supplies. A physical source of direct current is usually best simulated by an ideal *voltage source*, that is, a source of constant voltage independent of the source current, in series with a resistance. The terminal voltage of a representative source, therefore, is always less than its electromotive force by the amount of the voltage drop in the resistance * With a voltage source, solution by the loop method is made readily by including the internal series resistance with the loop resistance; *solution by the node method, however, requires knowledge of the current of the source which ordinarily is a function of the various resistances in* the network and hence generally unknown. It is sometimes desirable, therefore, to be able to convert a voltage source to an equivalent *current source* as shown in Fig. 4. It is emphasized, however, that this is not an essential preliminary to the use of the node method; the source current can remain as one of the unknowns if such choice of unknowns is advantageous.



(a) voltage source (b) current source

FIG. 4. Equivalent voltage and current sources.

$$\left.\begin{aligned} V_t &= E - RI_t, \\ I_t &= \frac{E - V_t}{R} = I - GE_t. \end{aligned}\right\} \quad [36a] \qquad \left.\begin{aligned} I_t &= I - GV_t, \\ V_t &= \frac{I - I_t}{G} = E - RI_t. \end{aligned}\right\} \quad [36b]$$

V_t = terminal voltage of source (function of I_t).
I_t = terminal current of source (function of V_t).
E = voltage-source emf (independent of V_t and I_t).
I = current-source internal current (independent of V_t and I_t).
R = voltage-source series resistance.
G = current-source shunt conductance.

Stated in words, Fig. 4 shows that any voltage source consisting of an electromotive force E independent of the terminal current I_t, having an internal series resistance R, can be replaced by a current source consist-

* If the current in a source is in the opposite direction to the increase in potential generated by the source, the terminal voltage is greater than the electromotive force. Under these circumstances, however, the source actually receives power and hence sometimes is designated as a *sink*.

ing of a current I independent of the terminal voltage V_t, having an internal shunt conductance G. Conversely, the current source can be replaced by the voltage source. That is, for identical terminal conditions of voltage and current, the terminal voltage derived for Fig. 4a can be equated to that derived for Fig. 4b, and the terminal current derived for Fig. 4b can be equated to that derived for Fig. 4a. The conversion relations which follow from this procedure are:

$$E = \frac{I}{G}, \tag{37a}$$

$$I = \frac{E}{R}, \tag{37b}$$

$$R = \frac{1}{G}. \tag{38}$$

It is emphasized that these three relations make a voltage and a current source equivalent only in that terminal voltages and currents are matched. The equivalent sources do not convert the same amounts of energy. For example, an efficient voltage source has a small internal series resistance in comparison with the resistance of the load connected to it. Hence the equivalent current source has a high internal shunt conductance in comparison with the conductance of the load connected to it and is therefore very inefficient. A voltage source which loses 10 per cent of the converted energy in its internal series resistance and delivers 90 per cent to its load is represented by an equivalent current source which delivers only 10 per cent of its converted energy to its load and loses 90 per cent in its internal shunt conductance. In order to supply the same energy to the load as the voltage source, the current source, therefore, must convert nine times as much energy as the voltage source. (In both cases all losses other than those which occur in the internal resistance or conductance are neglected.) For an efficient current source compared with its equivalent voltage source, the situation is the converse.

Sometimes, especially in communications circuits, the condition of maximum efficiency is not so much desired as the condition of maximum power output from a given source. For a voltage source, the power output is

$$P = EI - I^2R_S = I^2R_L, \tag{39}$$

where R_S is the internal resistance of the source and R_L is the resistance of the connected load. By rewriting Eq. 39 to eliminate I,

$$\left.\begin{aligned} P &= \frac{E^2}{R_S + R_L} - \frac{E^2}{(R_S + R_L)^2}R_S = \frac{E^2R_L}{(R_S + R_L)^2} \\ &= \frac{E^2}{R_S}\left(\sqrt{\frac{R_S}{R_L}} + \sqrt{\frac{R_L}{R_S}}\right)^{-2}. \end{aligned}\right\} \tag{39a}$$

The power is a maximum when $\sqrt{R_S/R_L} + \sqrt{R_L/R_S}$ is a minimum. This occurs for

$$\sqrt{\frac{R_S}{R_L}} = 1; \tag{40}$$

whence

$$R_L = R_S, \tag{41}$$

or for maximum power output the resistance of the connected load should equal the internal resistance of the source. For this condition,

$$P_{max} = \frac{E^2}{4R_S} = \frac{E^2}{4R_L}, \tag{39b}$$

from which it is evident that the efficiency is 50 per cent. By a similar procedure it can be shown that the maximum power output from a current source occurs when

$$G_L = G_S. \tag{41a}$$

Only under these circumstances are equivalent current and voltage sources equivalent with respect to internal power requirements as well as for terminal conditions.

The foregoing demonstrates that the maximum power obtainable from a particular source is inversely proportional to its resistance. In communications work, where this is an important item, the productiveness of a source is judged on the basis of its internal resistance.

A voltage source is idle when the external circuit is open; that is, when infinite resistance or zero conductance connects its terminals. When it is in operation, its terminal voltage is a function of the loss of potential in the internal series resistance, as previously stated. A current source is idle when its terminals are short-circuited; that is, when they are connected by zero resistance or infinite conductance. When it is in operation, its terminal current is a function of the loss of current through the internal shunt conductance. To open the circuit connected to the terminals of a current source of high efficiency may be as undesirable as to short-circuit the terminals of a voltage source of high efficiency. The former may lead to trouble owing to excessively high terminal voltage caused by the entire source current passing through the internal shunt conductance; the latter may lead to trouble owing to excessively high current caused by the entire source voltage being across the internal series resistance. However, there is no more disturbance caused by opening the external circuit of a current source which is the equivalent of a high-efficiency voltage source than by opening the circuit of the voltage source itself. The same is true for short-circuiting a voltage source which is the equivalent of a high-efficiency

current source. In other words, if a voltage source is of low efficiency, little change in current is caused by short-circuiting its normal load, and if a current source is of low efficiency, little change in voltage is caused by opening its normal load.

In the circuit of Fig. 3, the source element and the element bc may be considered to represent a physical voltage source. Since the source current I_{ab} hence ordinarily is unknown, in order to solve the network by the node method the voltage source can be converted to an equivalent current source, though this is not mathematically essential.

5. Solution of a Simple Resistance Network by the Loop and by the Node Methods

In Fig. 5 a simple resistance network is shown in two forms, (a) with voltage sources, and (b) with the equivalent current sources. The object is to find the branch currents I_{ab}, I_{cb}, and I_{be} and the corresponding voltage drops. By the loop method, Fig. 5a,

$$15.1I_1 - 5.0I_2 = 2.0, \qquad [42]$$

$$-5.0I_1 + 25.2I_2 = -4.0, \qquad [43]$$

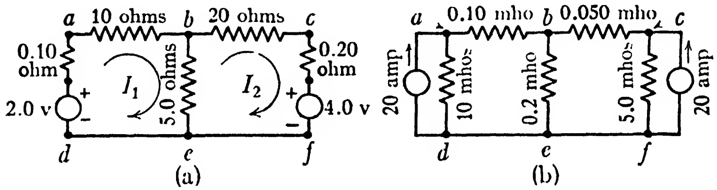


FIG. 5. Illustrative network for solution by various methods.

whence

$$I_{ab} = I_1 = 0.0855 \text{ amp}, \qquad [44]$$

$$I_{cb} = -I_2 = 0.142 \text{ amp}, \qquad [45]$$

$$I_{be} = 0.086 + 0.142 = 0.228 \text{ amp}. \qquad [46]$$

The node potentials with respect to e are

$$V_a = 2.0 - 0.10 \times 0.086 = 1.99 \text{ v}, \qquad [47]$$

$$V_b = 5.0 \times 0.228 = 1.14 \text{ v}, \qquad [48]$$

$$V_c = 4.0 - 0.20 \times 0.142 = 3.97 \text{ v}. \qquad [49]$$

By the node method, Fig. 5b,

$$10.1V_a - 0.10V_b = 20, \qquad [50]$$

$$-0.10V_a + 0.35V_b - 0.050V_c = 0, \qquad [51]$$

$$-0.05V_b + 5.05V_c = 20; \qquad [52]$$

whence

$$V_a = 1.99 \text{ v}, \tag{47a}$$

$$V_b = 1.14 \text{ v}, \tag{48a}$$

$$V_c = 3.97 \text{ v}, \tag{49a}$$

as before. The branch currents are

$$I_{ab} = 0.10\ (1.99 - 1.14) = 0.085 \text{ amp}, \tag{44a}$$

$$I_{cb} = 0.050\ (3.97 - 1.14) = 0.142 \text{ amp}, \tag{45a}$$

$$I_{be} = 0.20 \times 1.14 = 0.228 \text{ amp}, \tag{46a}$$

as before, within slide-rule precision. The branch voltage drops are

$$V_{ab} = 10 \times 0.085 = 1.99 - 1.14 = 0.85 \text{ v}, \tag{53}$$

$$V_{cb} = 20 \times 0.142 = 3.97 - 1.14 = 2.83 \text{ v}, \tag{54}$$

$$V_{be} = 5.0 \times 0.228 = 1.14 - 0 = 1.14 \text{ v}, \tag{55}$$

within slide-rule precision.

In this problem, solution by the loop method happens to be shorter than solution by the node method; the former requires the simultaneous solution of two equations, the latter, three. In other problems the advantage may be on the side of the node method. In this problem, since it is so elementary, direct solution in terms of branch currents is in fact relatively simple. The equations are:

$$10.1 I_{ab} + 5.0 I_{be} = 2.0, \tag{56}$$

$$20.2 I_{cb} + 5.0 I_{be} = 4.0, \tag{57}$$

$$I_{ab} + I_{cb} = I_{be}. \tag{58}$$

One current is readily eliminated from Eqs. 56 and 57 by means of Eq. 58; the former two equations then can be solved simultaneously. This method, however, rapidly becomes very cumbersome as the network becomes complicated, as was pointed out at the beginning of the section, unless considerable care is used in establishing and adhering to an orderly procedure comparable to other methods presented.

6. Application of the Principle of Superposition

When several sources are present, it is frequently advantageous to solve a network problem by finding the currents or voltages in the network resulting from the presence of one source at a time and then superposing the various currents or the various voltages for each loop, branch, or node. In the loop method, the voltage sources omitted (exclusive of internal series resistances) are replaced by connections of zero resistance; in the node method the current sources (exclusive of internal shunt conductances) are replaced by connections of zero conductance, or open circuits. The internal series resistances or shunt conductances always must be retained in their respective places.

Applying the principle of superposition to Fig. 5a, first considering only the left-hand source to be present, gives

$$I'_{ab} = \frac{2.0}{10.1 + \dfrac{5.0 \times 20.2}{5.0 + 20.2}} = 0.141 \text{ amp,} \tag{59}$$

$$I'_{cb} = \frac{-5.0}{25.2} \times 0.141 = -0.0280 \text{ amp,} \tag{60}$$

$$I'_{be} = \frac{20.2}{25.2} \times 0.141 = 0.113 \text{ amp.} \tag{61}$$

Considering only the right-hand source to be present gives

$$I''_{cb} = \frac{4.0}{20.2 + \dfrac{5.0 \times 10.1}{5.0 + 10.1}} = 0.170 \text{ amp,} \tag{62}$$

$$I''_{ab} = -\frac{5.0}{15.1} \times 0.170 = -0.0563 \text{ amp,} \tag{63}$$

$$I''_{be} = \frac{10.1}{15.1} \times 0.170 = 0.114 \text{ amp} \tag{64}$$

Combining the respective results gives

$$I_{ab} = 0.141 - 0.056 = 0.085 \text{ amp,} \tag{44b}$$

$$I_{cb} = -0.028 + 0.170 = 0.142 \text{ amp,} \tag{45b}$$

$$I_{be} = 0.113 + 0.114 = 0.227 \text{ amp,} \tag{46b}$$

as before, within slide-rule precision. From these the various voltages can be found readily.

Applying the principle to Fig. 5b, considering only the left-hand source to be present, gives

$$V'_a = \frac{20}{10 + \dfrac{0.10\left(0.20 + \dfrac{0.050 \times 5.0}{0.050 + 5.0}\right)}{0.10 + 0.20 + \dfrac{0.050 \times 5.0}{0.050 + 5.0}}} = 1.99 \text{ v,} \tag{65}$$

$$V'_{ab} = \frac{0.20 + \dfrac{0.05 \times 5.0}{0.05 + 5.0}}{0.30 + \dfrac{0.05 \times 5.0}{0.05 + 5.0}} \times 1.99 = 1.42 \text{ v,} \tag{66}$$

$$V'_{be} = \frac{0.10 \times 1.42}{0.20 + \dfrac{0.050 \times 5.0}{0.050 + 5.0}} = \frac{0.10}{0.25} \times 1.42 = 0.568 \text{ v,} \tag{67}$$

$$V'_{cb} = -\frac{5.0}{0.05 + 5.0} \times 0.568 = -0.562 \text{ v.} \tag{68}$$

Considering only the right-hand source to be present gives

$$V_c'' = \frac{20}{5.0 + \dfrac{0.050\left(0.20 + \dfrac{0.10 \times 10}{0.10 + 10}\right)}{0.050 + 0.20 + \dfrac{0.10 \times 10}{0.10 + 10}}} = 3.97 \text{ v}, \qquad [69]$$

$$V_{cb}'' = \frac{0.20 + \dfrac{0.10 \times 10}{0.10 + 10}}{0.25 + \dfrac{0.10 \times 10}{0.10 + 10}} \times 3.97 = 3.40 \text{ v}, \qquad [70]$$

$$V_{be}'' = \frac{0.050 \times 3.40}{0.20 + \dfrac{0.10 \times 10}{0.10 + 10}} = \frac{0.050}{0.299} \times 3.40 = 0.569 \text{ v}, \qquad [71]$$

$$V_{ab}'' = \frac{-10}{0.10 + 10} \times 0.569 = -0.563 \text{ v}. \qquad [72]$$

Combining the respective results gives

$$V_{ab} = 1.42 - 0.56 = 0.86 \text{ v}, \qquad [53a]$$

$$V_{cb} = -0.56 + 3.40 = 2.84 \text{ v}, \qquad [54a]$$

$$V_{be} = 0.568 + 0.569 = 1.14 \text{ v}, \qquad [55a]$$

as before, within slide-rule precision. From these the various currents can readily be found.

Whether the solution of a network problem by application of the principle of superposition is easier than by application of the loop or node method depends upon whether the combination of the resistances or conductances of the network by the rules for series and parallel combinations and perhaps by means of $\Delta - Y$ or $Y - \Delta$ transformations* is easier than solving a number of equations simultaneously. Application of the principle is particularly advantageous in determining the effect of an extraneous electromotive force in a network, such as a thermal electromotive force in a Wheatstone bridge or in other situations discussed in Art. 11.

7. Solution by determinants†

In the general case for a network having ℓ independent loops, the loop equations are

$$I_1R_{11} + I_2R_{12} + I_3R_{13} + \cdots + I_\ell R_{1\ell} = E_1, \qquad [73a]$$

$$I_1R_{21} + I_2R_{22} + I_3R_{23} + \cdots + I_\ell R_{2\ell} = E_2, \qquad [73b]$$

* Article 10.

† Details of the method of determinants are in App. B, which should be read by students not acquainted with the method.

$$I_1R_{31} + I_2R_{32} + I_3R_{33} + \cdots + I_tR_{3t} = E_3, \qquad [73c]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$I_1R_{t1} + I_2R_{t2} + I_3R_{t3} + \cdots + I_tR_{tt} = E_t. \qquad [73t]$$

The solution for any loop current such as I_2 is, by Cramer's rule,

$$I_2 = \frac{M_{12}}{D_R}E_1 + \frac{M_{22}}{D_R}E_2 + \frac{M_{32}}{D_R}E_3 + \cdots + \frac{M_{t2}}{D_R}E_t, \qquad [74]$$

where D_R is the determinant of the R's of Eqs. 73a to 73t and M_{32}, for example, is the cofactor of the third row and second column, including the sign factor $(-1)^{(3+2)}$. That is,

$$D_R = \begin{vmatrix} R_{11} & R_{12} & R_{13} & \ldots & R_{1t} \\ R_{21} & R_{22} & R_{23} & \ldots & R_{2t} \\ R_{31} & R_{32} & R_{33} & \ldots & R_{3t} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{t1} & R_{t2} & R_{t3} & \ldots & R_{tt} \end{vmatrix}, \qquad [75]$$

$$M_{32} = (-1)^5 \begin{vmatrix} R_{11} & R_{13} & R_{14} & \ldots & R_{1t} \\ R_{21} & R_{23} & R_{24} & \ldots & R_{2t} \\ R_{41} & R_{43} & R_{44} & \ldots & R_{4t} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{t1} & R_{t3} & R_{t4} & \ldots & R_{tt} \end{vmatrix}. \qquad [76]$$

In the general case for a network having n independent nodes, the node equations are

$$V_aG_{aa} + V_bG_{ab} + V_cG_{ac} + \cdots + V_nG_{an} = I_a, \qquad [77a]$$
$$V_aG_{ba} + V_bG_{bb} + V_cG_{bc} + \cdots + V_nG_{bn} = I_b, \qquad [77b]$$
$$V_aG_{ca} + V_bG_{cb} + V_cG_{cc} + \cdots + V_nG_{cn} = I_c, \qquad [77c]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$V_aG_{na} + V_bG_{nb} + V_cG_{nc} + \cdots + V_nG_{nn} = I_n. \qquad [77n]$$

The solution for any node voltage such as V_b is, by Cramer's rule,

$$V_b = \frac{M_{ab}}{D_G}I_a + \frac{M_{bb}}{D_G}I_b + \frac{M_{cb}}{D_G}I_c + \cdots + \frac{M_{nb}}{D_G}I_n, \qquad [78]$$

where D_G is the determinant of the G's of Eqs. 77a to 77n and M_{cb} is the cofactor of the cth (third) row and bth (second) column, including the sign factor $(-1)^{b+c}$ or $(-1)^{2+3}$. That is,

$$D_G = \begin{vmatrix} G_{aa} & G_{ab} & G_{ac} & \ldots & G_{an} \\ G_{ba} & G_{bb} & G_{bc} & \ldots & G_{bn} \\ G_{ca} & G_{cb} & G_{cc} & \ldots & G_{cn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ G_{na} & G_{nb} & G_{nc} & \ldots & G_{nn} \end{vmatrix}, \qquad [79]$$

$$M_{cb} = (-1)^5 \begin{vmatrix} G_{aa} & G_{ac} & G_{ad} & \cdots & G_{an} \\ G_{ba} & G_{bc} & G_{bd} & \cdots & G_{bn} \\ G_{da} & G_{dc} & G_{dd} & \cdots & G_{dn} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ G_{na} & G_{nc} & G_{nd} & \cdots & G_{nn} \end{vmatrix} . \qquad [80]$$

The actual labor of evaluating the determinants and cofactors may in the general case be very considerable, though in particular cases many terms vanish, either because not all loops contain voltage sources, because not all nodes are connected to current sources, or because certain mutual resistances or mutual conductances are absent. Furthermore, there are numerous short-cut methods which can be utilized to shorten the arithmetic.*

For illustration, the circuit of Fig. 5b is utilized again; the solution of the circuit of Fig. 5a is omitted, because for a simple circuit containing only two loops there is no need for the degree of organization which the method of determinants introduces. The three-node case is simple enough to illustrate the method without becoming lost in excessive detail. From Eqs. 50, 51, and 52,

$$D_G = \begin{vmatrix} 10.1 & -0.10 & 0 \\ -0.10 & 0.35 & -0.050 \\ 0 & -0.050 & 5.05 \end{vmatrix}$$

$$= (-1)^{1+1}(10.1) \begin{vmatrix} 0.35 & -0.050 \\ -0.050 & 5.05 \end{vmatrix} + (-1)^{2+1}(-0.10) \begin{vmatrix} -0.10 & 0 \\ -0.050 & 5.05 \end{vmatrix}$$

$$= 10.1[(0.35)(5.05) - (-0.050)(-0.050)] + 0.10\,[(-0.10)(5.05)] = 17.8. \qquad [81]$$

$$M_{aa} = (-1)^{1+1} \begin{vmatrix} 0.35 & -0.050 \\ -0.050 & 5.05 \end{vmatrix} = 1.77, \qquad [82]$$

$$M_{ab} = M_{ba} = (-1)^{2+1} \begin{vmatrix} -0.10 & 0 \\ -0.050 & 5.05 \end{vmatrix} = 0.505, \qquad [83]$$

$$M_{ac} = M_{ca} = (-1)^{3+1} \begin{vmatrix} -0.10 & 0 \\ 0.35 & -0.050 \end{vmatrix} = 0.005 \text{ (negligible)}, \qquad [84]$$

$$M_{bb} = (-1)^{2+2} \begin{vmatrix} 10.1 & 0 \\ 0 & 5.05 \end{vmatrix} = 51.0, \qquad [85]$$

$$M_{bc} = M_{cb} = (-1)^{2+3} \begin{vmatrix} 10.1 & 0 \\ -0.10 & -0.050 \end{vmatrix} = 0.505, \qquad [86]$$

$$M_{cc} = (-1)^{3+3} \begin{vmatrix} 10.1 & -0.10 \\ -0.10 & 0.35 \end{vmatrix} = 3.52. \qquad [87]$$

* Appendix B.

Hence

$$V_a = \frac{1.77}{17.8}(20) + 0 + \frac{0.005}{17.8}(20) = 1.99 \text{ v}, \quad [47b]$$

$$V_b = \frac{0.505}{17.8}(20) + 0 + \frac{0.505}{17.8}(20) = 1.14 \text{ v}, \quad [48b]$$

$$V_c = \frac{0.005}{17.8}(20) + 0 + \frac{3.52}{17.8}(20) = 3.96 \text{ v}, \quad [49b]$$

as before, within slide-rule precision.

8. The reciprocity theorem[1]

An inspection of Eq. 74 suggests that the respective coefficients of the E's may be regarded as generalized conductances. For example,

$$g_{22} = \frac{M_{22}}{D_R} \quad [88]$$

and

$$g_{12} = \frac{M_{12}}{D_R}. \quad [89]$$

The former, g_{22}, is termed the *short-circuit self-conductance* of loop 2. This means that, if all source voltages are short-circuited except that in loop 2, the self-conductance multiplied by the source voltage in loop 2 gives the current in loop 2. The conductance g_{12} is termed the *short-circuit transfer conductance* between loops 1 and 2. This means that, if all source voltages are short-circuited except that in loop 1, the transfer conductance multiplied by the source voltage in loop 1 gives the current in loop 2. In terms of these symbols, Eq. 74 becomes

$$I_2 = E_1g_{12} + E_2g_{22} + E_3g_{32} + \cdots + E_lg_{l2}. \quad [74a]$$

Since

$$M_{21} = M_{12}, \text{ etc.,} \quad [90]^*$$

therefore

$$g_{21} = g_{12}, \text{ etc.} \quad [91]$$

▶ Hence the current in any loop such as loop 2 caused by an electromotive force in any other loop such as loop 1 is identical with the current in loop 1

[1] J. W. Strutt (Baron Rayleigh), "Some General Theorems Relating to Vibrations," *Proc. Lond. Math. Soc.*, IV (June, 1873), 357–368.

* App. B; Art. 6, Ch. VIII.

caused by the same electromotive force in loop 2, all other source voltages being short-circuited. In other words, a pure voltage source and an ideal (resistanceless) ammeter can be interchanged without altering the indication of the instrument. ◀

This is called the reciprocity theorem. The theorem can be stated also in node terminology. By rewriting Eq. 78

$$V_b = I_a r_{ab} + I_b r_{bb} + I_c r_{cb} + \cdots + I_n r_{nb}, \tag{78a}$$

r_{bb} can be defined as the *open-circuit self-resistance* of node b, and r_{ab} as the *open-circuit transfer resistance* between nodes a and b. The meaning of r_{bb} is that, if all current sources are open-circuited except the one connected to node b, the self-resistance multiplied by the source current gives the potential of node b. The meaning of r_{ab} is that, if all current sources are open-circuited except the one connected to node a, the transfer resistance multiplied by the source current connected to node a gives the potential of node b. Since

$$M_{ba} = M_{ab}, \text{ etc.}, \tag{92}$$

therefore

$$r_{ba} = r_{ab}, \text{ etc.} \tag{93}$$

Hence the potential of any node a caused by a current source connected to any other node b is identical with the potential of node b caused by the same current source connected to node a, all other current sources being open-circuited. In other words, a pure current source and an ideal (conductanceless) voltmeter can be interchanged without altering the indication of the instrument.

It is important to remember that

$$g_{22} \neq \frac{1}{R_{22}} \neq \frac{1}{r_{bb}} \neq G_{bb}, \text{ etc.}, \tag{94}$$

$$g_{12} \neq \frac{1}{R_{12}} \neq \frac{1}{r_{ab}} \neq G_{ab}, \text{ etc.} \tag{95}$$

An illustration of the use of the reciprocity theorem is in Art. 11.

9. Thévenin's theorem

▶ Thévenin's theorem states that any network of resistance elements and voltage sources if viewed from any two points in the network may be

replaced by a voltage source and a resistance in series between those two points.[2]◀

For example, in Fig. 6a, any network is represented, and a–b are any two points in it. Figure 6b represents the equivalent of the network as viewed from a–b. In order to use the equivalent, R_n and E must be de-

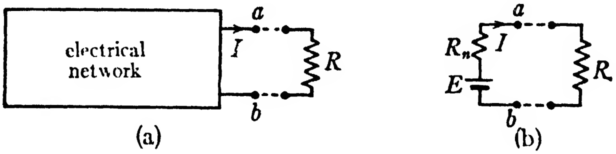


FIG. 6. Derivation of equivalent circuit by Thévenin's theorem.

termined, which requires two conditions. The conditions of open and short circuit across a–b serve as well as any. For open circuit,

$$E = \text{voltage drop } V_{ab} \text{ of original network.}$$ ▶[96]

For short circuit,

$$I = \frac{E}{R_n} = \frac{E}{\text{net resistance of original network between } a\text{-}b}; \qquad [97]$$

hence

$$R_n = \text{net resistance of original network between } a\text{-}b,$$ ▶[98]

that is, the resistance measured at the terminals of the original network with all its voltage sources replaced by connections of zero resistance. If a resistance R is connected across a–b, the current in it is

$$I_R = \frac{V_{ab}}{R_n + R}.$$ ▶[99]

It must be remembered that V_{ab} is the voltage across a–b when R is absent. If the resistance R is in series with a voltage source E',

$$I_R = \frac{V_{ab} \pm E'}{R_n + R}, \qquad [99a]$$

[2] M. L. Thévenin, " Sur un nouveau théorème d'électricité dynamique," *Comptes rendus*, XCVII (1883), 159-161. Some 30 years earlier (1853) Helmholtz proved that he could calculate the current in a resistance connected between two points a and b of a large carbon rod, in which a battery produced current, by assuming that the carbon-rod-battery system could be replaced with respect to a and b by a fictitious generator having a constant internal voltage and resistance. H. Helmholtz, " Über einige Gesetze der Vertheilung elektrischer Ströme in körperlichen Leitern, mit Anwendung auf die thierisch-elektrischen Versuche," *Ann. d. Phys. u. Chem.*, Series III, XXIX (June 1853), 222, 359-363. The theorem is actually more general than stated for the special case of resistance networks. It can be stated in terms of current source and conductance if desired.

or, if two networks are interconnected, the interchange current is

$$I = \frac{V_{ab} \pm V_{a'b'}}{R_n + R_n'}. \qquad [99b]$$

The proof of Eqs. 96 to 99b follows readily from Eq. 74a. If current I_2 of this equation is assigned to the external loop of the network of Fig. 6a, and E_2 is taken as the increase in potential from a to b,

$$-E_2 = \left(E_1 \frac{g_{12}}{g_{22}} + E_3 \frac{g_{32}}{g_{22}} + \cdots + E_t \frac{g_{t2}}{g_{22}}\right) - \frac{I_2}{g_{22}}. \qquad [74b]$$

The corresponding equation in terms of the symbols of Fig. 6b is

$$IR = E - IR_n. \qquad [100]$$

Hence

$$E = E_1 \frac{g_{12}}{g_{22}} + E_3 \frac{g_{32}}{g_{22}} + \cdots + E_t \frac{g_{t2}}{g_{22}}, \qquad [96a]$$

$$R_n = \frac{1}{g_{22}}; \qquad [98a]$$

that is, Eq. 96a follows from Eq. 74b if I_2 is zero, and Eq. 98a follows from Eq. 74b if $E_1, E_3 \ldots E_t$ are zero. It should be noted that $-E_2$ can be visualized as a source connected across a–b, or as the resistance drop in R. In other words, the theorem is independent of the nature of what is connected across a–b externally and relates only to an equivalent substitute for what is connected to a–b inside the box of Fig. 6a.

As an illustration of the application of the theorem, the current I_{bc}, Fig. 5a, is recomputed. In the absence of the 5-ohm resistance,

$$V_{bc} = 4.0 - \frac{4.0 - 2.0}{30.3} \times 20.2 = 2.67 \text{ v}, \qquad [101]$$

$$R_n = \frac{10\,1 \times 20.2}{10.1 + 20.2} = 6.73 \text{ ohms}; \qquad [102]$$

whence

$$I_{bc} = \frac{2.67}{6.73 + 5.0} = 0.228 \text{ amp}. \qquad [46c]$$

The application of Thévenin's theorem is frequently advantageous when the current in one element of a network is particularly desired or when the current in an added element is desired.

10. Delta-wye and wye-delta transformations[3]

In the application of the principle of superposition or of Thévenin's theorem, conditions sometimes arise wherein the net resistance between

[3] A. E. Kennelly, " The Equivalence of Triangles and Three-pointed Stars in Conducting Networks," *Electrical World and Engineer*, XXXIV (1899), 413–414.

two points cannot be computed by means of series and parallel combinations. For example, in computing the galvanometer current of a Wheatstone bridge, Fig. 7a, by Thévenin's theorem, it is necessary to compute the net resistance between a–b, Fig. 7b. If one of the triangles or deltas,

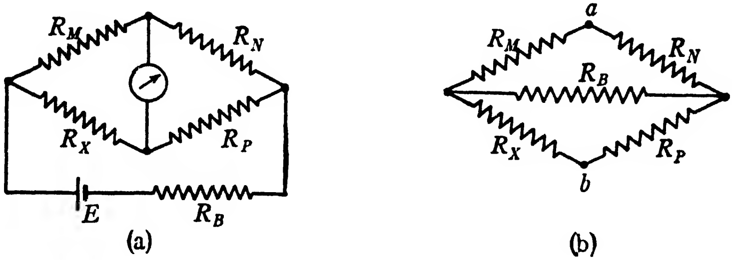


FIG. 7. Wheatstone bridge network.

for example, the one formed by R_X, R_P, and R_B, can be replaced by a wye as illustrated by Fig. 8, the net resistance between points a–b then can be computed as a series-parallel combination.

In order for a wye to be equivalent to a delta, the resistance viewed

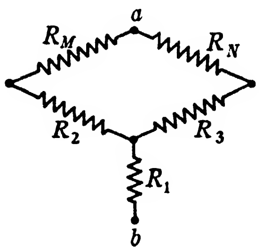


FIG. 8. Substitution of wye for delta.

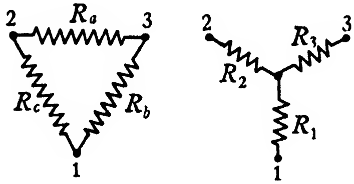


FIG. 9. Equivalent wye and delta.

from any pair of wye terminals must be the same as the resistance viewed from the corresponding pair of delta terminals.* Therefore, from Fig. 9, with no external connections to any of the terminals,

$$R_1 + R_2 = \frac{(R_a + R_b)R_c}{R_a + R_b + R_c}, \qquad [103]$$

$$R_2 + R_3 = \frac{(R_b + R_c)R_a}{R_a + R_b + R_c}, \qquad [104]$$

$$R_3 + R_1 = \frac{(R_c + R_a)R_b}{R_a + R_b + R_c}; \qquad [105]$$

* There is, of course, no delta terminal corresponding to the common connection of the wye resistances.

whence

$$R_1 = \frac{R_bR_c}{R_a + R_b + R_c}, \qquad [106]$$

$$R_2 = \frac{R_cR_a}{R_a + R_b + R_c}, \qquad [107]$$

$$R_3 = \frac{R_aR_b}{R_a + R_b + R_c}. \qquad [108]$$

The derivation of the delta parameters in terms of the wye parameters is simpler in terms of conductances. In the derivation of the wye resistances in terms of the delta resistances, no external connections were made to any of the terminals of Fig. 9. However, if the wye and the delta are actually equivalent so far as measurements at their terminals can determine, any connections whatever can be made externally at the terminals without affecting this equivalence. By taking advantage of this fact and of the fact that conductances in parallel are additive, conductance equations identical in form to Eqs. 103 to 108 can be obtained by connecting one pair of terminals at a time:

$$G_a + G_b = \frac{(G_1 + G_2)G_3}{G_1 + G_2 + G_3} \text{ (with terminals 1 and 2 connected)}, \qquad [109]$$

$$G_b + G_c = \frac{(G_2 + G_3)G_1}{G_1 + G_2 + G_3} \text{ (with terminals 2 and 3 connected)}, \qquad [110]$$

$$G_c + G_a = \frac{(G_3 + G_1)G_2}{G_1 + G_2 + G_3} \text{ (with terminals 3 and 1 connected)}; \qquad [111]$$

whence

$$G_a = \frac{G_2G_3}{G_1 + G_2 + G_3}, \qquad [112]$$

$$G_b = \frac{G_3G_1}{G_1 + G_2 + G_3}, \qquad [113]$$

$$G_c = \frac{G_1G_2}{G_1 + G_2 + G_3}. \qquad [114]$$

Equations 106 to 108 and 112 to 114 can be expressed respectively in terms of conductances and resistances,

$$G_1 = \frac{G_aG_b + G_bG_c + G_cG_a}{G_a}, \qquad [106a]$$

$$G_2 = \frac{G_aG_b + G_bG_c + G_cG_a}{G_b}, \qquad [107a]$$

$$G_3 = \frac{G_aG_b + G_bG_c + G_cG_a}{G_c}; \qquad [108a]$$

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}, \qquad [112a]$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}, \qquad [113a]$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}. \qquad [114a]$$

An inspection of these relations reveals an orderliness by which they can readily be remembered. For those which have for their denominators the sum of the three parameters, the two parameters composing the product in the numerator represent elements which terminate at a point or points common to the element represented by the parameter being sought. For those which have for their numerators the products of the parameters taken two at a time, the parameter composing the denominator represents an element which lies opposite to the element represented by the parameter being sought. The relations may be abbreviated

$$R_1 = \frac{R_bR_c}{\Sigma R}, \qquad ▶[106b]$$

$$G_1 = \frac{\Sigma GG}{G_a}, \quad \text{etc.}; \qquad ▶[106c]$$

$$R_a = \frac{\Sigma RR}{R_1}, \qquad ▶[112b]$$

$$G_a = \frac{G_2G_3}{\Sigma G}, \quad \text{etc.} \qquad ▶[112c]$$

Since the delta and the wye networks each involve only three component resistances, the three resistances viewed from the three possible terminal pairs formed from the terminals 1, 2, and 3 (the third terminal in each case being free) just suffice to characterize uniquely either network. For example, Eqs. 103, 104, and 105 represent the resistances viewed from the respective terminal pairs, the left-hand members being in terms of wye resistances, and the right-hand in terms of delta resistances. Likewise since the delta and wye networks may each be considered in terms of three component conductances, the three conductances viewed from each terminal (the other two terminals in each case being grounded) just suffice to characterize either network. For example, Eqs. 109, 110, and 111 represent the conductances viewed from the respective terminals, the

left-hand members being in terms of delta conductances, and the right-hand in terms of wye conductances.

From these facts it is evident that any three-terminal network, however complex, containing only fixed-resistance elements, can be replaced by an equivalent delta or an equivalent wye network. It is necessary merely to make resistance measurements between terminal pairs and solve for the component resistances of the equivalent delta or wye from Eqs. 103, 104, and 105 or to make conductance measurements from the respective terminals and solve for the component conductances of the equivalent delta or wye from Eqs. 109, 110, and 111.

11. Further illustrative examples of resistance networks

A power-distribution network is shown in Fig. 10, the generators being represented by circles and the loads by rectangles. The network is in a ring form so that any load may be supplied even though there is a break

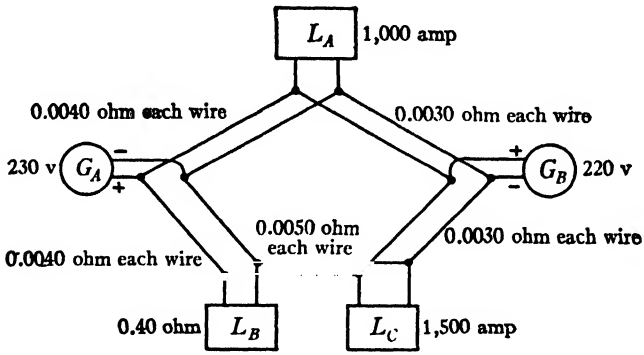


FIG. 10.* Ring-bus distribution network.

in the transmission lines. The circuit is symmetrical in that each wire of each pair has the same resistance. Therefore, as far as voltage drops across the loads are concerned, the network may be simplified by assuming that one wire ring has zero resistance, and that each wire in the other ring has twice its actual resistance. The problem is to determine the line currents, load currents, voltages, resistances, and generator currents.

* When it is necessary to diagram a wire which crosses another but is not electrically connected with it, the crossing is sometimes indicated by means of a jumper, thus: ——⌒—— . However, in complicated diagrams, the multiplicity of jumpers becomes confusing and tedious to draw. In this book, electrical connection of wires is indicated by a dot or small circle at their intersection. No other intersecting lines are to be regarded as electrically connected.

Solution: The network may be split into two independent parts, the simpler of which is shown in Fig. 11. By Thévenin's theorem this may be reduced to the circuit of Fig. 12. In this figure, the values of E and R become

$$E = 220 + 10 \times \frac{6}{14} = 224 \text{ v.} \qquad [115]$$

$$R = \frac{6.0 \times 8.0}{14} \times 10^{-3} = 3.4 \times 10^{-3} \text{ ohm;} \qquad [116]$$

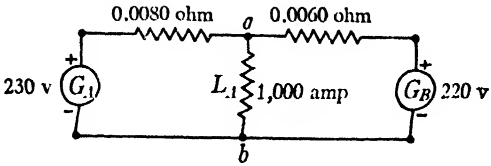


FIG. 11. Upper portion of network of Fig. 10.

whence

$$V_{L_A} = 224 - 1{,}000 \times 3.4 \times 10^{-3} = 221 \text{ v} = V_{ab}, \qquad [117]$$

$$R_{L_A} = \frac{221}{1{,}000} = 0.221 \text{ ohm,} \qquad [118]$$

$$I_{G_A-L_A} = \frac{230 - 221}{0.0080} = 1{,}100 \text{ amp,} \qquad [119]$$

$$I_{G_B-L_A} = \frac{220 - 221}{0.0060} = -200 \text{ amp.} \qquad [120]$$

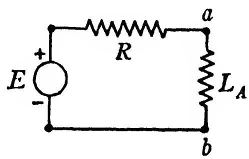


FIG. 12. Reduction of Fig. 11 by Thévenin's theorem.

The other independent part of the circuit is shown in Fig. 13. It has been rearranged slightly from what might appear to be the obvious way of drawing it in order that I_{L_C}, which is known, becomes one of the loop currents, and the number of equations to be solved is therefore reduced.

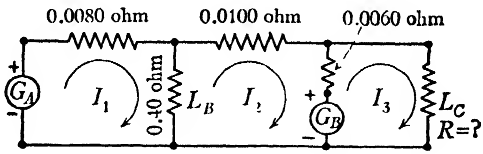


FIG. 13. Lower portion of network of Fig. 10.

This circuit is readily solved by means of loop equations, thus

$$0.408I_1 - 0.40I_2 = 230, \qquad [121]$$

$$0.416I_2 - 0.40I_1 - 1{,}500 \times 0.006 = -220, \qquad [122]$$

$$1{,}500 \times 0.0060 + 1{,}500R - 0.0060I_2 = 220. \qquad [123]$$

The solution of the first and second equations gives

$$I_2 = 620 \text{ amp} = I_{L_B-L_C}, \qquad [124]$$

$$I_1 = 1{,}170 \text{ amp} = I_{G_A-L_B}, \qquad [125]$$

whence

$$I_{GB-LC} = 1{,}500 - 620 = 880 \text{ amp}, \quad [126]$$

$$I_{LB} = 1{,}170 - 620 = 550 \text{ amp}, \quad [127]$$

$$V_{LB} = 230 - 1{,}170 \times 0.0080 = 221 \text{ v}, \quad [128]$$

$$V_{LC} = 220 - 880 \times 0.0060 = 215 \text{ v}, \quad [129]$$

$$R_{LC} = \frac{215}{1{,}500} = 0.143 \text{ ohm}. \quad [130]$$

Checks upon the solution may be obtained as follows:

$$V_{LB} = 550 \times 0.40 = 220 \text{ v}, \quad [128a]$$

$$V_{LC} = 221 - 620 \times 0.010 = 215 \text{ v}. \quad [129a]$$

The total generator currents are then*

$$I_{GA} = 1{,}200 + 1{,}100 = 2{,}300 \text{ amp}, \quad [131]$$

$$I_{GB} = 900 - 200 = 700 \text{ amp}. \quad [132]$$

From these it appears that the two generators are not equally loaded. Generator loads are commonly expressed in terms of current because the heat loss in the generator caused by the load current is ordinarily the element which limits the output of the machine. Generators having equal current but unequal voltages, of course, do not have equal power loads. It is of interest to see what voltage generator B should have in order to make the two generator currents equal. To determine this the generator currents may be written in terms of the generator voltages as follows:

$$I_A = V_A g_{AA} + V_B g_{AB}, \quad [133]$$

$$I_B = V_A g_{AB} + V_B g_{BB}. \quad [134]$$

From the previous determinations all the V's and I's are known for a special case, so that the three g's are left to be determined. The determination of g_{AA} is illustrated in Fig. 14. Using this value of g_{AA} gives

$$g_{AB} = \frac{I_A - V_A g_{AA}}{V_B} = \frac{2300 - 230 \times 116}{220} = -111 \text{ mhos}, \quad [136]$$

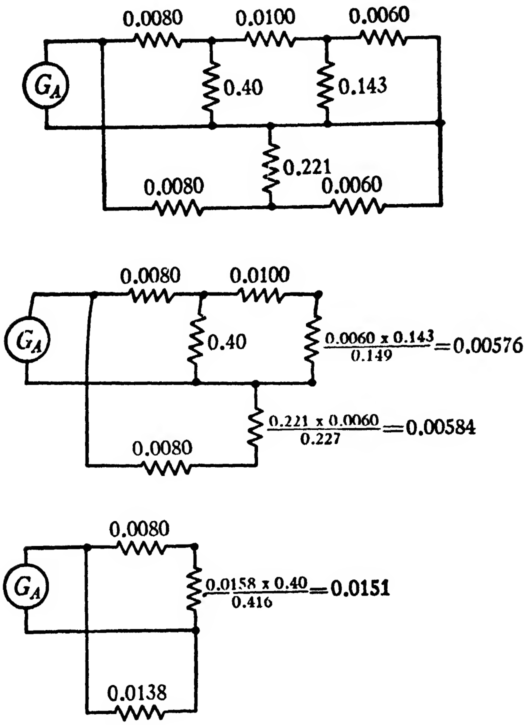
$$g_{BB} = \frac{I_B - V_A g_{AB}}{V_B} = \frac{700 + 230 \times 111}{220} = 119 \text{ mhos}; \quad [137]$$

when I_A and I_B are equal,

$$V_A g_{AA} + V_B g_{AB} = V_A g_{AB} + V_B g_{BB}, \quad [138]$$

$$V_B = V_A \frac{g_{AA} - g_{AB}}{g_{BB} - g_{AB}} = 230 \frac{116 + 111}{119 + 111} = 227 \text{ v}. \quad [139]$$

* Here the values taken from Eqs. 125 and 126 are rounded off to be consistent with the reliability of the values taken from Eqs. 119 and 120. This is indicative of the reliability of such computations in actual practice, because the precision with which the generator voltages can be measured is such that the uncertainty in the difference between nearly equal voltages is considerable.



$$g_{AA} = \frac{1}{0.0231} + \frac{1}{0.0138} = 116 \text{ mhos}$$

FIG. 14. Computation of input conductance g_{AA} for circuit of Fig. 10. The numbers on the diagrams represent ohms.

An alternative method of solving the example follows. For the problem as originally stated, with generator voltages as in Fig. 10, node equations can be applied to Fig. 11:

$$\frac{230 - V_{LA}}{0.008} + \frac{220 - V_{LA}}{0.006} = 1{,}000, \qquad [140]$$

whence

$$V_{L_A} = V_{ab} = \frac{6 \times 230 + 8 \times 220 - 48}{14} = 221 \text{ v.} \qquad [141]$$

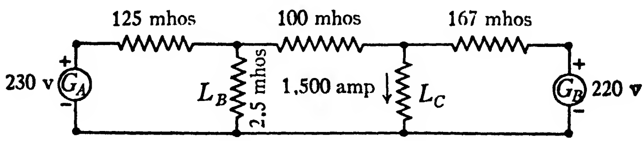


FIG. 13a. Modification of Fig. 13.

Node equations can be written also for Fig. 13 but can be utilized more readily if the figure is rearranged as Fig. 13a:

$$228V_{L_B} - 100V_{L_C} = 125 \times 230, \qquad [142]$$

$$1{,}500 - 100V_{L_B} + 267V_{L_C} = 167 \times 220; \qquad [143]$$

whence

$$V_{L_B} = \frac{288 \times 2.67 + 352}{5.09} = 221 \text{ v}, \quad [128b]$$

$$V_{L_C} = \frac{352 \times 2.28 + 288}{5.09} = 215 \text{ v}. \quad [129b]$$

The load voltages and other known quantities suffice for calculation of the unknown currents and resistances.

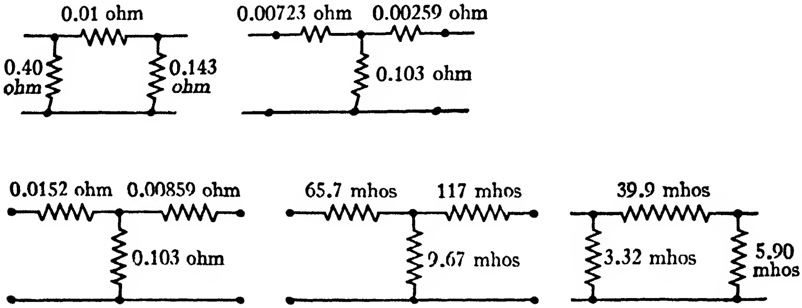


FIG. 13b. Reduction of network of Fig. 13a to equivalent π.

For determining the generator voltages required in order to make the generator currents equal, the network connecting the two generators of Fig. 10 can be replaced by an equivalent π.* The procedure is first to replace the central π of Fig. 13a by an

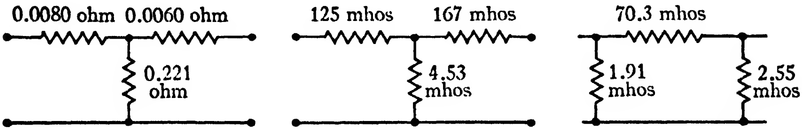


FIG. 11a Reduction of network of Fig. 11 to equivalent π.

equivalent T,* then to add the remainder of the connecting network to the respective arms of the T, and then to replace that resultant T by an equivalent π. The T of Fig. 11 then is replaced by an equivalent π, the branches of which then are paralleled with the corresponding branches of the equivalent π derived from Fig. 13a. The steps are illustrated in Figs. 13b, 11a, and 14a. The self- and transfer conductances are

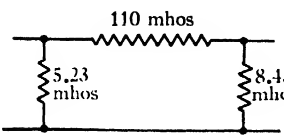


FIG. 14a. Equivalent π for those of Figs. 11a and 13b in parallel.

$$g_{AA} = 110 + 5.23 = 115 \text{ mhos}, \quad [135a]$$

$$g_{AB} = \quad g_{BA} = -110 \text{ mhos}, \quad [136a]$$

$$g_{BB} = 110 + 8.45 = 118 \text{ mhos}. \quad [137a]$$

The use of these admittances instead of those computed by the other method also gives 227 volts for V_B, within slide-rule precision.

* The π and T are identical electrically with the Δ and Y and acquire different symbols merely on account of the forms in which it is sometimes convenient to draw the diagrams.

As another illustrative problem, the circuit of the Kelvin double bridge, which is shown in Fig. 15, is considered. The unknown X and the standard P are both of very low resistance, and therefore they require potential terminals (represented by arrowheads) which are separate from the current terminals. Often the unknown is a cylindrical bar of metal of circular cross section immersed in a tank of oil whose temperature can be controlled. The connection A between X and P has a resistance which may be of the same order of magnitude as either X or P. In order that the balance condition should be independent of A, two sets of ratio arms (MN and mn) are used.

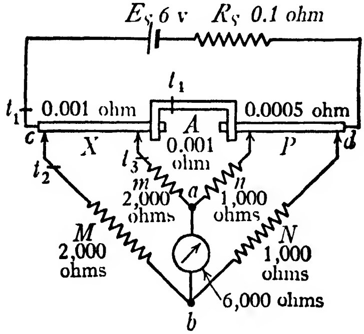


FIG. 15. Kelvin double-bridge network.

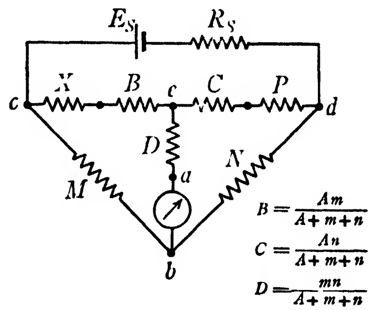


FIG. 16. Modification of Fig. 15, inner delta replaced by wye.

The balance condition can be determined as follows: First the delta composed of A, m, and n is transformed into a wye. The circuit of Fig. 16 results. This is the circuit of an ordinary Wheatstone bridge, which is in balance when the open-circuit voltage between e and b is zero. Thus

$$V_{eb} = V_{ec} + V_{cb} = V_{cd}\left[\frac{-(X+B)}{X+B+C+P} + \frac{M}{M+N}\right] = 0, \qquad [144]$$

$$MX + MB + MC + MP = MX + MB + NX + NB. \qquad [145]$$

Substituting the expressions for B and C,

$$\frac{MAn}{A+m+n} + MP = NX + \frac{NAm}{A+m+n}, \qquad [146]$$

$$M[An + AP + mP + nP] = N[AX + mX + nX + Am]. \qquad [147]$$

In order to satisfy this equation regardless of the value of A, the coefficient of A and the rest of the equation must be set equal to zero independently. Thus

$$M(n + P) = N(m + X), \qquad [148]$$

$$MP(m + n) = NX(m + n). \qquad [149]$$

Substituting Eq. 149 in Eq. 148 gives the simultaneous pair

$$Mn = Nm, \tag{150}$$

$$MP = NX, \tag{151}$$

both of which must be satisfied in order to obtain a balance which is independent of A, as is easily done by fixing M, N, m, and n to conform to the first requirement, and by adjusting P to satisfy the second requirement.

The sensitivity of the bridge, which is the next item to be studied, may be defined as the ratio of galvanometer current to a small change ΔX in X from the value for balance. This ratio is very closely given by

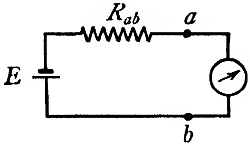


FIG. 17. Reduction of Fig. 15 by Thévenin's theorem.

$$\text{sensitivity} = \left.\frac{dI_G}{dX}\right|_{I_G=0}. \tag{152}$$

The value of this derivative can be found most readily by applying Thévenin's theorem to the whole bridge network exclusive of the galvanometer. The result is indicated in Fig. 17, where E is the same as V_{ab} previously discussed, and R_{ab} remains to be discussed later in the analysis. The galvanometer current is

$$I_G = \frac{E}{R_{ab} + R_G}; \tag{153}$$

whence

$$\left.\frac{dI_G}{dX}\right|_{I_G=0} = \frac{1}{R_{ab} + R_G}\left.\frac{dE}{dX}\right|_{I_G=0} + E\frac{d}{dX}\left.\left(\frac{1}{R_{ab} + R_G}\right)\right|_{I_G=0}. \tag{154}$$

At balance, E is, of course, equal to zero, so the second term drops out. The derivative of E is determined from Eq. 144 as follows:

$$\frac{dE}{dX} = V_{cd}\frac{-X - B - C - P + X + B}{(X + B + C + P)^2}$$

$$= \frac{-V_{cd}}{\left(\frac{X + B}{P + C} + 1\right)^2 (C + P)}. \tag{155}$$

At balance, by substituting from Eq. 145,

$$\left.\frac{dE}{dX}\right|_{I_G=0} = \frac{-V_{cd}}{\left(\frac{M}{X} + 1\right)^2 (C + P)}. \tag{156}$$

There remains the evaluation of R_{ab}. This can be determined readily by reference to Fig. 15. At balance, no current is in the galvanometer, so

by the reciprocity theorem, if E_S is replaced by a short circuit and if a voltage is applied between a and b for the measurement of R_{ab}, no current is in R_S. Consequently, R_S cannot influence the value of R_{ab}. Therefore, R_S may be made an open circuit for the evaluation of R_{ab}. The same conclusion can be reached concerning A by redrawing the circuit as is shown in Fig. 18. It is shown readily by considerations of symmetry that, if a

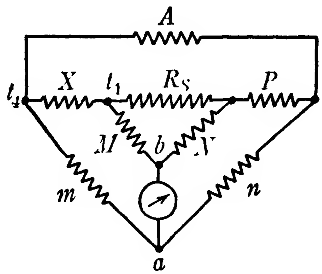


FIG. 18. Rearrangement of Fig. 15 to show that A is without influence in determination of R_{ab}.

power source is inserted in series with A, the original balance equations 150 and 151 still hold.

The result of these deductions is that R_{ab} can be written as

$$R_{ab} = \frac{(M + m + X)(N + n + P)}{(M + m + X) + (N + n + P)}. \tag{157}$$

Incidentally, this resistance R_{ab} is the resistance external to the galvanometer which is effective in damping the motion of the galvanometer when no shunt is used.

The sensitivity is then given by the equation

$$\text{sensitivity} = \frac{-V_{cd}}{(R_{ab} + R_g)\left(\frac{M}{N} + 1\right)^2 (C + P)}, \tag{158}$$

in which R_{ab} is determined from Eq. 157 and C is determined from Fig. 16.

As a specific case, a bridge with the following values of voltage and resistance marked on Fig. 15 is considered:

$$R_{ab} = \frac{(4{,}000)(2{,}000)}{6{,}000} = 1{,}333 \text{ ohms}, \tag{159}$$

$$V_{cd} = 6 \times \frac{0.0025}{0.1025} = 0.146 \text{ v}, \tag{160}$$

$$C = \frac{(0.001)(1{,}000)}{3{,}000} = 0.00033, \tag{161}$$

$$\text{sensitivity} = \frac{0.146}{(7{,}333)(9)(0.00083)} = 2.66 \times 10^{-3} \text{ amp/ohm}. \tag{162}$$

Thus, if a galvanometer which can detect 10^{-8} amp is used, the detectable change in resistance is

$$\Delta X = \frac{10^{-8}}{2.7 \times 10^{-3}} = 3.7 \times 10^{-6}\text{ ohm} \qquad [163]$$

$$\frac{\Delta X}{X} = \frac{3.7 \times 10^{-6}}{10^{-3}} = 0.37\text{ per cent.} \qquad [164]$$

The above treatment of sensitivity is entirely rigorous and contains no assumptions. By making some approximations, the same result can be very quickly obtained. If P is altered a small amount by changing the position of the potential terminal which is connected to n, it is equivalent to introducing a voltage $I_P\Delta P$ in series with n. By applying Thévenin's theorem to this circuit and considering that X, A, and P are all zero,

$$\Delta E_{ab} = \frac{m}{m+n} I_P \Delta P - \frac{2}{3} \times \frac{6}{0.1025} \Delta P = 39\Delta P. \qquad [165]$$

Resistance R_{ab} may be found by again considering X, A, and P to be zero, whence

$$R_{ab} = \frac{mn}{m+n} + \frac{MN}{M+N} = \frac{2 \times 10^6}{3{,}000} + \frac{2}{3} \times 10^3 = 1{,}333\text{ ohms.} \qquad [166]$$

From this ΔI is

$$\Delta I = \frac{39\Delta P}{7{,}333} = 5.32 \times 10^{-3}\Delta P. \qquad [167]$$

From the original balance equation ΔP is obtained in terms of ΔX as follows:

$$P = \frac{N}{M} X, \qquad [151a]$$

$$\Delta P = \frac{dP}{dX}\Delta X = \frac{N}{M}\Delta X. \qquad [168]$$

From this the sensitivity is

$$\frac{\Delta I}{\Delta X} = 5.32 \times 10^{-3} \times \frac{1}{2} = 2.66 \times 10^{-3}\text{ amp/ohm,} \qquad [169]$$

which is a fair approximation to the correct answer of 2.82×10^{-3} amp/ohm.

In view of the fact that the unknown is often at a temperature different from that of the rest of the circuit, the effect of thermal electromotive forces should be investigated. Four thermal electromotive forces (t_1, t_2, t_3, t_4) of equal magnitude are considered to be at the points indicated in Fig. 15. Because of its position, t_1 aids or opposes E_S and so cannot affect the balance condition. By redrawing the bridge circuit as is done in Fig. 18, and applying the considerations of symmetry, it can be shown that t_4 is also ineffectual. Either t_2 or t_3 alone causes an error, but that they compensate each other approximately can be shown as follows:

Since X, A, and P are small compared to M, N, m, and n, it seems justifiable to consider X, A, and P as zero. Then the value of E_t caused by t_2 and t_3 is given by superposition as

$$E_t = \frac{N}{M+N}t_2 - \frac{n}{m+n}t_3 \approx t\left(\frac{Nm + Nn - Mn - Nn}{(M+N)(m+n)}\right). \quad [170]$$

By applying the first balance equation, it is seen that E_t is 0.

The effect of any small unbalanced thermal electromotive forces may be eliminated practically by making two balances with E_S in the two possible directions and taking the average of the two resistance readings.

PROBLEMS

1. In the circuit of Fig. 19 the resistance of each lamp of group A is 180 ohms.
(a) What is the voltage across group of lamps B?
(b) What is the resistance of each lamp of group B (assuming the lamps to be alike?

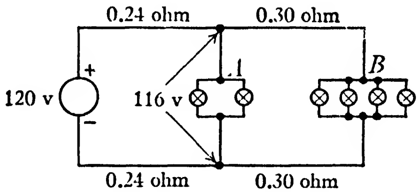


FIG. 19. Lighting circuit, Prob. 1.

(c) What is the line drop in volts between the generator and group A?
(d) What is the line drop in volts between groups A and B?
(e) What is the current in each lamp in group A and in each lamp in group B?

2. Figure 20 represents a simplification of a situation encountered in household wiring. The wires are 14 AWG. The lamp is to be considered as a constant resistance.

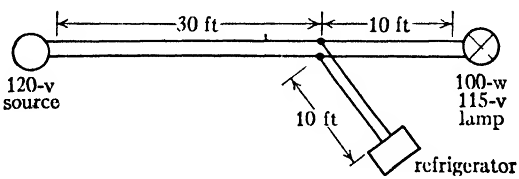


FIG. 20. For computation of residential voltage regulation, Prob. 2.

(a) What is the lamp voltage when the refrigerator is not running?
(b) What is the lamp voltage when the refrigerator is taking 1.5 amp? (Running condition.)
(c) What is the lamp voltage when the refrigerator is taking 10 amp? (Starting condition.)

3. The following questions refer to Fig. 21.
(a) What is the voltage between A and B?
(b) What is the current in each battery?
(c) What is the resistance between A and B?

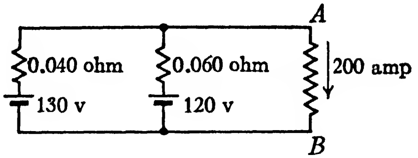


FIG. 21. Stand-by battery discharge circuit, Prob. 3.

4. Figure 22 shows a three-wire distribution system which enables either 115-v or 230-v power to be obtained. In the calculations, it may be assumed that the lamps are constant resistance and that the motor current is constant at 10 amp.

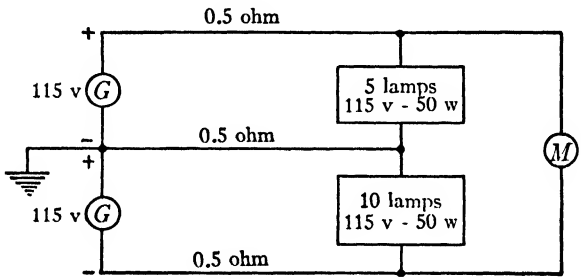


FIG. 22. Three-wire distribution system, Prob. 4.

(a) What is the current in each of the line wires?
(b) What is the voltage across each of the three loads?
(c) If the middle wire is broken, what is the voltage across each load?
(d) In which of the three transmission wires should fuses be placed?

5. What are the equations required for determining the currents and voltages of the network of Fig. 23
(a) on the node basis, using the ground as the reference node;
(b) on the loop basis?

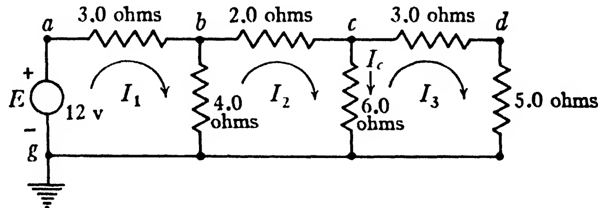


FIG. 23. Network for exercise in use of loop or node methods, Prob. 5.

What is the ratio of I_c to I_1 in terms of self- and mutual resistances? What is the ratio for the parameters given above?

6. Figure 24 is a proposed circuit for a certain type of police signal system. When K_1 is closed, the relay R closes the switch K_2, thereby connecting the load (various signal lamps) to the generator. The relay closes when its current rises to 9 ma and opens when the current drops to 3 ma.

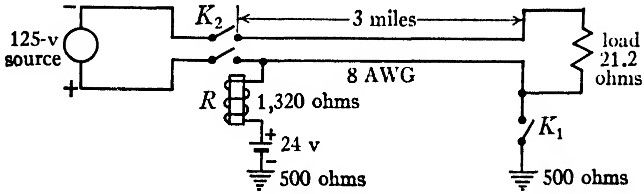


FIG. 24. Circuit for police signal system, Prob. 6.

(a) What is the relay current when K_1 is closed but before the relay closes? Will the relay close?
(b) What is the relay current after K_2 closes, with K_1 still closed? Will the relay remain closed?
(c) What is the relay current when K_2 is closed but when K_1 is open? Will the relay open?
(d) How can failure of the circuit to allow K_1 to exercise control over the operation of the relay be remedied?

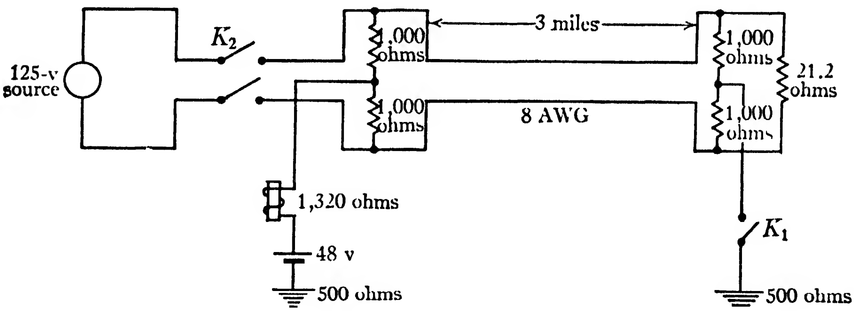


FIG. 25. Another circuit for police signal system, Prob. 7.

7. One method of making the operation of the circuit of Prob. 6 independent of the polarities of the sources is shown in Fig. 25. What are the answers to the questions of Prob. 6 if applied to this circuit?

8. The circuit of a private branch telephone switchboard is shown in Fig. 26, in which nearly all the elements have been drawn as resistances, their inductance and their coupling to mechanical circuits being omitted. Two relays, whose coils are shown, require 12 ma coil current in order to operate.

(a) If the resistances of the two loops are 0 and 400 ohms, respectively, which of the two relays operates?
(b) With the loop resistances given in (a), one of the condensers becomes short-circuited. Which of the two relays operates?

(c) If the possibility of condenser failure is omitted, what is the maximum resistance that a loop may have if its associated relay is to operate regardless of the *resistance of the other loop?*

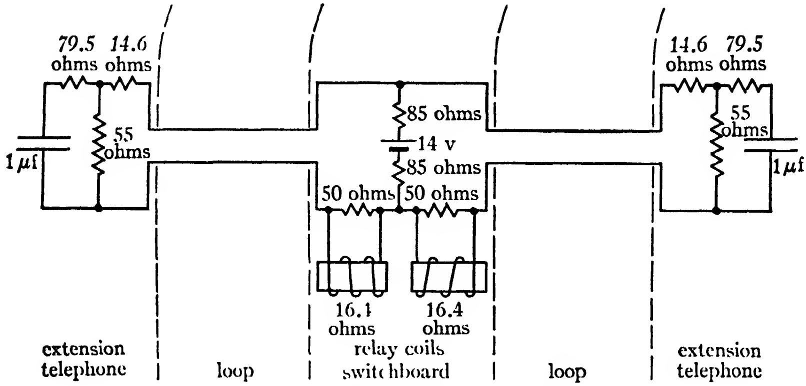


FIG. 26. Telephone circuit, Prob. 8.

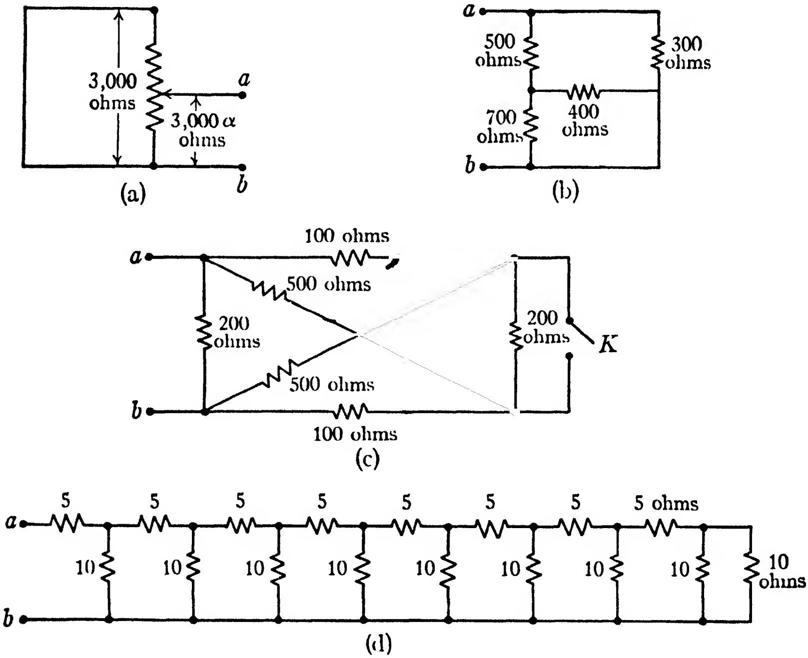


FIG. 27. Networks for reduction to single equivalent resistances, Probs. 9, 10, and 11.

9. What is the resistance between a and b in the accompanying circuits?
(a) Fig. 27a, (b) Fig. 27b, (c) Fig. 27d, (d) Fig. 27c with K closed.

What is the answer to (a) plotted as a function of α on rectangular co-ordinate paper? On log-log paper?

10. In Fig. 27d of Prob. 9 a voltage of 128 v is applied between terminals a and b. What are the voltages between the bottom node and each of the top nodes?

11. In Fig. 27c of Prob. 9, what is the resistance between a and b when the switch is open?

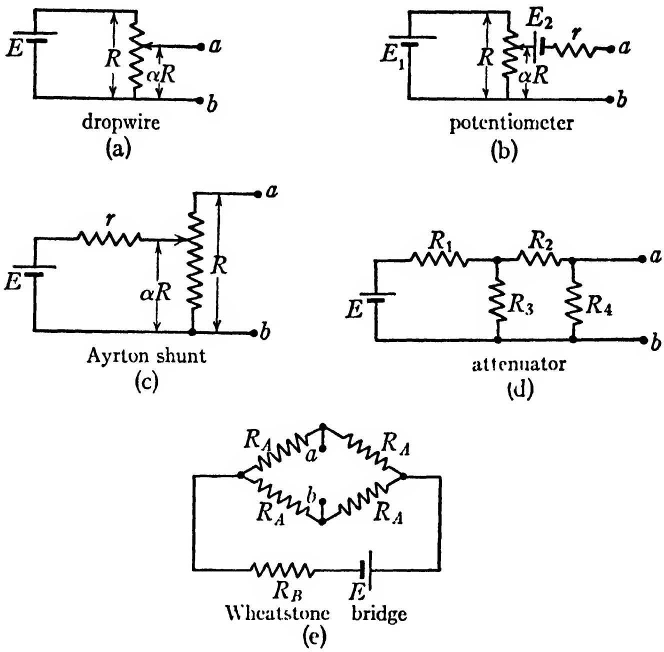


FIG. 28. Networks for reduction by Thévenin's theorem, Prob. 12.

12. The networks of Fig. 28 (viewed from terminals a–b) are to be reduced to the single electromotive force E and single resistance R_n of Thévenin's theorem. The symbol α is a numeric.

13. A certain direct-current power supply has an output voltage of 600 v when the output current is 400 ma, and an output voltage of 650 v when the output current is 200 ma. What are the simplest circuits that represent this power supply as far as the relation between output voltage and output current is concerned? Values are to be given for each circuit element.

14. The following questions refer to Fig. 29.

(a) What are the network equations required to solve for the three mesh currents by the loop method?

(b) With the two voltage sources in the above network transformed to equivalent current sources, what are the equations required to solve for the node voltages by the node method?

(c) What is the current in the 40-v battery obtained by applying Thévenin's theorem to the part of the circuit to the left of a–b?

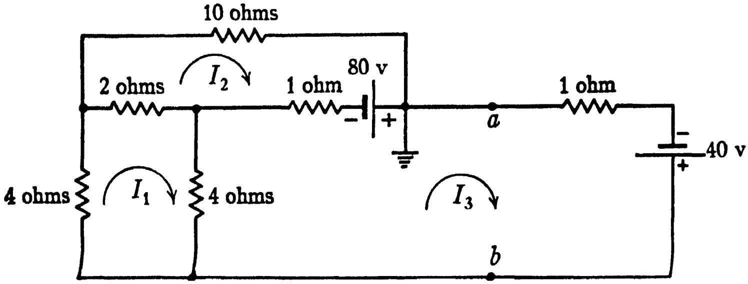


FIG. 29. Network for exercise in application of the loop and node methods, and of Thévenin's theorem, Prob. 14.

15. A complicated network of resistances has four terminals, a, b, c, and d. The resistance measured between a and c is 0, and the resistance measured between any of the remaining pairs is 6 ohms. What are the simplest circuits which may be used to replace this network if no conditions internal to the network are to be studied? A value is to be given for each resistance of the equivalent circuits.

16. A pyrometer bridge network is connected as shown in Fig. 30. The arm P is made of copper and its resistance at 70 F is 30 ohms. All other resistors are made of manganin, and all except the slide wire have a resistance of 30 ohms. The slide wire has a total resistance of 2 ohms. The galvanometer has a resistance of 30 ohms.

(a) When the bridge is balanced, how does the displacement of the slider from the midpoint vary as a function of the deviation of temperature from 70 F?
(b) What is the temperature range of the instrument?
(c) When the temperature is 70 F, and the slider is placed at a, what is the ratio of galvanometer current to battery current?

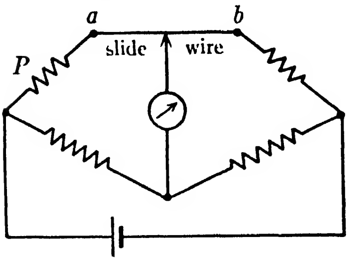


FIG. 30. Wheatstone bridge pyrometer circuit, Prob. 16.

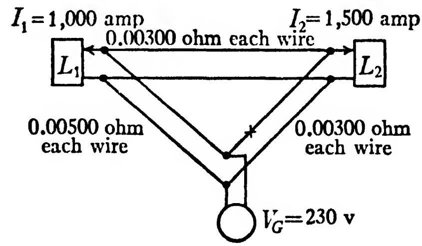


FIG. 31. Ring-bus distribution network, Prob. 17.

17. For the purpose of insuring continuity of service, power companies frequently make use of loop distribution circuits. Such a circuit is illustrated in Fig. 31.

(a) What are the voltages at the loads, the resistances at the loads, and the currents in the lines L_1-L_2, L_2-G, and G-L_1?
(b) A break at point × occurs in one of the wires of the circuit L_2-G interrupting the current therein. What are the new currents drawn by the loads and the voltages at the loads under these new conditions? Each load may be assumed to be a fixed resistance.

CHAPTER III

Transient Response of Simple Circuits

1. Introduction

The conceptions of the lumped-circuit elements — resistance, inductance, and capacitance including the relations between their currents and terminal voltages or the derivatives or integrals of these currents and voltages, are developed in Ch. I. It is also shown that when such elements are connected to form a network, the element currents and voltages satisfy two circuit relations, the Kirchhoff voltage law and the Kirchhoff current law. Utilizing the element relations and these two circuit relations makes it possible to write the differential equations from which the currents and voltages in a network can be determined. Equation 1d, p. 114, is an example of such a differential equation.

This chapter considers the solution of such equations for simple circuits to which constant voltages or currents are suddenly applied, beginning with very simple circuits, analysis of which is already somewhat familiar. Though the solutions obtained are useful to the electrical engineer, the methods by which they are obtained are even more useful, for they serve as a foundation for the study of more complicated circuits. Emphasis therefore is laid on the general principles by which lumped-parameter circuits are analyzed and, incidentally, useful results are obtained. Once these principles, in both their physical and mathematical aspects, are clearly understood as applied to the simpler cases, their application to more complicated networks is relatively simple in principle though often laborious arithmetically.

The parameters R, L, and S as used here are subject to the limitations imposed by the assumptions used in deriving them in Ch. I. These limitations do not affect the mathematical analysis by which the solution of a given differential equation is obtained but rather the accuracy with which the differential equation describes an actual physical circuit, and therefore the agreement to be expected between mathematical and experimentally observed results. With the conditions and apparatus used in elementary experimental aspects of this subject, however, the experimenter's technique, or lack of it, and not the limitations of lumped-parameter theory, is the probable source of any observed discrepancies between theory and experiment.

In this chapter the use of ideal elements is assumed; that is, that the resistance is resistance alone, that the inductance has no resistance or capacitance effects, and that the elastance has no resistance or inductance

effects. In nearly all cases a series resistance element accompanies an inductance element to represent the resistance of the coil, which is usually not negligible. The assumption is also made that voltage sources have *negligible internal resistance and negligible inductance and capacitance effects — assumptions that can often be realized in practice provided* care is taken. Appreciable internal series resistance or inductance of a source can be added to the parameters of the external circuit for purposes of analysis.

Four simple circuits are first considered, namely, the series resistance-inductance or RL circuit, the series resistance-elastance or RS circuit, the series resistance-inductance-elastance or RLS circuit, and the parallel resistance-capacitance or RC circuit.

2. Directions of voltages and currents when functions of time

At this point it is advisable to review the conventions of direction for voltage and current mentioned in Art. 3, Ch. I, and more particularly set forth in Art. 2, Ch. II. These preceding discussions apply to the work of this chapter if it is merely understood (1) that the time function which represents a potential difference always represents the amount by which a designated point is higher in potential than another designated point *algebraically*, in accordance with the polarity markings on the diagram, or the double-subscript notation, and (2) that the time function which represents a current always represents the amount of current in a particular part of a circuit *algebraically*, in accordance with the arrow mark on the diagram, or the double-subscript notation. For example, in Fig. 1, $i(t)$, or $i_{ab}(t)$, is the amount of current in the arrow or $a\ b$ direction as a function of time. If $i(t)$ is known in terms of a mathematical expression which can be evaluated for any instant of time t, the result gives the amount of current in the arrow direction at that instant.

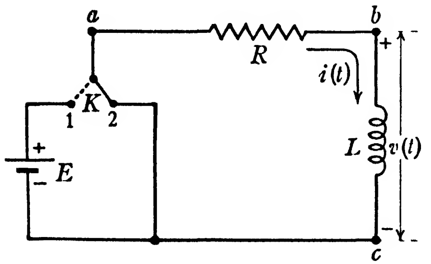


Fig. 1. Series RL circuit.

This result may conceivably be a negative number, perhaps -10 amperes. A current of -10 amperes in the arrow direction of course merely means a current of $+10$ amperes in the opposite direction. Similarly $v(t)$, or $v_{bc}(t)$, is the amount by which the potential of point $b(+)$ exceeds that of point $c(-)$ as a function of time. If $v(t)$ is known in terms of a mathematical expression which can be evaluated for any instant of time t,

the result gives the amount by which the potential of point b is higher than that of point c at that instant. This result also may conceivably be a negative number, perhaps -50 volts. In spite of the polarity marks, a *potential difference of -50 volts of course merely means that point c is $+50$ volts with respect to point b* at the particular instant under consideration.

It should be recalled from Arts. 2, 3, and 22, Ch. I, that the expressions Ri, $L\frac{di}{dt}$, and $S\int idt$ represent *algebraically* voltage drops in the arrow direction assigned to the current in the particular branch.

3. Notation

The symbols used for current $i(t)$ and potential difference $v(t)$ can be described as *functional notation*. They mean respectively that current i and potential difference v are functions of time t only For example, $i(x,y)$ means that i is a function of both x and y. However, since in this and several succeeding chapters time is the only independent variable, the lower-case letters are generally used alone: v, i, and so on representing the dependent variables as functions of time, except that occasionally functional notation is used for clarity or emphasis. For example, $i(5)$ means the current when t is five units of time and $i(0)$ means the current when time is zero. In a switching operation $i(0-)$ means the current just before the switching operation, $i(0+)$ means the current just after the switching operation. This notation is useful when the function is discontinuous at the instant of switching. Capital letters are in general used to represent constants. As a further development of the convention established in Ch. II, the symbol $e(t)$ always stands for the electromotive force of an active element (a source), and the symbol $v(t)$ always stands for the voltage drop through passive elements (resistance, inductance, capacitance) *in the direction of the appended subscripts* if double-subscript notation is used.

4. Series *RL* circuit

The series combination of resistance and inductance illustrated in Fig. 1 is discussed first. It is assumed that the switch K is so arranged that the constant-voltage source may be inserted or removed instantaneously, always leaving the circuit closed.

When the electromotive force is first inserted, a current is established gradually, since the rate of increase of the current is limited by the inductance L. The inductance property of the circuit is associated with energy storage in the magnetic field. Upon removal of the voltage source this energy momentarily remains, and so it is reasonable to suppose that the

current does not instantly cease but dies out gradually as the stored energy is dissipated in the form of heat in the resistance element. The problem is to determine more precisely the manner in which this process of build-up and decay goes on, and how it depends upon the circuit parameters, the value of the applied electromotive force E, and the sequence of switching operations which may be carried out.

The simplest situation is probably the one in which the switch K is originally in position 2 and the current i is zero. This is called an *initial-rest condition. At a given instant, which is designated the initial instant at which t is zero and from which time is counted, the switch is thrown to* position 1, thus suddenly inserting the electromotive force E. Thereafter the switch remains in position 1. The problem is to determine the manner in which a current is established.

The potential drop across a resistance is Ri, and the potential drop across an inductance is $L\frac{di}{dt}$, in the direction of the current. In the direction assumed positive for current, a potential rise of E occurs in the source. This may be described equally well as a potential drop of $-E$ in the same direction. Applying Kirchhoff's voltage rule to the potential drops around the circuit gives the differential equation

$$L\frac{di}{dt} + Ri - E = 0. \tag{1}$$

It is shown in elementary physics courses that if the electromotive force E is applied to the circuit when t is zero, as in this case, the solution of this differential equation is

$$i(t) = \frac{E}{R} - \frac{E}{R}\epsilon^{-(R/L)t}. \tag{2}$$

Equation 1, it should be observed, is a linear differential equation with constant coefficients. The solution of such an equation can be resolved into a steady-state term (denoted by the subscript s) and a transient term (denoted by the subscript t). Thus in this solution, the term E/R represents the steady-state current, and the term $-(E/R)\epsilon^{-(R/L)t}$ represents the transient current. The current $i(t)$ is at all times equal to the algebraic sum of these two components, as is shown in Fig. 2, p. 176, in which the curve for the current $i(t)$ is the sum of the curves for E/R and $-(E/R)\epsilon^{-(R/L)t}$.

Examination of both Eq. 2 and the curve shows that:

(a) When t is zero, the resultant current $i(0)$ is zero; so the transient current at this instant must have a negative value equal to the positive value of the steady-state current.

(b) As t increases, the transient current, which is numerically negative, diminishes in magnitude, and thus the resultant current i increases in magnitude.
(c) The magnitude of the transient current decreases at an exponential time rate, so that theoretically it takes an infinite time for the transient to reach zero and for the resultant current to reach the steady-state value.

A study of the differential equation 1 and its solution discloses that:

(a) When t and i are zero, the potential drop across the resistance is zero. Since the total voltage drop around the circuit is always zero, all the applied voltage at the first instant is impressed as a voltage drop $L\dfrac{di}{dt}$ across the inductance L. In other words, the entire electromotive force of the source is being used to accelerate the flow of charge through the inductance, or to give the current a rate of change.
(b) When t is infinity, and i is E/R, the voltage drop across the resistance equals the electromotive force E; that is, all the applied voltage is used in maintaining a current through the resistance. Since the current is no longer changing in value, the voltage drop across the inductance L is zero.

5. A GENERAL METHOD OF SOLUTION

In the simple case considered in the preceding article, it is possible to separate the variables so that the differential equation 1 can be integrated simply by direct methods. More usually, this method is not applicable and a somewhat broader view regarding the solution of differential equations is necessary.

Generally the solution of a differential equation is not obtained by an explicit process, such as simple differentiation or integration, that can be performed by a simple formula. Rather it is the process of finding — by any means whatsoever — a function that satisfies both the differential equation and at the same time any suitable initial or boundary conditions that can be arbitrarily specified. This conception of a solution is that used in the application of differential equations to physical problems and is in contrast to that of the mathematician who usually considers any function that has the necessary number of arbitrary constants or functions to be a solution. It applies to differential equations both of the ordinary type that contain only one independent variable and of the partial type that contain more than one independent variable. Only the restricted form of the ordinary type that arises from the study of simple linear circuits is now concerned.

The mathematical formulation of dynamic-equilibrium conditions in lumped-parameter circuits that contain the kinds of elements discussed in Ch. I leads to a restricted type of ordinary differential equation. This equation has the general form

$$a_n \frac{d^n y}{dt^n} + \cdots a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = f(t) \qquad [3]$$

which has the following restrictions: First, the a's are all constants. Second, the dependent variable y and its derivatives are functions of t only and appear only in the first power; no term contains a product of y and a derivative, or the product of two derivatives. Because of the second restriction, the equation is said to be linear. Because of the first *it is said to have constant coefficients. Equation 3 is therefore said to be* an *ordinary linear differential equation with constant coefficients.* Its order is that of the highest derivative appearing in it, Eq. 3 being of the nth order and Eq. 1 of the first order. Equations reducible to this type are dealt with in this and following chapters. In this chapter $f(t)$ is merely a constant. In Ch. IV and subsequently it is commonly a sinusoidal function of the independent variable, time.

The nature of the equations that are to follow having been indicated, the solution for the simple series RL circuit is developed next by the same methods that are used for the higher-order equations of more complicated circuits. The reasoning used is given in considerable detail in order that the process of thought may be clear before situations more involved in their details are discussed.

The first step in the solution of an ordinary linear differential equation with constant coefficients, as typified by Eq. 1, is usually a consideration of the term not containing i, which here is the constant E. For this purpose Eq. 1 is rewritten as

$$L \frac{di}{dt} + Ri = E. \qquad [1a]$$

Here E represents the agency, or energy source, or driving force causing current in the circuit. Generally the current may be expected to vary with time in the same general fashion as the driving force, at least after any initial disturbances have died down. In this case the driving force is a constant after it is applied. Since the current may well be expected to be constant, at least eventually, this assumption is tried: i is taken as I_s, a constant. Substituting I_s in Eq. 1a, remembering that its derivative is zero, gives

$$I_s = \frac{E}{R}, \qquad [4]$$

which is an entirely correct solution of Eq. 1a. Although Eq. 4 satisfies the differential equation, it can easily be shown not to satisfy arbitrary initial conditions. It is, therefore, not a complete solution of Eq. 1a but is only the steady-state term or, in mathematical terminology, the ***particular integral.***

The subject of initial conditions as related to this problem can be approached by solving Eq. 1a for di/dt, giving

$$\frac{di}{dt} = \frac{E - Ri}{L}. \qquad [5]$$

Equation 5 shows that if L is not zero, di/dt is finite, unless E or Ri is infinite, a condition which does not occur in practice. In other words, the current in this circuit, because of the inductance, can change only an infinitesimal amount di during an infinitesimal time interval dt. *In fact, the current in any inductance cannot in general change instantaneously.* This statement can be justified also from energy considerations, since magnetic stored energy $\frac{1}{2}Li^2$ is associated with a current i in an inductance L. An instantaneous jump in current requires an infinite rate of change of stored energy, which is physically impossible. This is an important property of inductance that should be noted carefully, as it is a necessary prerequisite to understanding initial conditions in an inductive circuit.

As applied to the present situation the finite rate of change of current in an inductance has the following effect: If at the instant before the switch is moved from position 2 to position 1 in Fig. 1 the current in L is $i(0-)$ then, assuming that the switching occurs in a negligibly short time, the current in L and therefore in the entire circuit is still $i(0-)$ at the instant after the switch reaches position 1. But there is no reason for $i(0-)$ to be equal to I_s as given by Eq. 4 because $i(0-)$ can be given any reasonable value by a suitable experimental procedure in an actual circuit. Thus the paradox develops that the solution of Eq. 4 calls for one value of current even though it has been demonstrated that the current may have any value at the instant of switching.

The paradox can be resolved by showing that Eq. 4 is merely *a* solution but not the ***complete*** solution of Eq. 1a. This can be done as follows: If i_t is a solution of

$$L\frac{di}{dt} + Ri = 0, \qquad [6]$$

which is the differential equation of the circuit of Fig. 1 with E removed and replaced with a short circuit, then $I_s + i_t$ is a solution of Eq. 1a, as

can be demonstrated by substituting it in Eq. 1a.

$$i = I_s + i_t, \tag{7}$$

$$\left.\begin{aligned} L\frac{di}{dt} + Ri &= L\frac{d(I_s + i_t)}{dt} + R(I_s + i_t) \\ &= L\frac{dI_s}{dt} + L\frac{di_t}{dt} + RI_s + Ri_t \\ &= L\frac{dI_s}{dt} + RI_s + \left(L\frac{di_t}{dt} + Ri_t\right) = E. \end{aligned}\right\} \tag{8}$$

Since I_s is a constant, and by Eq. 6 the expression in parentheses is zero, Eq. 8 reduces to Eq. 4, and Eq. 7 is a solution of Eq. 1a.

Next to be found is the current i_t, which is a solution of Eq. 6. Putting i_t in Eq. 6 gives

$$L\frac{di_t}{dt} + Ri_t = 0, \tag{9}$$

which can be integrated by separating the variables. Thus

$$\frac{di_t}{i_t} = -\frac{R}{L}dt, \tag{10}$$

$$\ln i_t = -\frac{R}{L}t + k, \tag{11}$$

$$i_t = \epsilon^{-(R/L)t+k} = \epsilon^k\epsilon^{-(R/L)t} = A\epsilon^{-(R/L)t}, \tag{12}$$

which is written in functional notation for subsequent convenience as

$$i_t(t) = A\epsilon^{-(R/L)t}. \tag{12a}$$

In mathematical terminology, the equation in which the force-function term is replaced by zero – in this case Eq. 6 is known as the *reduced** equation. Its solution -- in this case Eq. 12 — is known mathematically as the *complementary function*, while physically it is known as the *force-free* or *transient* component of current.

The constant A in Eq. 12 is equivalent to the constant of integration k in Eq. 11 and can have any finite value whatsoever, as far as the differential equation 6 is concerned. It is the undetermined nature of this constant A that enables i as given by Eq. 7 to satisfy an arbitrary initial value of the current as well as the final steady-state value given by Eq. 4. This is an important idea which is illustrated in practically every complete circuit solution in this and subsequent chapters.

* The terms *auxiliary* and *homogeneous* also are in use.

Substitution of Eqs. 12 and 4 in Eq. 7 — using functional notation — gives

$$i(t) = I_s + i_t(t) = \frac{E}{R} + A\epsilon^{-(R/L)t}. \quad [13]$$

This relation has some interesting properties. By the properties of the exponential

$$i_t(0) = A, \quad [14]$$

$$i_t(\infty) = 0 \quad [15]$$

and, consequently,

$$i(0) = \frac{E}{R} + A, \quad [16]$$

$$i(\infty) = I_s + 0 = \frac{E}{R} \quad [17]$$

Thus the final or steady-state current $i(\infty)$ has the value given by Eq. 4, whereas the initial current $i(0)$ may have any finite value whatever. This is precisely the kind of $i(t)$ that is needed to satisfy both the original differential equation 4 and the arbitrary initial value of the current. Thus if t is taken as zero at the instant when K is switched from 2 to 1 in Fig. 1, and the current $i(0-)$ in the inductance immediately prior to switching is known, it can be said that

$$i(0) = i(0-), \quad [18]$$

since the inductance current cannot change instantaneously. Then

$$A = -\frac{E}{R} + i(0) = -\frac{E}{R} + i(0-), \quad [19]$$

and the final complete solution of Eq. 1a, including the specified initial conditions, is

$$i(t) = \frac{E}{R} + \left[i(0-) - \frac{E}{R}\right]\epsilon^{-(R/L)t}. \quad ▶[20]$$

If the initial current in the inductance is zero, Eq. 20 reduces to Eq. 2, which can be obtained by direct integration.

It is worth while to note that A is the initial value of i_t, and that

$$i(0) = I_s + i_t(0) \quad [21]$$

or

$$i_t(0) = i(0) - I_s. \quad [21a]$$

Thus at the initial instant the transient current $i_t(0)$ equals the discrepancy between the actual current $i(0)$ and the value I_s demanded by the steady state. Thus the transient current $i_t(t)$ serves as a shock absorber for whatever initial discrepancy may exist and gradually (exponentially) retires this discrepancy so as to leave only the forced current demanded by the intensity of the source. *This physical significance of the transient may be applied to all problems in circuit-transient behavior.* In other words, the initial value of the transient component must be such that when it is added to the initial value* of the steady-state component, the sum is the actual value that physical considerations require to exist at that instant.

It should be noted that in the circuit of Fig. 1 one physical initial condition can be arbitrarily specified, namely, the current. The solution of the reduced equation also provides one arbitrary constant of integration. That the number of initial conditions of the circuit that can be specified is equal to the number of arbitrary constants of integration in the solution of the reduced equation is not accidental. For any lumped circuit — or lumped-parameter physical system, for that matter — the general solution of the differential equation of the system must provide as many arbitrary constants of integration as there are independently specifiable initial conditions in the system. This is an important and essential correlation between the physical system and the mathematics describing it that is emphasized frequently henceforth.

In order that the main thread of the argument used in this solution may stand out clearly from the details of manipulation, the steps in the solution — which in principle are similar to those used in the solution of all ordinary linear differential equations with constant coefficients when the applied forces are constants — are recapitulated:

(a) The current that exists in the circuit after steady conditions have been established is determined.

(b) The reduced differential equation is solved. This solution, which gives the transient component of current, must include one arbitrary constant.

(c) The value of the constant of integration, that is, the initial value of the transient current, is determined from the known initial values of the actual and of the steady-state component of current.

The application of the method of this section to a numerical example is given in Art. 7.

* When the applied force is not a constant, the particular integral or steady-state term is not always a constant.

6. Time constant of series RL circuit

One characteristic of the transient component i_t of the current is *independent of all conditions of the problem except the circuit parameters.* This characteristic is the rate at which i_t disappears, or dies out, and is often of more significance than the magnitude of the transient. At first thought one might expect to measure this characteristic by the time for the transient to die out completely, but it is immediately seen that the exponential factor $\epsilon^{-(R/L)t}$, which causes the transient term to decrease with time, approaches zero only asymptotically. In other words, theoretically it takes an infinite time for the transient to disappear. Practically, i_t, or the difference between the steady component I_s and the actual current i, usually becomes too small to detect experimentally within a relatively short time.

A convenient measure of the duration of the transient is the *time constant* T given by

$$t = T \equiv \frac{L}{R}, \qquad [22]$$

at which time the transient term reduces to ϵ^{-1}, or 0.368 of its initial value. Thus

$$i_t(T) = i_t(0)\epsilon^{-T/T} = i_t(0)\epsilon^{-1} = 0.368i_t(0). \qquad [23]$$

In the interval between T and $2T$ the transient is also reduced by a factor ϵ^{-1}, so that

$$i_t(2T) = i_t(0)\epsilon^{-2} = i_t(0)0.368 \times 0.368 = 0.135i_t(0). \qquad [24]$$

The same reduction by a factor of ϵ^{-1} occurs in each succeeding time interval T. A table showing the fraction of the initial value persisting at any integral multiple of T is readily prepared as follows:

t	0	T	$2T$	$3T$	$4T$	$5T$
$\frac{i_t(t)}{i_t(0)}$	1	0 368	0 1353	0 0498	0 0183	0 0067
		$6T$	$7T$	$8T$	$9T$	$10T$
		0 0025	0 00091	0 00034	0 00012	0 000045

Alternatively, the number of time constants required to reduce the value of the transient by various negative integral powers of ten is as follows:

$\frac{i_t(t)}{i_t(0)}$	1.0	0 1000	0.0100	0.0010	0 0001
t	0	2 303T	4.605T	6.908T	9.210T

Thus the transient is approximately one per cent of its initial value at a time equal to 4.6 time constants and is 0.01 per cent at 9.2 time constants.

Another useful interpretation of the time constant is indicated in Fig. 2. This is the time that would be required for the transient to disappear if its rate of change at the initial instant continued. This can be

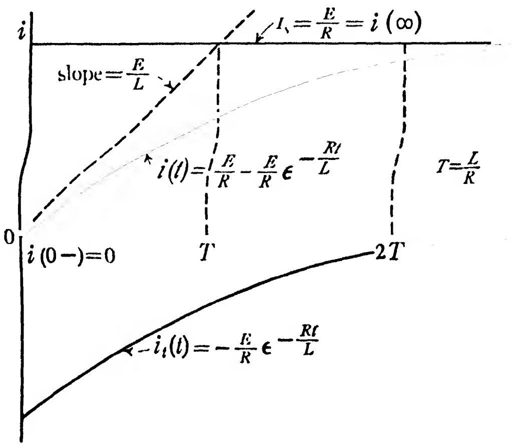


FIG. 2. Current build up in series RL circuit.

shown from Eq. 12, which is the general form of the transient term. The initial value of i_t is A, and the initial value of di_t/dt is

$$\left.\frac{di_t}{dt}\right|_{t=0} = -\frac{RA}{L}\epsilon^{-(R/L)0} = -\frac{RA}{L}, \qquad [25]$$

the negative sign merely indicating a decrease. Thus the time Δt required for i_t to disappear at this rate is

$$\Delta t = \frac{A}{\frac{RA}{L}} = \frac{L}{R} = T. \qquad [26]$$

It is seen easily that if in an RL circuit the inductance is made vanishingly small compared to R, the duration of any transient in it is vanishingly small, and the steady state is approached with exceeding rapidity.

Figure 3 shows the effect of different values of T on the shape of the curve of the transient component of current.

The order of magnitude of time constant obtainable practically in circuits consisting of inductance coils without any added series resistance is of interest. The air-core coil which has the highest ratio of L/R that can be obtained with a given weight of conductor is a circular coil with a square winding cross section. Its mean diameter is approximately 3.0

times one side of the winding section.[1] Such a coil is commonly called a Maxwell coil or, more accurately, a Maxwell-Shawcross-Wells coil. For this or any air-core inductance coil of given shape, weight, and conductor material, the L/R ratio is independent* of the size of conductor used, since both L and R increase by the same factor when the conductor size is reduced and the number of turns is increased. Furthermore, it can be shown readily that increasing *all linear dimensions of an air-core coil by a factor k increases* L/R by a factor k^2 and increases the weight by k^3; L/R therefore increases as (weight)$^{2/3}$. A Maxwell coil containing slightly over two pounds of copper has a time constant of 0.010 second. For T equal to 0.10 second about 70 pounds of copper are required.

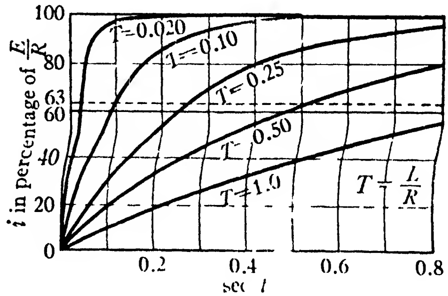


FIG. 3. Effect of time constant on current build-up, series RL circuit.

Even with an iron core the L/R ratio can be made as high as 10 seconds only by using many hundreds of pounds of material.

7. Illustrative example of series RL circuit

Figure 4 shows the essential features of the field circuit of a special direct-current generator called an *exciter*, which is used to furnish direct current to the field coils of a large alternating-current generator. Under

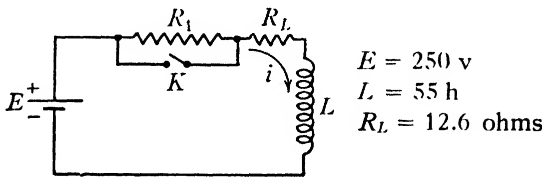


FIG. 4. Field circuit of exciter

certain conditions it is necessary to build up the exciter field current i very rapidly from a relatively small normal value. This rapid increase is accomplished by having normally in series with the field coils a relatively large resistance R_1, which is short-circuited by the contactor K in the event that a rapid increase of i is required. An actual circuit, of course,

[1] H. B. Brooks, "Design of Standards of Inductance and the Proposed Use of Models in the Design of Air-Core and Iron-Core Reactors," *J. Res. Nat. Bur. Stand.*, VII (Aug., 1931), 289–328, gives an excellent and comprehensive treatment of the subject. An abridged treatment is in the volume on magnetic circuits and transformers of this textbook series, ch. vii.

* Except for the influence of thickness of insulation.

contains other features, but these are irrelevant to the particular problem considered here, which involves these questions:

(a) If the circuit is operating under normal, steady conditions with K open and R_1 so adjusted that i is 5.40 amperes, what is the rate of increase of i immediately after K is closed?

(b) How many seconds after K is closed are required for the current to reach 14.00 amperes?

(c) If, 5.00 seconds after K is closed, the abnormal condition requiring a large coil current is removed, how many seconds after K is opened are required for i to return to within 10 per cent of its final steady value?

(d) What power in watts must R_1 be capable of dissipating continuously in the form of heat?

(e) How much energy is converted into heat in R_1 from the instant when K is initially closed until some time t when the transient of part (c) following the opening of K has substantially subsided? How does this energy compare with that so converted over the same period of time under normal operation, that is, with no switching?

(f) For what maximum potential difference must contactor K be designed to operate, assuming that under some conditions K may not be opened until i has reached its final value?

Solution: In this solution, instead of merely substituting numbers in formulas, the reasoning of Art. 3 is paralleled numerically to emphasize principles.

(a) Since after K is closed, the current remains momentarily unchanged in L at 5.40 amp, a potential drop is caused in R_L in the direction of i of 5.40×12.6, or 68.0 v, and $250 - 68.0$, or 182 v drop is left across the inductance. That is

$$L\frac{di}{dt} = 182 \text{ v}, \tag{27}$$

or

$$\frac{di}{dt} = \frac{182}{55} = 3.31 \text{ amp/sec}. \tag{28}$$

(b) For the condition with K closed the final, steady-state current is

$$I_s = \frac{250}{12.6} = 19.84 \text{ amp}. \tag{29}$$

The initial current is 5.40 amp; hence the transient component i_t must have a value immediately after K is closed such that

$$i(0) = I_s + i_t(0), \tag{30}$$

or

$$5.40 = 19.84 + i_t(0). \tag{30a}$$

Hence

$$i_t(0) = 5.40 - 19.84 = -14.44 \text{ amp}. \tag{31}$$

The form of the transient is

$$i_t = A\epsilon^{-(R/L)t}. \qquad [12]$$

From the foregoing results,

$$A = -14.44 \text{ amp}, \qquad [32]$$

$$\frac{R}{L} = \frac{12.6}{55} = 0.229 \text{ sec}^{-1}. \qquad [33]$$

Hence

$$i(t) = I_s + i_t(t) = 19.84 - 14.44\epsilon^{-0.229t}. \qquad [34]$$

To find the time t_1 such that $i(t_1)$ is 14.00 amp, one writes

$$i(t_1) = 14.00 = 19.84 - 14.44\epsilon^{-0.229t_1}, \qquad [34a]$$

$$\epsilon^{-0.229t_1} = \frac{19.84 - 14.00}{14.44} = 0.404, \qquad [35]$$

$$0.229t_1 = \ln\frac{1}{0.404} = \ln 2.48 = 0.906, \qquad [36]$$

$$t_1 = \frac{0.906}{0.229} = 3.96 \text{ sec}. \qquad [37]$$

(c) When time t_2 is 5.00 sec,

$$i(5.00) = 19.84 - 14.44\epsilon^{-0.229\times 5.00} = 19.84 - 14.44 \times 0.317 = 15.25 \text{ amp}. \qquad [38]$$

This is also the current immediately after K is opened. The new steady-state current, with K open, is 5.40 amp. Since the actual current is 15.25 amp and the steady-state value for the new condition (K open) is 5.40 amp, there must be an initial transient component

$$i_t(0') = 15.25 - 5.40 = 9.85 \text{ amp}, \qquad [39]$$

measuring the time t' from the instant K is opened. For this condition, the circuit resistance is

$$R' = \frac{250}{5.40} = 46.3 \text{ ohms} \qquad [40]$$

and

$$\frac{R'}{L} = \frac{46.3}{55.0} = 0.841 \text{ sec}^{-1}. \qquad [41]$$

Then

$$i(t') = 5.40 + 9.85\epsilon^{-0.841t'}. \qquad [42]$$

The time t_1' at which $i(t')$ is within 10 per cent of its final value of 5.40 amp is obtained from

$$1.1 \times 5.40 = 5.40 + 9.85\epsilon^{-0.841t_1'} \qquad [43]$$

$$\epsilon^{-0.841t_1'} = \frac{0.100 \times 5.40}{9.85} = 0.0547. \qquad [44]$$

Thus

$$0.841t_1' = \ln\frac{1}{0.0547} = 2.915 \qquad [45]$$

$$t_1' = 3.46 \text{ sec.} \qquad [46]$$

The current as determined in (a), (b), and (c) is plotted in Fig. 5.

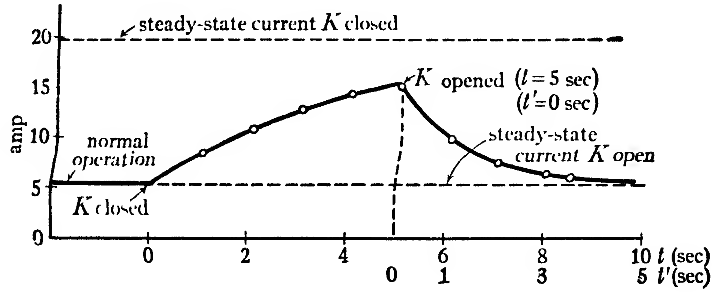


Fig. 5. Plot of current in field circuit of Fig. 4.

(d) The series resistance R_1 is

$$R_1 = 46.3 - 12.6 = 33.7 \text{ ohms.} \qquad [47]$$

Power dissipated in R_1 at 5.40 amp is

$$P = i^2R = 5.40^2 \times 33.7 = 984 \text{ w.} \qquad [48]$$

(e) If no switching takes place, the energy converted into heat in R_1 is $984t$ joule. As a result of the switching, the energy so converted, after K is initially closed, is

$$\left.\begin{aligned} W &= \int_{t=0}^{t=5.00} 0 \cdot R\,dt + \int_{t'=0}^{t'} i^2R\,dt' = 33.7\int_0^{t'} (5.40 + 9.85\epsilon^{-0.841t'})^2 dt' \\ &= 33.7\int_0^{t'} [5.40^2 + 2 \times 5.40 \times 9.85\epsilon^{-0.841t'} + 9.85^2\epsilon^{-1.682t'}]dt' \\ &= 33.7\left[5.40^2t' - \frac{2 \times 5.40 \times 9.85}{0.841}(\epsilon^{-0.841t'} - 1) - \frac{9.85^2}{1.682}(\epsilon^{-1.682t'} - 1)\right]. \end{aligned}\right\} \qquad [49]$$

The transient has substantially subsided when the exponential terms have become negligible. Thus when t' is large compared with the time constant T_1, which is $1/0.841$ sec,

$$\left.\begin{aligned} W &= 33.7\left[5.40^2t' + \frac{2 \times 5.40 \times 9.85}{0.841} + \frac{9.85^2}{1.682}\right] \\ &= 984t' + 4{,}250 + 1{,}940 \text{ joules} \end{aligned}\right\} t' \gg T_1. \qquad [49a]$$

If t' is expressed in terms of t, as $t - 5.00$, the expression for W—for t greater than 5.00 sec—becomes,

$$\left.\begin{aligned} W &= 984(t - 5.00) + 6{,}190 \\ &= 984t - 4{,}920 + 6{,}190 \\ &= 984t + 1{,}270 \text{ joules} \end{aligned}\right\} t > 5.00 \text{ and } t - 5.00 \gg T_1. \qquad [49b]$$

This energy exceeds that which would have been converted into heat during the same time (under normal operation) by 1,270 joules.

(f) The maximum potential difference occurring across R_1 is the iR_1 drop for the maximum possible i, since the inductance maintains i momentarily unchanged after K is opened. Thus

$$\text{Maximum voltage across } R_1 = 19.84 \times 33.7 = 668 \text{ v.} \tag{50}$$

8. Series RS circuit

The circuit considered in this section is shown in Fig. 6; it consists of a resistance R and elastance S in series that can be connected by switch K either to a constant-voltage source E (position 1) or to a short circuit (position 2). The problem is to find the current and condenser charge resulting from any sequence of operations of K that may be specified. This problem proves to be mathematically identical with that of the series RL circuit considered in the foregoing articles, but the corresponding dependent variables and coefficients represent quite different physical quantities. It is worth while, therefore, to reason through the solution in detail both because of the different physical concepts involved and for further practice in solving ordinary linear differential equations with constant coefficients.

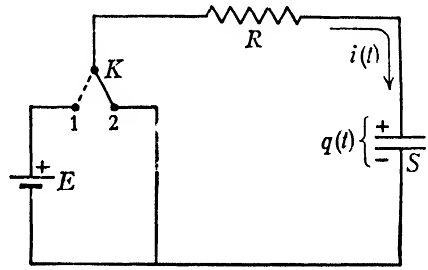


Fig. 6. Series RS circuit.

The differential equation for the circuit of Fig. 6 can be written from the current-voltage, or charge-voltage, equations of Art. 20, Ch. I, and Kirchhoff's voltage law. For switch K in position 1 the differential equation in terms of condenser charge is

$$R\frac{dq}{dt} + Sq = E, \tag{51}$$

while in terms of current the equation can be written

$$Ri + S\int idt = E, \tag{51a}$$

in which $\int idt$ is understood to mean $\int_0^t idt + q(0)$. It is important to note also that $q(t)$ represents the amount of positive charge on the top side of the condenser, and that a charge of equal magnitude but of opposite sign always resides on the other side. Hence when $i(t)$ is positive, $q(t)$ is increasing; that is, the top side is acquiring positive charge (or

losing negative charge) and vice versa. If K is in position 2, E becomes zero in Eqs. 51 and 51a. Equation 51a contains an integral and no derivative and is sometimes called an integral equation. In reality it contains the same information as Eq. 51 but is written in terms of i, or dq/dt, instead of in terms of q.

As pointed out in discussing the RL circuit, the solution of a differential equation for a particular system is a function that satisfies both the differential equation and the initial conditions. For the present problem, Eq. 51 is the differential equation. The initial conditions may be specified as follows: At an instant from which time is measured, switch K is moved *instantaneously from position 2 to position 1. At the instant before K is* moved, the elastance has a charge $q(0-)$ resulting from some previous operation of K. The details of this previous operation need not be known provided the charge $q(0-)$ resulting from it is known. The problem is to find the function $q(t)$ that satisfies Eq. 51 or Eq. 51a and these initial conditions.

Because of its similarity in form to Eq. 1a, Eq. 51, rather than Eq. 51a is treated first, by exactly the same procedure as with Eq. 1a. The initial step is to find the charge Q_s on the condenser after steady conditions have been established. Since the applied force E is a constant, a reasonable assumption for Q_s, to be verified by substitution in Eq. 51, is that it is a constant. The expression

$$Q_s = \frac{E}{S} = \text{a constant} \tag{52}$$

satisfies Eq. 51, and therefore is taken as the steady-state component of q.

As in the previous problem, Q_s and $q(0-)$ are generally not equal because Q_s is fixed by Eq. 52 and $q(0-)$ can have any specified value. Unless the condenser charge can change instantaneously, Q_s alone cannot constitute a solution of the problem. By solving Eq. 51 for

$$\frac{dq}{dt} = \frac{E - Sq}{R}, \tag{53}$$

it is seen that the actual condenser charge q can change only at a finite rate if R is not zero and the other quantities are finite — conditions that are true in any physical circuit. The conclusion is that *q cannot change instantaneously* and that therefore Q_s is not the complete solution but merely a part of it.

By reasoning as in Art. 6, it can be seen that if q_t satisfies the reduced equation

$$R\frac{dq}{dt} + Sq = 0, \tag{54}$$

$Q_s + q_t$ satisfies Eq. 51. It can be verified readily by separating the variables in Eq. 54 and integrating, or by substituting Eq. 55 in Eq. 54, that

$$q_t = A\epsilon^{-(S/R)t} \tag{55}$$

satisfies Eq. 54, A being any constant. Taking q as the sum of Q_s from Eq. 52 and q_t from Eq. 55 gives

$$q = \frac{E}{S} + A\epsilon^{-(S/R)t}, \tag{56}$$

in which A has not been determined. Since the charge q cannot change instantaneously,

$$q(0) = q(0-). \tag{57}$$

Therefore for the initial instant

$$q = q(0-) = \frac{E}{S} + A \tag{58}$$

or

$$A = q(0-) - \frac{E}{S}. \tag{58a}$$

Substituting Eq. 58a in Eq. 56 gives

$$q(t) = Q_s + q_t = \frac{E}{S} + \left[q(0-) - \frac{E}{S}\right]\epsilon^{-(S/R)t}. \tag{▶59}$$

That Eq. 59 is the desired solution can be verified readily by showing that it satisfies Eq. 51 and that $q(0)$ satisfies the initial conditions. This verification is suggested as an exercise for the student.

Equation 59 has exactly the same mathematical form as Eq. 20 with q corresponding to i, S to R_L, and R_S to L, R_L being the R of Eq. 20 and R_S the R of Eq. 59. This being the case, the curves of Figs. 2 and 3 when relabeled in terms of this problem are applicable without change. The student may find it instructive to rephrase the discussion of the curves in these figures in terms of the RS circuit.

It is well to note that in this example the transient component of charge again serves as a shock absorber gradually to retire the initial discrepancy between the charge actually present on the condenser and that called for by the steady state. Transients in lumped-parameter linear circuits in which constant forces are instantaneously applied, removed, or changed in magnitude result from the fact that in nearly all cases the charge on a capacitance and the current in an inductance cannot be changed instantaneously. Any solution of such a circuit must there-

fore provide for a gradual change of each capacitance charge and each inductance current from initial values to final steady values. The final value of each is its steady-state component alone. The initial value of each is the sum of its steady-state component and the initial value of its transient component. This is a fundamental conception of transient behavior.

9. Time constant of series *RS* circuit

From the parallelism between the series *RL* and *RS* circuits pointed out in the preceding article it is evident that the mathematical discussion *relating to the time constant T_L, or L/R, applies throughout to the time* constant

$$T_S = \frac{R}{S} \qquad [60]$$

for the *RS* circuit and need not be repeated here. However, a few comments on the physical aspects of T_S are appropriate.

In general, the series resistance associated with a condenser is so small as to be negligible in all but very refined work, so that the marked time lag indicated by curves similar to Figs. 2 and 3 is obtainable ordinarily with a series combination of a condenser and separate resistor. This condition is in marked contrast to that of the inductance coil, whose associated series resistance is seldom negligible. In a high-quality condenser the time constant T_S, as obtained by short-circuiting it when charged, is often under 10^{-6} second. The actual time limit for discharge is as likely to be set by inductance effects as by series resistance.

A condenser also has another and quite unrelated time constant determined by its capacitance and its leakage, or shunt, resistance. This time constant is a measure of the time of self-discharge of a condenser left on open circuit. In Art. 15, Ch. I, it is stated that a leakage resistance of 1,000 ohms per daraf is not uncommon. The corresponding time constant of self-discharge is 1,000 seconds, or roughly, 15 minutes. Under exceptional conditions values measured in days are obtainable. This leakage resistance, however, is ordinarily not a constant but is a function very complex in behavior.*

From the considerations of the two preceding paragraphs it can be seen that the series resistance of a high-quality condenser is likely to be negligible unless the condenser is used in a circuit having a time constant less than 10^{-6} second. Similarly its shunt leakance is likely to be negligible unless the condenser is used in a circuit whose time constant is greater than one second. These limits are rough but indicate orders of magnitude

* Dielectrics are discussed in the reference volume, Ch. I.

and show that resistance effects associated with condensers are much less important than those associated with inductance coils.

10. Solution of series *RS* circuit in terms of current

If the current instead of the charge is desired for the series *RS* circuit, it can be obtained from Eq. 59 by differentiation. Thus

$$i = \frac{dq}{dt} = \frac{E - Sq(0-)}{R} \epsilon^{-(S/R)t}. \qquad [61]$$

Figures 7 and 8 are analogous to Figs. 2 and 3, respectively, for current, if $q(0)$ is zero.

It is worth while, however, to solve the differential equation 51a

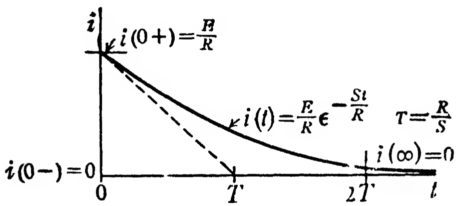


Fig. 7. Current decay in series *RS* circuit.

including initial conditions directly for the current, since the process introduces certain ideas that are used subsequently. The basic procedure remains the same as before.

By considering first the steady-state solution, it can be seen that zero current is a solution of Eq. 51a, for, once $\int i dt$ has reached the value E/S, the condenser is charged to the full voltage of the battery and no potential difference remains across the resistance to cause current in it.

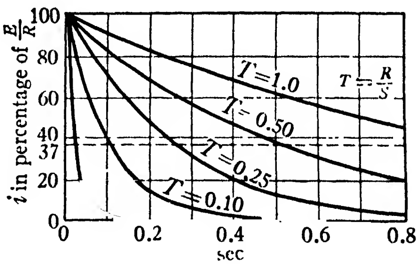


Fig. 8. Effect of time constant on current decay in series *RS* circuit.

At the instant of switching, however, the condenser voltage is $Sq(0-)$, which in general is not the same as E, because $q(0-)$ can be specified arbitrarily. Since q cannot change instantaneously, this same condenser voltage is present immediately after switching. The initial current can therefore be calculated from

$$i(0+) = \frac{E - Sq(0-)}{R}. \qquad [62]$$

For the true RS circuit there is nothing to prevent an instantaneous change of current from any previous value to $i(0+)$. Physically it is, of course, impossible to have a circuit without some inductance, though in the usual RS circuit the inductance is likely to be extremely small. The effect of this small inductance is to cause a finite, but very small, time to elapse before the current completes its change from whatever value $i(0-)$ it may have had prior to switching, to $i(0+)$. Usually this time is so small compared to the time constant T_S that when the RS transient is plotted this initial current change appears on the plot as indistinguishable from a vertical line. The current $i(0+)$, therefore, is considered to be that given by Eq. 62 regardless of the value of $i(0-)$.

Since I_s is zero, evidently $i(0+)$ must consist entirely of transient component i_t, which must satisfy the reduced equation

$$Ri + S\int idt = 0. \tag{63}$$

To prevent confusion of its meaning by constants of integration, Eq. 63 is differentiated once to remove the sign of integration. Thus

$$R\frac{di}{dt} + Si = 0 \tag{64}$$

is the reduced form of the differential equation for the circuit. A new procedure useful in more complicated cases is employed to solve Eq. 64, instead of the direct method of solution. As a trial, it is assumed that an exponential function satisfies Eq. 64. The previous solution of the same problem gives good reason for the assumption. As a matter of fact, it is found subsequently that an exponential function satisfies the reduced form of the general, ordinary linear differential equation with constant coefficients, Eq. 3. As a trial, then, subject to verification, it is assumed that

$$i_t = B\epsilon^{pt}, \tag{65}$$

in which B and p are unknown constants. Substituting Eq. 65 and its indicated derivative into Eq. 64 gives

$$(pR + S)B\epsilon^{pt} = 0. \tag{66}$$

This is called the *conditional equation*, since it expresses the conditions which must be satisfied if the exponential solution is to succeed. Since ϵ^{pt} cannot be zero at all times, Eq. 66 can be satisfied only if

$$B = 0 \tag{67}$$

or

$$p = -\frac{S}{R}. \tag{68}$$

It is emphasized that Eq. 67 is a correct solution of Eq. 66; it means that the circuit is at rest and in the absence of any driving force continues at rest. This solution is ordinarily of no particular interest and so in mathematical terminology is called *trivial.* The other possible solution of Eq. 66, namely, Eq. 68, leaves B undetermined. Thus Eq. 65 is a solution of Eq. 64 provided p is equal to $-S/R$, and B may have any value that may be necessary to satisfy the initial conditions. When this value of p is put in, Eq. 65 becomes

$$i_t = B\epsilon^{-(S/R)t}. \qquad [69]$$

But when t is zero (just after switching) $i(0+)$ is as given by Eq. 62. Hence evaluating B from Eqs. 69 and 62 gives

$$i = i_t = \frac{E - Sq(0-)}{R} \epsilon^{-(S/R)t}, \qquad ▶[61]$$

which is the complete solution by this method. It is identical with the solution obtained by the first method, as of course it should be.

As a corollary to this method of solution, it is pointed out that if the algebraic quantity p is written in place of the differential operator d/dt in the reduced equation 64 of the conditional equation

$$(Rp + S)i = 0 \qquad [66a]$$

is obtained. Hence if the *characteristic equation*

$$Rp + S = 0 \qquad [66b]$$

is written immediately, the value of p can be determined immediately without the intermediate steps of testing the validity of the exponential solution once it is known that such a function satisfies a particular type of equation. This same short cut or trick can be applied to the RL case and, as appears later, to the LS case, to the RLS case, and, in fact, to all ordinary linear differential equations having constant coefficients, of any order whatever. It is strongly emphasized, however, that in all cases p is an algebraic quantity, that it is not identical with the differential operator d/dt, and that the validity of putting p in place of d/dt in order to solve for p by writing the characteristic equation immediately by inspection of the reduced equation rests solely upon the knowledge that an experimental procedure such as that carried out in this article has previously succeeded.

Equation 66b is called the characteristic equation because it determines the character of the transient solution. *The form of the transient depends only upon the circuit parameters* (and the circuit connections) *and in no way upon the driving force applied to the circuit.* This statement is true of any linear circuit, however complex.

11. Illustrative example of series RS circuit

In Fig. 9a is shown a circuit used to obtain a small known time delay, which operates as follows: With switch K open, condenser C is discharged to zero voltage and point a has a potential of -55 volts with respect to point c. When K is closed, the condenser becomes charged and its voltage builds up, making the potential of a positive with respect to b. When a reaches a potential of $+50$ volts with respect to b, its potential with

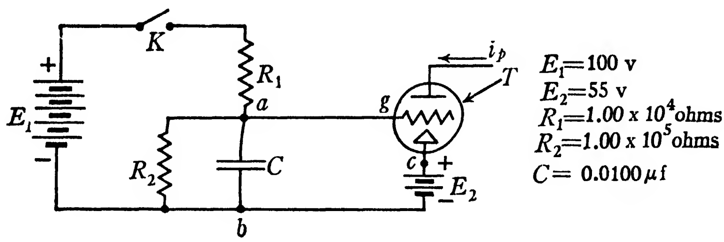


Fig. 9a. Circuit for obtaining known time delay.

respect to c becomes -5 volts and the gas-filled thermionic tube T suddenly becomes conducting, producing a finite current i_p. The interval between the closing of K and the sudden appearance of i_p is desired. It can be assumed safely that the current in lead ag is negligible and that i_p appears at the instant that the potential difference V_{ab} between a and b reaches 50 volts. The current i_p has no effect on the circuit to be considered.

Solution: The steady-state solution for the charge q on the condenser is readily obtained by inspection, by noting that R_1 and R_2 are in series and that C has no steady-state component of current. Thus

$$V_{ab} = E_1 \frac{R_2}{R_1 + R_2} = 100 \frac{1.00 \times 10^5}{1.10 \times 10^5} = 90.9 \text{ v} \tag{70}$$

and the steady-state condenser charge Q_s is

$$Q_s = CV_{ab} = 0.0100 \times 10^{-6} \times 90.9 = 0.909 \times 10^{-6} \text{ coulomb.} \tag{71}$$

Since the actual charge q on the condenser is zero before K is closed, and cannot change suddenly, $q(0+)$ is 0, counting time from the closing of K. Thus the initial value $q_t(0+)$ of the transient component of charge must be $-Q_s$, or -0.909×10^{-6} coulomb.

Next the form of the transient component of charge is considered. This must satisfy the reduced or force-free differential equation of the circuit. Such an equation can be written in either of two ways. In the first, the impressed force — the voltage E_1 — is included in the equation (or in this case two simultaneous equations), and E_2 is then replaced by zero. In the second and, in this case, easier way, the circuit dia-

gram is redrawn, omitting the driving force, and the equation for the force-free circuit is written. Figure 9b shows this force-free circuit in which E_1 is zero; that is, the battery is replaced by a short circuit. This immediately makes clear that as far as determining q_t is concerned, R_1 and R_2 are in parallel and can be replaced by a single

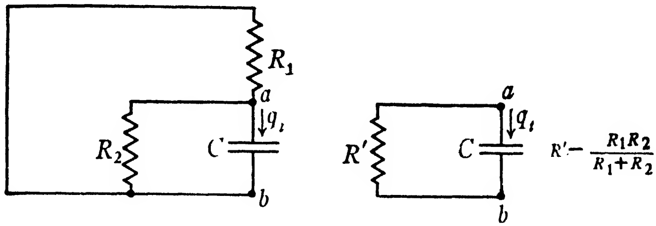


FIG. 9b. Simplified force-free circuits from Fig. 9a

resistance R', or $R_1R_2\ (R_1 + R_2)$, which is the resistance of the parallel combination of R_1 and R_2. For this simple circuit the differential equation is

$$R'\frac{dq_t}{dt} + Sq_t = 0, \qquad [72]$$

whose solution is

$$q_t = A\epsilon^{-(S/R')t}, \qquad [73]$$

A being an arbitrary constant. When t is 0, $q_t(0)$ is -0.909×10^{-6} coulomb and equals A. Also

$$\frac{S}{R'} = \frac{S(R_1 + R_2)}{R_1R_2} = \frac{10^6 \times 1.10 \times 10^5}{0.0100 \times 1.00 \times 10^9} = 1.10 \times 10^4\ \text{sec}^{-1}. \qquad [74]$$

Hence

$$q_t(t) = -0.909 \times 10^{-6}\epsilon^{-1.10\times 10^4 t} \qquad [75]$$

and

$$q(t) = Q_s + q_t(t) = 0.909 \times 10^{-6}(1 - \epsilon^{-1.10\times 10^4 t}). \qquad [76]$$

Therefore

$$V_{ab}(t) = Sq(t) = \frac{0.909 \times 10^{-6}}{0.0100 \times 10^{-6}}(1 - \epsilon^{-1.10\times 10^4 t}) = 90.9(1 - \epsilon^{-1.10\times 10^4 t})\ \text{v}. \qquad [77]$$

The time t_1 at which $V_{ab}(t_1)$ is 50.0 v is found from

$$50 = 90.9(1 - \epsilon^{-1.10\times 10^4 t_1}). \qquad [78]$$

Whence

$$\epsilon^{-1.10\times 10^4 t_1} = 1 - \frac{50}{90.9} = 0.450, \qquad [79]$$

$$1.10 \times 10^4 t_1 = 0.798, \qquad [80]$$

$$t_1 = \frac{0.798}{1.10} \times 10^{-4} = 7.25 \times 10^{-5}\ \text{sec}. \qquad [81]$$

12. Series LS circuit

The RL and RS circuits for which solutions have been developed in the preceding articles both occur widely in practice. They are important for the present purposes both because of this fact and because they afford a good starting point for transient-circuit analysis. The circuit next to be considered does not have this practical justification, for it cannot be realized physically. It is the series inductance-elastance circuit without resistance, shown in Fig. 10, which is not realizable actually because self-inductance is unavoidably accompanied by an appreciable and usually a significant amount of resistance. The justification for considering this ideal problem, apart from its theoretical interest as a limiting case, lies in its usefulness as a medium for the development of certain very important *ideas relating to complex numbers and functions which prove so valuable in circuit theory. In the LS circuit these ideas appear with a minimum of* complicating detail.

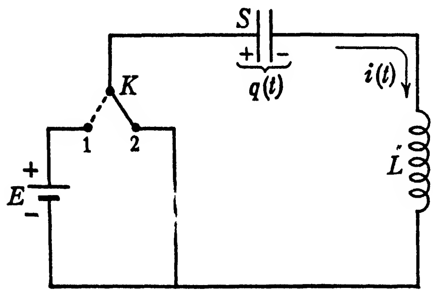


Fig 10. Series LS circuit.

The result of suddenly moving switch K from position 2 to position 1 in the circuit of Fig. 10 is now considered, assuming that this is accomplished without interrupting the current in the main circuit, and that, immediately prior to the switching, the charge $q(0-)$ on the condenser and the current $i(0-)$ in the inductance can have arbitrary values resulting from the previous history of the circuit. These assumptions constitute the initial conditions.

By application of the Kirchhoff voltage law, the differential equation of the circuit with K in position 1 is readily written, in terms of the charge q on the condenser, as

$$L\frac{d^2q}{dt^2} + Sq = E. \qquad [82]$$

Charge rather than current is selected for the dependent variable as a matter of convenience.

First is sought the steady-state component of charge, which, as is seen presently, has little physical meaning unless there is a slight amount of resistance present. However,

$$Q_s = \frac{E}{S} \qquad [83]$$

and

$$I_s = \frac{dQ_s}{dt} = 0 \qquad [84]$$

are solutions of Eq. 82, and the circuit is in equilibrium with zero current with this charge on the condenser.

Turning next to the transient solution and its relation to the initial conditions, inspection of Eq. 82 shows that d^2q/dt^2 or di/dt, must be finite, and that the current cannot change instantaneously. Neither can q, which is the integral of i, change instantaneously. Therefore

$$q(0+) = q(0-) \qquad [85]$$

and

$$i(0+) = i(0-), \qquad [86]$$

and the transient solution must contain two undetermined constants, one to allow the initial current $i(0+)$ to be different from the steady current I_s, the other to allow the initial charge $q(0+)$ to be different from the steady charge Q_s.

The reduced or force-free equation to which the transient charge q_t must be a solution is

$$L\frac{d^2q}{dt^2} + Sq = 0. \qquad [87]$$

As stated before, the exponential function is always found to satisfy such a force-free equation. The reason is rather evident if one notes that any derivative of an exponential function contains the same exponential as a factor and that when an exponential function for q is substituted into such an equation as Eq. 87 it reduces to an algebraic polynomial times the exponential. Carrying out this process with the assumption

$$q_t = \mathrm{A}\epsilon^{\mathrm{p}t}, \qquad [88]$$

in which A and p are undetermined constants, gives the conditional equation

$$L\mathrm{p}^2\mathrm{A}\epsilon^{\mathrm{p}t} + S\mathrm{A}\epsilon^{\mathrm{p}t} = 0 = (L\mathrm{p}^2 + S)\mathrm{A}\epsilon^{\mathrm{p}t}. \qquad [89]$$

A nontrivial solution of this equation requires that

$$L\mathrm{p}^2 + S = 0, \qquad [89a]$$

of which the roots are

$$\mathrm{p}^2 = -\frac{S}{L} \qquad [90]$$

or

$$\mathrm{p} = \pm j\sqrt{\frac{S}{L}}, \tag{91}$$

in which

$$j \equiv \sqrt{-1}. \tag{92}$$

Equation 89a is the characteristic equation of this circuit which as a consequence of proof just given can be written immediately by inspection of the reduced equation 87.

In mathematics and physics the symbol i is used instead of j to represent the square root of minus one, but electrical engineers use j to avoid possible confusion with the symbol i used for current. Thus far a number which is the square root of a negative number has not been encountered and before it is used its properties should be examined to determine whether or not the usual processes of algebra are applicable to it. Yet in order not to interrupt the continuity of the argument it is assumed tentatively that j can be manipulated like any other algebraic quantity and the solution of the problem is carried on.

From Eq. 91 it is seen that two values of p,

$$\mathrm{p}_1 = +j\omega_0 \tag{93a}$$

and

$$\mathrm{p}_2 = -j\omega_0, \tag{93b}$$

using the convenient abbreviation

$$\omega_0 \equiv \sqrt{\frac{S}{L}}, \tag{94}$$

satisfy Eq. 89a. From this it may be inferred that both

$$q_1 = \mathrm{A}_1\epsilon^{\mathrm{p}_1 t} \tag{88a}$$

and

$$q_2 = \mathrm{A}_2\epsilon^{\mathrm{p}_2 t} \tag{88b}$$

are solutions of Eq. 87, A_1 and A_2 each being an arbitrary constant. Furthermore, if either q_1 or q_2 substituted in the left-hand side of Eq. 87 reduces it to zero, their sum $q_1 + q_2$ does likewise. The transient component of charge q_t is therefore tentatively taken as

$$q_t = q_1 + q_2 = \mathrm{A}_1\epsilon^{\mathrm{p}_1 t} + \mathrm{A}_2\epsilon^{\mathrm{p}_2 t}, \tag{95}$$

which — and this is very significant — provides two independent arbitrary constants A_1 and A_2 that will allow both of the two arbitrary initial

conditions to be satisfied. It is always found that the mathematical solution if correctly carried out provides exactly the same number of undetermined constants as the number of initial conditions that can be arbitrarily specified in the circuit.

Combining Eqs. 83 and 95, using Eqs. 93a and 93b in the latter, gives the general solution

$$q = Q_s + q_t = \frac{E}{S} + A_1\epsilon^{j\omega_0 t} + A_2\epsilon^{-j\omega_0 t}. \tag{96}$$

Differentiating gives

$$i = 0 + j\omega_0 A_1\epsilon^{j\omega_0 t} - j\omega_0 A_2\epsilon^{-j\omega_0 t}, \tag{97}$$

which must be made to satisfy the initial conditions, Eqs. 85 and 86. Thus, putting Eq. 85 in Eq. 96, and Eq. 86 in Eq. 97, gives

$$q(0-) = \frac{E}{S} + A_1 + A_2 \tag{98}$$

and

$$i(0-) = 0 + j\omega_0 A_1 - j\omega_0 A_2. \tag{99}$$

Solving this pair of equations gives

$$A_1 = \frac{1}{2}\left[q(0-) - \frac{E}{S} + \frac{i(0-)}{j\omega_0}\right] \tag{100}$$

and

$$A_2 = \frac{1}{2}\left[q(0-) - \frac{E}{S} - \frac{i(0-)}{j\omega_0}\right]. \tag{101}$$

Putting these values in Eqs. 96 and 97 gives the final solution, with all constants evaluated, as

$$\left.\begin{aligned} q = \frac{E}{S} + \left[q(0-) - \frac{E}{S}\right]\frac{\epsilon^{j\omega_0 t} + \epsilon^{-j\omega_0 t}}{2} \\ + \frac{i(0-)}{\omega_0}\frac{\epsilon^{j\omega_0 t} - \epsilon^{-j\omega_0 t}}{j2} \end{aligned}\right\}, \tag{▶102}$$

$$\left.\begin{aligned} i = 0 + i(0-)\left(\frac{\epsilon^{j\omega_0 t} + \epsilon^{-j\omega_0 t}}{2}\right) \\ - \omega_0\left[q(0-) - \frac{E}{S}\right]\left(\frac{\epsilon^{j\omega_0 t} - \epsilon^{-j\omega_0 t}}{j2}\right) \end{aligned}\right\}, \tag{▶103}$$

or, for the special case of initial rest conditions, that is, $q(0-)$ and $i(0-)$ each zero,

$$q = \frac{E}{S} - \frac{E}{S}\left(\frac{\epsilon^{j\omega_0 t} + \epsilon^{-j\omega_0 t}}{2}\right), \qquad [104]$$

$$i = \frac{E}{\sqrt{SL}}\left(\frac{\epsilon^{j\omega_0 t} - \epsilon^{-j\omega_0 t}}{j2}\right). \qquad [105]$$

These equations were obtained from the tentative form, Eq. 95. That they satisfy the differential equation can be verified by substituting them in it. That they also satisfy the initial conditions can be seen by letting t become zero. Therefore Eqs. 102 and 103 constitute a solution of the problem, and the assumed form Eq. 95 is justified.

To give these expressions meaning one must examine the nature, properties, and interpretation of the quantity j which appears for the first time in this problem but which is used extensively in electric-circuit theory from this point on.

13. Complex numbers[2]

In solving the characteristic equation 89a it is necessary to remove the factor (-1) from $-S/L$ before taking its square root; and then merely to indicate $\sqrt{-1}$. Thereafter $\sqrt{-1}$, indicated by j, is used exactly as any other algebraic factor such as a or x. It is now necessary to determine whether or not this procedure is justified and, if so, how a result expressed in terms of j is to be interpreted. Some of the steps by which the conceptions of numbers arose are briefly reviewed in order to clarify the significance of j.

The simplest, and undoubtedly the first, historical concept of number was that of a positive integer developed by the process of counting. To such numbers the processes of addition and multiplication could be applied readily, as could also division and subtraction under certain conditions. Division could be performed provided the result was also an integer. To permit the idea of division to be applied in all cases, the idea of fractional numbers had to be invented. In a somewhat similar way, subtraction could be carried out always only if negative numbers were per-

[2] A very readable reference on this subject, which might be entitled "The story of $\sqrt{-1}$," is in W. F. Osgood, *Advanced Calculus* (New York: The Macmillan Company, 1929) Ch. xx. Accepted engineering terminology uses the term *vector* in the same sense as complex number which exists only in a plane and hence cannot be manipulated according to the rules for space vectors. Plane vectors, however, have the same rule for addition as complex numbers and can be conveniently located in the plane by the use of the exponential or polar form. In some cases the rule for the dot product has useful meaning when applied to complex numbers. In other cases the application of the rules of vectors to complex numbers, or vice versa, is either meaningless or impossible.

mitted as a result. To this growing concept of number the early mathematicians added the notion of irrational numbers such as $\sqrt{2}$, π, and ϵ that cannot be expressed as the product or quotient of the foregoing, or rational, numbers. Every addition to the concept of number required testing by the rules of arithmetic to see that it could be treated by the methods previously used. A next step occurred when mathematicians wished to solve an equation such as

$$x^2 + 4 = 0 \qquad [106]$$

(the problem that occasions this discussion), the result of which again required an extension of the concept of number, the result being

$$x = \pm 2\sqrt{-1}, \qquad [107]$$

or in engineering notation

$$x = \pm j2. \qquad [107a]$$

Along with the concept of number had grown the geometrical interpretation of number as a position on a line. Thus all positive numbers—integer, fraction, and irrational—can be plotted along a line such as *ba*

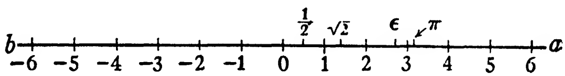


FIG. 11. Scale of real numbers.

of Fig. 11. Negative numbers are easily taken care of by extending this same line to the left of zero. But, asked the perturbed mathematicians first trying to use j, where on this scale of numbers can j be placed? Evidently there was no place for it, and they therefore concluded that it must be not a real number but an imaginary number, using " real " and " imaginary " in their usual nontechnical senses.

Thus arose the term *imaginary number*. Since that time imaginary numbers have become so well understood that anyone dealing with even elementary mathematics finds them no more difficult to use than real numbers. In fact the term " imaginary," while still applied to numbers containing $\sqrt{-1}$, has long since lost its meaning of unreality in this connection and has become a purely technical term parallel in usage to " negative " or " irrational." The theory of numbers has of course passed far beyond this point; for the purpose in hand, however, more detailed consideration of the properties and uses of only imaginary numbers and combinations of real and imaginary numbers is necessary.

A real advance in the theory of imaginary numbers occurred when Argand suggested in 1806 that geometrically numbers need not be con-

fined to a line but might be allowed to occupy a plane, the real numbers being plotted along the x axis and imaginary numbers along the y axis as shown in Fig. 12.

This suggests the following idea: The negative number scale can be obtained by rotating the scale for positive numbers about zero through 180 degrees from $0x$ to $0x'$. This rotation is taken as positive in a counter-clockwise direction, in accordance with established convention. By this conception, multiplying a number by -1 is equivalent to rotating it $+180$ degrees. The next idea is that multiplying a number by $\sqrt{-1}$, or j, may be represented by rotation through half of 180 degrees or 90 degrees. Thus $j4$ becomes the line segment 04 rotated through 90 degrees. Also

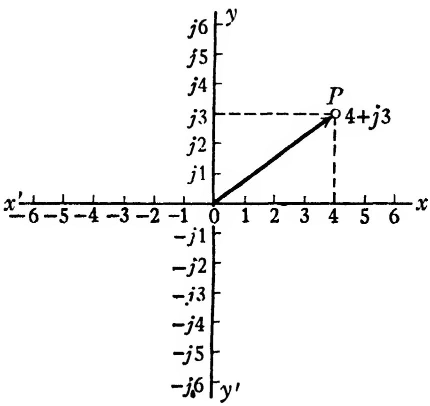


FIG. 12. The complex plane.

$$j(j4) = j^2 4 = (-1)4 = -4 \qquad [108]$$

is 04 rotated through 180 degrees. Similarly

$$j^3 4 = j^2(j4) = (-1)j4 = -j4 \qquad [109]$$

is 04 rotated through 270 degrees, and

$$j^4 4 = (j^2)(j^2)4 = (-1)(-1)4 = 4 \qquad [110]$$

is 04 rotated through 360 degrees, or back to its original position.

Thus far numbers have been placed only on the four lines $0x$, $0y$, $0x'$, and $0y'$. The next question is: Why not utilize the entire plane and have numbers such as $4 + j3$, which is interpreted as four units of real number combined with three units of j? Such a number may be associated with point P in the plane and also with the line or *plane vector* $0P$. By this procedure a new kind of number called a *complex* number is invented, so that three principal kinds of numbers — real, imaginary, and complex — are now devised. The plane in which these three kinds are plotted is called the *complex plane*.

The ideas relating to imaginary and complex numbers and their geometric interpretation have been presented only as conceptions, without justification. Complex numbers are justified if they prove useful. They are useful if they can be manipulated mathematically, especially if the

rules of manipulation are the same as those for real numbers. These conditions are met if the rules are suitably interpreted to fit the circumstances.

Dealing first with addition and with the inverse process, subtraction, two complex numbers $a + jb$ and $c + jd$ are considered, a, b, c, and d being real. They are added by adding like components; that is, their sum is defined by

$$(a + jb) + (c + jd) \equiv (a + c) + j(b + d) \qquad \blacktriangleright[111]$$

and is a complex number with a real part $(a + c)$ and an imaginary part $j(b + d)$. Numerically, for example,

$$(4 + j3) + (-2 + j1) = (4 - 2) + j(3 + 1) = 2 + j4. \qquad [112]$$

When these two complex numbers and their sum are plotted in the complex plane, they add geometrically according to the force, or parallelogram, law, their sum being representable by the point R or the plane vector $0R$. Subtraction is carried out in the obvious manner as

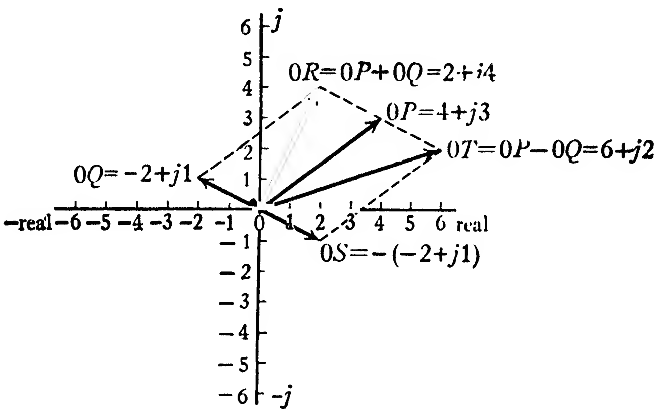


FIG. 13. Addition and subtraction of complex numbers.

$$(a + jb) - (c + jd) = (a - c) + j(b - d), \qquad \blacktriangleright[113]$$

or, numerically,

$$(4 + j3) - (-2 + j1) = (4 + 2) + j(3 - 1) = 6 + j2. \qquad [114]$$

These operations are illustrated geometrically in Fig. 13.

Multiplication of complex numbers is done according to the usual laws of algebra as follows:

$$\left.\begin{aligned}(a + jb)(c + jd) &\equiv ac + jad + jbc + j^2bd \\ &= (ac - bd) + j(ad + bc)\end{aligned}\right\}, \qquad \blacktriangleright[115]$$

or, numerically,

$$(4 + j3)(-2 + j1) = -8 + j4 - j6 + j^2 3 = -11 - j2. \qquad [116]$$

This result is illustrated in Fig. 14.

Another rule for multiplication that is sometimes simpler numerically is given subsequently.

A special and important instance of multiplication is that in which the two factors are *conjugate* complex numbers. Two complex numbers are said to be conjugates if their real parts are equal and their imaginary

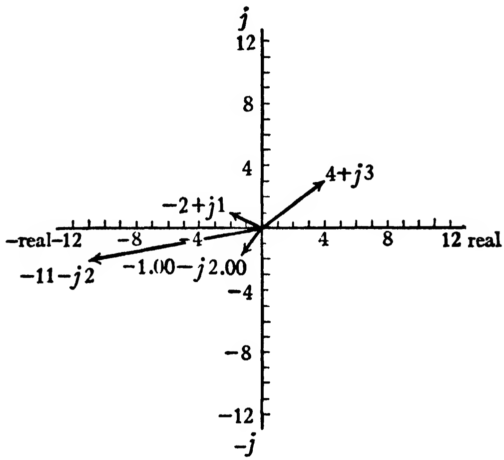


FIG. 14. Multiplication and division of complex numbers.

parts are equal in magnitude but opposite in sign. Thus $c + jd$ and $c - jd$ are conjugates. Their product,

$$(c + jd)(c - jd) = c^2 + d^2, \qquad [117]$$

is seen to be a real number.

Division is indicated by

$$\frac{a + jb}{c + jd}, \qquad [118]$$

but as the result stands it is not in the simple form of a complex number used thus far. It can be made so by the device of multiplying both numerator and denominator by the conjugate of $c + jd$, which is $c - jd$. Carrying out this operation on Eq. 118 gives

$$\left.\begin{aligned} \frac{a + jb}{c + jd} &= \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{ac - jad + jbc - j^2 bd}{c^2 - jcd + jdc - j^2 d^2} \\ &= \frac{(ac + bd) + j(-ad + bc)}{c^2 + d^2}. \end{aligned}\right\} \qquad ▶[119]$$

Numerically,

$$\left.\begin{aligned}\frac{4+j3}{-2+j1} &= \frac{(4+j3)(-2-j1)}{(-2+j1)(-2-j1)} = \frac{-8-j6-j4-j^2 3}{4+j2-j2-j^2 1}\\ &= \frac{-8+3+j(-4-6)}{4+1} = \frac{-5-j10}{5} = -1.00-j2.00.\end{aligned}\right\} \quad [120]$$

This result is also shown in Fig. 14. As with multiplication, another rule which is often simpler to apply numerically is given subsequently. It is suggested that the student verify the fact that the result of Eq. 119 when multiplied by $c + jd$ gives back $a + jb$.

In the foregoing discussion a complex number has been described in terms of its real and imaginary components or, geometrically, in terms of Cartesian co-ordinates. It can equally well be described, on the basis of its geometric representation, in polar co-ordinates. This description assumes that the geometric interpretation of a complex number proves to be useful and self-consistent. Thus the $a + jb$ plane vector has a magnitude $\sqrt{a^2 + b^2}$ and makes an angle $\tan^{-1} b/a$ with the abscissa axis, or axis of reals. This is often written

$$A = a + jb = (\sqrt{a^2 + b^2})\underline{/\theta} = A\underline{/\theta}, \qquad [121]$$

$$\theta = \tan^{-1}\frac{b}{a}, \qquad [122]$$

$$A \equiv \sqrt{a^2 + b^2}, \qquad [123]$$

the symbol $\underline{/\theta}$ meaning "at an angle θ." In Eq. 123, A is called the *magnitude* or *absolute value* of the complex number, while θ is called its *angle*. Both A and θ are real numbers. The equality signs in Eq. 121 do not represent an analytically demonstrated equality but merely a sameness which is based upon geometrical considerations and an understanding as to the meaning of A and $\underline{/\theta}$ in terms of a and jb. The following is a numerical example of this operation:

$$4 + j3 = \sqrt{4^2 + 3^2}\,\underline{/\tan^{-1}\tfrac{3}{4}} = 5\underline{/36.9^\circ}. \qquad [124]$$

The form $a + jb$ is called the ***rectangular*** form, and the form $A\underline{/\theta}$ is called the ***polar*** form of expressing a complex number. Reversing the process — that is, converting from polar to rectangular form — is done in the geometrically obvious way:

$$A\underline{/\theta} = A\,(\cos\theta + j\sin\theta) = a + jb. \qquad [125]$$

By means of the rectangular form in terms of trigonometric functions the useful polar form for multiplication referred to above can be derived.

For example, a second complex number also can be written in the same forms

$$c + jd = B\underline{/\phi} = B\ (\cos\phi + j\sin\phi). \qquad [126]$$

Then taking the product of the trigonometric forms of Eqs. 125 and 126 by the rule of Eq. 115 gives

$$\left.\begin{aligned}(a + jb)(c + jd) &= AB\,[(\cos\theta\cos\phi - \sin\theta\sin\phi)\\ &\quad + j\,(\sin\theta\cos\phi + \cos\theta\sin\phi)]\end{aligned}\right\} \qquad [127]$$

$$= AB\,[\cos(\theta + \phi) + j\sin(\theta + \phi)] \qquad [127a]$$

$$= AB\underline{/\theta + \phi}. \qquad [127b]$$

The last step follows from the fact that the angle in the polar form Eq. 121 is also the argument for the sine and cosine in Eq. 125. Equation 127b supplies the additional rule for multiplication of complex numbers: The magnitude of the product of two complex numbers is the product of their individual magnitudes, and the angle of the product is the sum of their individual angles. It can be written

$$(a + jb)(c + jd) = (A\underline{/\theta})(B\underline{/\phi}) = AB\underline{/\theta + \phi}. \qquad \blacktriangleright[128]$$

By considering the same numerical examples,

$$4 + j3 = 5\underline{/36.9^\circ}, \qquad [129]$$

$$-2 + j1 = \sqrt{(-2)^2 + 1^2}\ \underline{/\tan^{-1}\frac{1}{-2}} = \sqrt{5}\underline{/153.4^\circ}, \qquad [130]$$

$$(4 + j3)(-2 + j1) = 5\sqrt{5}\underline{/36.9^\circ + 153.4^\circ} = 5\sqrt{5}\underline{/190.3^\circ}. \qquad [131]$$

This can be checked with the result of the numerical illustration of Eq. 116. Thus

$$\left.\begin{aligned}-11 - j2 &= \sqrt{(-11)^2 + (-2)^2}\ \underline{/\tan^{-1}\frac{-2}{-11}}\\ &= \sqrt{125}\underline{/190.3^\circ} = 5\sqrt{5}\underline{/190.3^\circ}.\end{aligned}\right\} \qquad [132]$$

The use of the polar forms for division follows easily from the rule for multiplication and can be written

$$\frac{a + jb}{c + jd} = \frac{A\underline{/\theta}}{B\underline{/\phi}} = \frac{A}{B}\underline{/\theta - \phi}; \qquad \blacktriangleright[133]$$

that is, magnitudes are divided and angles are subtracted.

Yet to be indicated is one of the most useful and elegant forms in which to express a complex number, especially for the analytical work

associated with electric circuits. This is the exponential form which can readily be derived from Eq. 125. For convenience, only the part of Eq. 125 that expresses angle, $\cos\theta + j\sin\theta$, is considered, and written in terms of the power series, for the sine and cosine. It is known that

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots, \tag{134}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots, \tag{135}$$

$$j\sin\theta = j\theta - j\frac{\theta^3}{3!} + j\frac{\theta^5}{5!} - j\frac{\theta^7}{7!} + \cdots. \tag{136}$$

But by substituting $\sqrt{-1}$ for j, Eqs. 134 and 136 can be written

$$\cos\theta = 1 + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^6}{6!} + \cdots, \tag{134a}$$

$$j\sin\theta = j\theta + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^7}{7!} + \cdots, \tag{136a}$$

and their sum is

$$\cos\theta + j\sin\theta = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \cdots = \epsilon^{j\theta}. \tag{137}$$

The last step follows provided the assumption is made that the series definition of the exponential function holds for imaginary, as well as for real, values of the argument. That it does is substantiated by the fact that, when defined in this way, the exponential function with an imaginary argument (or complex as is seen presently) forms a part of a consistent mathematics of complex numbers.

When the geometrical interpretation of Eq. 137 is examined, it is readily seen by applying Eq 123 that $\cos\theta + j\sin\theta$ has a magnitude of unity, and that $\cos\theta + j\sin\theta$ is represented in the complex plane by a plane vector of unit length at a positive or counterclockwise angle of θ measured from the axis of reals. Furthermore and this is a very important idea — $\epsilon^{j\theta}$ is also represented by a unit-length plane vector at an angle $+\theta$ with the axis of reals. For example,

$$\epsilon^{j\pi} = -1, \tag{138}$$

$$\epsilon^{j(\pi/2)} = j, \tag{139}$$

$$\epsilon^{-j(\pi/2)} = -j, \tag{140}$$

$$\epsilon^{-j(3\pi/2)} = j, \tag{141}$$

$$\epsilon^{j2\pi} = 1. \tag{142}$$

The last expression is perhaps the most remarkable of all from one point of view: In it two irrational or unmeasurable numbers, ϵ and π, are combined with the imaginary number j to equal the simplest possible number, one. The nonmathematically minded can scarcely conceive $\epsilon^{j2\pi}$ as being more useful than its equivalent, one. Yet the exponential form may be much more useful.

From Eqs. 125 and 137 it can be seen that the complex number $a + jb$ can also be written

$$a + jb = A(\cos\theta + j\sin\theta) = A\epsilon^{j\theta}. \qquad [143]$$

The polar product form Eq. 127b fits in nicely with the exponential law of additive exponents. Thus if

$$a + jb = A\epsilon^{j\theta} \qquad [144]$$

and

$$c + jd = B\epsilon^{j\phi}, \qquad [145]$$

$$(a + jb)(c + jd) = AB\epsilon^{j\theta}\epsilon^{j\phi} = AB\epsilon^{j(\theta+\phi)}. \qquad ▶[146]$$

Much detail has been inevitable in developing the ideas and processes associated with complex numbers. The principal features may be summarized thus:

(a) A complex number may be written in the following forms:

$$\mathrm{A} = a + jb = A(\cos\theta + j\sin\theta) = A\epsilon^{j\theta} = A\underline{/\theta}, \qquad ▶[147]$$

in which

$$\theta = \tan^{-1}\frac{b}{a} \qquad [122]$$

and

$$A = \sqrt{a^2 + b^2}. \qquad [123]$$

(b) Two complex numbers $(a + jb)$ and $(c + jd)$ are added or subtracted:

(1) $$(a + jb) \pm (c + jd) = (a \pm c) + j(b \pm d), \qquad ▶[111, 113]$$

(2) graphically according to the force, or parallelogram, law.

(c) Two complex numbers $(a + jb)$, or $A\underline{/\theta}$, and $(c + jd)$ or $B\underline{/\phi}$, are multiplied:

(1) in rectangular form by

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc), \qquad ▶[115]$$

(2) in polar form by

$$(A\underline{/\theta})(B\underline{/\phi}) = AB\underline{/\theta + \phi}, \qquad ▶[128a]$$

(3) in exponential form by

$$A\epsilon^{j\theta}B\epsilon^{j\phi} = AB\epsilon^{j(\theta+\phi)}. \qquad ▶[146a]$$

(d) Two complex numbers $a + jb$, or $A\underline{/\theta}$, and $c + jd$, or $B\underline{/\phi}$, are divided:

(1) in rectangular form by

$$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{(c+jd)(c-jd)} = \frac{(ac+bd)+j(-ad+bc)}{c^2+d^2}, \qquad ▶[119a]$$

(2) in polar form by

$$\frac{A\underline{/\theta}}{B\underline{/\phi}} = \frac{A}{B}\underline{/\theta - \phi}, \qquad ▶[133a]$$

(3) in exponential form by

$$\frac{A\epsilon^{j\theta}}{B\epsilon^{j\phi}} = \frac{A}{B}\epsilon^{j(\theta-\phi)}. \qquad ▶[148]$$

(e) A complex number may be represented graphically:

(1) In plane Cartesian co-ordinates by a point P whose abscissa is the real part a and whose ordinate is the magnitude b of the imaginary part. For this purpose the abscissa axis is known as the axis of reals, the ordinate axis as the axis of imaginaries. The plane defined by these co-ordinates is known as the complex plane, that is, a plane on which to plot complex numbers.

(2) In polar co-ordinates in the complex plane by the same point P which is at the end of a radius vector of length A, drawn from the origin, and making an angle θ with the axis of reals. Positive values of θ are measured counterclockwise.

The radius vector $0P$ as well as the point P may be considered to represent the complex number.

14. Complex functions of a real variable and rotating plane vectors*

In the foregoing article the ideas relating to complex numbers are developed and their arithmetic is related to the arithmetic of real numbers. But in the realm of real mathematical quantities the idea of fixed numbers has long been too confining, so that men have learned to think in terms of variables and functions. To these are applied the laws of algebra, which are the same as the laws of arithmetic. So, too, in the realm of complex quantities it is necessary to extend the ideas developed for complex numbers to complex functions. Thus the conceptions and

* The latter part of footnote on p. 194 should be reread.

processes associated with the complex number $a + jb$ are extended without change to the complex function $a(x) + jb(x)$, x being a real variable, and $a(x)$ and $b(x)$ real functions. In cases to be encountered shortly, x may also be a complex variable. Evidently if a and b are functions of x, the corresponding polar variables A and θ are also functions of x, as given by Eqs. 121, 122, and 123.

When the ideas relating to complex numbers are extended to complex functions, the processes of manipulation remain unaltered, so that the only additional aspects that justify discussion are the properties of these functions. These are developed as they are needed. The functions that occasion this consideration of complex numbers and functions are the exponential ones of Eqs. 104 and 105. Since these are very important in circuit theory, it is worth while to consider them in some detail.

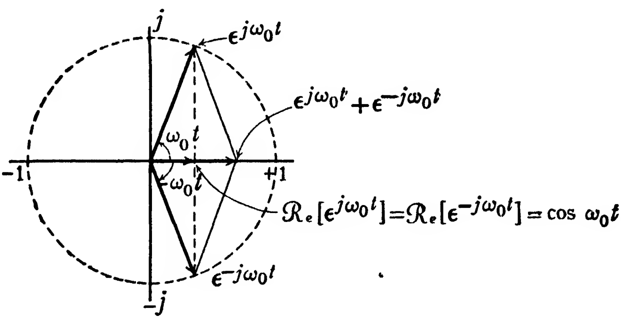


FIG. 15. Plot of vector components of cosine function.

From the preceding article it is evident that $\epsilon^{j\omega_0 t}$ can be plotted in the complex plane as a plane vector of unit length making an angle of $\omega_0 t$ radians with the axis of reals. This is in fact the most useful interpretation of this function for the present purpose. Similarly $\epsilon^{-j\omega_0 t}$ is represented by a unit-length plane vector at an angle of $-\omega_0 t$ radians. These two vectors are shown in the complex plane of Fig. 15. The angle $\omega_0 t$ of the $\epsilon^{j\omega_0 t}$ vector increases linearly with time; that is, this vector rotates at a constant angular velocity of ω_0 radians per second. Likewise the $\epsilon^{-j\omega_0 t}$ vector rotates at a constant angular velocity of $-\omega_0$ radians per second. Between t equal to zero and such a time that $\pm\omega_0 t$ is equal to $\pm 2\pi$, each of these vectors makes one complete revolution and has returned to its initial position along the axis of reals. This time for one complete revolution is called the *period* T of the function and is given by

$$\omega_0 T = 2\pi \tag{149}$$

or

$$T = \frac{2\pi}{\omega_0}. \tag{149a}$$

Thus the exponential function of an imaginary variable is periodic in the same way that the trigonometric functions are periodic; namely, its values are identical for values of its argument that differ by any integral multiple of 2π. Thus

$$\epsilon^{j(\theta+k2\pi)} = \epsilon^{j\theta} \qquad [150]$$

or

$$\epsilon^{j(\omega_0 t+k2\pi)} = \epsilon^{j\omega_0 t}, \qquad [151]$$

in which k is any integer.

This similarity between the exponential function with an imaginary argument and the trigonometric functions is not accidental; in fact, these two functions are very intimately related as suggested already by Eqs. 125 and 137. This relation can be seen graphically from Fig 15. The sum of $\epsilon^{j\omega_0 t}$ and $\epsilon^{-j\omega_0 t}$, as can be seen by adding them graphically, evidently lies along the axis of reals and has the value $2\cos\omega_0 t$, or

$$\frac{\epsilon^{j\omega_0 t} + \epsilon^{-j\omega_0 t}}{2} = \cos\omega_0 t \qquad [152]$$

This expression also follows readily from Eq. 137. Thus if

$$\epsilon^{j\theta} = \cos\theta + j\sin\theta, \qquad [137a]$$

it follows that

$$\epsilon^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta. \qquad [137b]$$

Adding these two expressions and dividing by two result in

$$\frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2} = \cos\theta, \qquad [152a]$$

which is the same as Eq. 152 when θ is given the value $\omega_0 t$. This result can also be readily verified from the series expansions for the functions.

One further relation is useful at this point. It is often convenient to consider only the real part of $\epsilon^{j\theta}$, which is $\cos\theta$. This can be written

$$\mathcal{R}_e[\epsilon^{j\theta}] = \cos\theta \qquad [153]$$

and is read "the real part of $\epsilon^{j\theta}$." The notation $\mathcal{R}_e[\epsilon^{j\theta}]$ appears at first sight to be a clumsy substitute for $\cos\theta$, but it is found very convenient presently. It should be noticed that

$$\mathcal{R}_e[\epsilon^{j\theta}] = \mathcal{R}_e[\epsilon^{-j\theta}]. \qquad [154]$$

Thus there are several ways of expressing the cosine function, each of which is found of use for certain purposes. They are

$$\cos\theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2} = \mathcal{R}_e[\epsilon^{j\theta}] = \mathcal{R}_e[\epsilon^{-j\theta}]. \qquad [155]$$

By use of Eqs. 154 and 155, Eq. 104 can now be rewritten as

$$\left.\begin{aligned} q &= \frac{E}{S} - \frac{E}{S}\left(\frac{\epsilon^{j\omega_0 t} + \epsilon^{-j\omega_0 t}}{2}\right) = \frac{E}{S} - \frac{E}{S}\cos\omega_0 t \\ &= \frac{E}{S}(1 - \cos\omega_0 t) = \frac{E}{S}(1 - \mathcal{R}_e[\epsilon^{j\omega_0 t}]), \end{aligned}\right\} \qquad [156]$$

which is readily interpreted physically. Before this equation is discussed further, however, the combination of exponential functions occurring in Eq. 105 should be examined.

If the exponential terms in Eq. 105 are plotted for some convenient

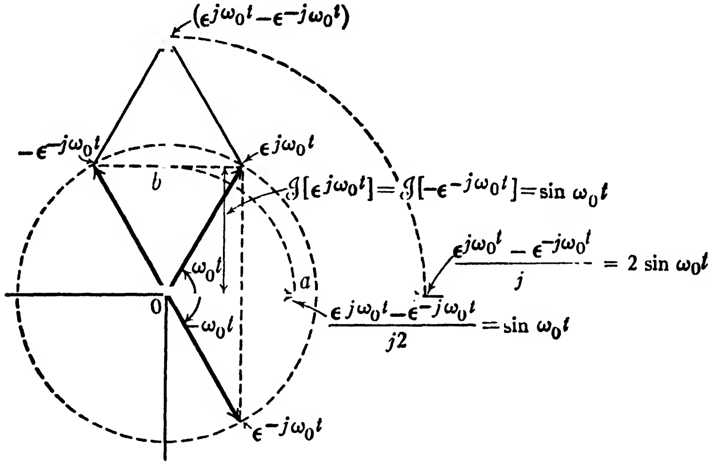


FIG. 16. Plot of vector components of sine function.

value of $\omega_0 t$, the result may be shown as in Fig. 16. The sum of $\epsilon^{j\omega_0 t}$ and $-\epsilon^{-j\omega_0 t}$ is seen to be $j2\sin\omega_0 t$, that is, a vector of length $2\sin\omega_0 t$ located along the j axis. If this vector is rotated backward 90 degrees, a manipulation which is equivalent to dividing by j, and then is divided by two, it is the real function $\sin\omega_0 t$. It is often convenient, however, to dispense with this -90-degree rotation, as the value of the projection of $\epsilon^{j\omega_0 t}$ on the j axis is $\sin\omega_0 t$. Thus, for representing $\sin\omega_0 t$,

$$\mathcal{J}[\epsilon^{j\omega_0 t}] \equiv \sin\omega_0 t \qquad [157]$$

may be used, $\mathcal{J}[\epsilon^{j\omega_0 t}]$ being read "the imaginary part of $\epsilon^{j\omega_0 t}$," and understood to mean the coefficient of j; that is, $\mathcal{J}[\epsilon^{j\omega_0 t}]$ is a real function. Its *value* is understood to be a real number and to include the sign preceding the term. For example, the value of $\mathcal{J}[10\epsilon^{-j(\pi/6)}]$ is -5. That

$$\frac{\epsilon^{j\omega_0 t} - \epsilon^{-j\omega_0 t}}{j2} = \sin\omega_0 t \qquad [158]$$

is true can also be seen analytically by subtracting Eq. 137b from Eq. 137a, dividing the difference by $j2$, and replacing θ by $\omega_0 t$, or from the series expansions for the functions.

The sine of θ can thus be expressed in several ways, each useful for certain purposes. They are

$$\sin\theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{j2} = \mathcal{I}[\epsilon^{j\theta}] = \mathcal{I}[-\epsilon^{-j\theta}]. \qquad [159]$$

By returning again to Eq. 105 it is seen that i can be expressed as

$$i = \frac{E}{\sqrt{SL}}\left(\frac{\epsilon^{j\omega_0 t} - \epsilon^{-j\omega_0 t}}{j2}\right) = \frac{E}{\sqrt{SL}}\sin\omega_0 t = \frac{E}{\sqrt{SL}}\mathcal{I}[\epsilon^{j\omega_0 t}]. \qquad [160]$$

15. Series *LS* circuit initially at rest

By means of the theory of complex numbers and functions, Eqs. 104 and 105 are converted into the forms of Eqs. 156 and 160. Thus in a series *LS* circuit, which is initially at rest and to which a constant voltage is instantaneously applied, the charge q on the condenser and the current i are given in the forms most useful for interpretation by

$$q = \frac{E}{S}(1 - \cos\omega_0 t) = \frac{E}{S} - \mathcal{R}_e\left[\frac{E}{S}\epsilon^{j\omega_0 t}\right], \qquad \blacktriangleright[156a]$$

$$i = \frac{E}{\sqrt{SL}}\sin\omega_0 t = \mathcal{I}\left[\frac{E}{\sqrt{SL}}\epsilon^{j\omega_0 t}\right]. \qquad \blacktriangleright[160a]$$

In the exponential forms the coefficients E/S and $E/\sqrt{SL}$ have been put inside the brackets and have become a part of the complex functions of which the real or imaginary part is wanted. That multiplication by a *real* number and taking the real or imaginary part of a complex number are commutative operations can readily be seen either graphically or by trial with Eq. 143. For example, the real part of $10\epsilon^{j(\pi/4)}$ is also ten times the real part of $\epsilon^{j(\pi/4)}$, or $10/\sqrt{2}$. Multiplication by a *complex* number and taking the real or imaginary part of a *complex* number are *not* commutative operations, however. Thus, using the rectangular form,

$$(a + jb)\mathcal{R}_e[c + jd] = c(a + jb) \neq \mathcal{R}_e[(a + jb)(c + jd)] = ac - bd. \qquad [161]$$

In Fig. 17 the trigonometric and exponential forms of Eqs. 156a and 160a are plotted in order to show more clearly the relations between them and also to facilitate their physical interpretation. Equation 160a, plotted in the lower part of this figure, shows how the sine curve for i is related to the rotating plane vector $(E/\sqrt{SL})\epsilon^{j\omega_0 t}$ which is shown at the instant $\omega_0 t$ is equal to $\pi/6$. The value of its imaginary part is the ordinate

of the i curve. That is, as this vector rotates in the complex plane the value of its imaginary part at each position is used as the ordinate of a point whose abscissa is the value of $\omega_0 t$, the result being the sine curve shown.

Next the plot of q, which is a cosine curve of amplitude E/S displaced upward by an amount equal to its amplitude, is considered. Its geometrical construction from the rotating vector is most easily accomplished by locating the center of the complex plane at the distance E/S above the

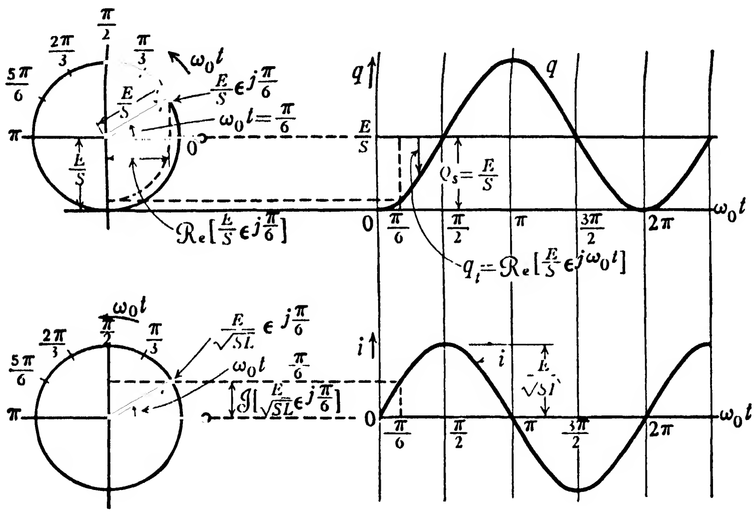


FIG. 17. Plots of charge and current, series LS circuit.

$\omega_0 t$ axis of the q versus $\omega_0 t$ co-ordinate system. The real part of the $(E/S)\epsilon^{j\omega_0 t}$ vector when rotated backwards to a vertical axis becomes the ordinate of the cosine curve. This construction is shown for the instant $\omega_0 t$ is equal to $\pi/6$.

With Eqs. 156a and 160a plotted, they are considered from the physical point of view, noting first that $q(0)$ and $i(0)$ are both zero as specified for this particular solution. Both current and charge then oscillate about their steady-state values — zero for the current and E/S for the charge. It is easily seen geometrically that the current curve has the form of the time derivative of the charge. These oscillations continue undiminished in this idealized circuit containing no resistance. Any physical coil always has a finite resistance the effect of which, as appears later in the chapter, is to cause the transient oscillations to die out gradually, leaving eventually only q equal to E/S and i equal to zero.

The angular frequency ω_0, or $\sqrt{S/L}$, of the characteristic oscillation is a very important parameter of this circuit. It is called the *characteristic*

angular frequency or *natural angular frequency* of the circuit, since it is determined entirely by the constants of the circuit and not in any way by the nature of the driving force applied to the circuit. This characteristic oscillation has a period $2\pi/\omega_0$ which is entirely analogous to the period of a freely swinging pendulum. Just as the period of a freely swinging pendulum is quite independent of the nature of the disturbance or force that sets it in motion, so the period of free oscillation of charge in an LS circuit is independent of the nature of the disturbing force that initiates the oscillations.

16. Series LS circuit with initial charge and current

Further consideration of Eqs. 102 and 103 for the LS circuit with initial charge and current leads to certain additional ideas relating to the exponential function and its use in circuit analysis The rather complicated Eqs. 102 and 103 can be expressed almost as compactly as Eqs. 104 and 105 are condensed in Eqs. 156a and 160a.

As a first step in the simplification of Eqs. 102 and 103, they can be written

$$q = \frac{E}{S} + B_1 \cos \omega_0 t - B_2 \sin \omega_0 t, \qquad [162]$$

$$i = -\omega_0 B_2 \cos \omega_0 t - \omega_0 B_1 \sin \omega_0 t, \qquad [163]$$

by using Eqs. 155 and 158. The constants B_1 and B_2 have the values

$$B_1 \equiv q(0-) - \frac{E}{S}, \qquad [164]$$

$$B_2 \equiv \frac{-i(0-)}{\omega_0}. \qquad [165]$$

Although the forms given in Eqs. 162 and 163 can be interpreted readily, they can be made more useful if the sine and cosine terms are combined. Thus in the equation for q, if these terms alone are considered, one can write

$$B_1 \cos \omega_0 t - B_2 \sin \omega_0 t = B \cos (\omega_0 t + \delta), \qquad [166]$$

provided

$$B_1 \equiv B \cos \delta, \qquad [167]$$

$$B_2 \equiv B \sin \delta, \qquad [168]$$

as can be verified by putting Eqs. 167 and 168 in Eq. 166. From Eqs. 167 and 168, by squaring and adding, one obtains

$$B_1^2 + B_2^2 = B^2 (\cos^2 \delta + \sin^2 \delta) = B^2, \qquad [169]$$

$$\tan \delta = \frac{\sin \delta}{\cos \delta} = \frac{B_2}{B_1}. \qquad [170]$$

Thus q can be written, using Eq. 166 with B and δ as given by Eqs. 169 and 170, and B_1 and B_2 as defined in Eqs. 167 and 168,

$$q = \frac{E}{S} + B\cos(\omega_0 t + \delta). \qquad [171]$$

The student can verify that

$$i = -\omega_0 B \sin(\omega_0 t + \delta). \qquad [172]$$

The change back to an exponential form is next made by writing Eq. 171

$$\left.\begin{aligned} q &= \frac{E}{S} + \mathcal{R}_e[B\epsilon^{j(\omega_0 t+\delta)}] = \frac{E}{S} + \mathcal{R}_e[B\epsilon^{j\delta}\epsilon^{j\omega_0 t}] \\ &= \frac{E}{S} + \mathcal{R}_e[\mathbf{B}\epsilon^{j\omega_0 t}], \end{aligned}\right\} \qquad [173]$$

$$\mathbf{B} \equiv B\epsilon^{j\delta} = B_1 + jB_2. \qquad [174]$$

The term $\mathcal{R}_e[\mathbf{B}\epsilon^{j\omega_0 t}]$ is the real part of the product of a complex number $\mathbf{B}$ and a complex function $\epsilon^{j\omega_0 t}$. The first has a magnitude B and a phase angle δ. The second, $\epsilon^{j\omega_0 t}$, has a magnitude one and an angle $\omega_0 t$ that increases linearly with time at the rate of ω_0 radians per second. Since the product of two complex numbers has a magnitude equal to the product of the magnitudes of the individual numbers, the magnitude of $\mathbf{B}\epsilon^{j\omega_0 t}$ is B. Since the product has an angle equal to the sum of the angles of the factors, the angle of $\mathbf{B}\epsilon^{j\omega_0 t}$ is $\omega_0 t + \delta$. Thus $\mathbf{B}\epsilon^{j\omega_0 t}$ can be plotted in the complex plane as a vector of magnitude B and angle $\omega_0 t + \delta$ as shown in Fig. 18.

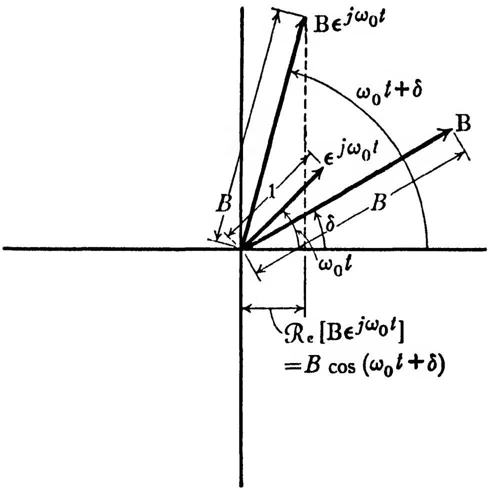


FIG. 18. Vector interpretation of transient charge in series *LS* circuit.

When $\omega_0 t$ is zero,

$$\mathbf{B}\epsilon^{j\omega_0 t} = \mathbf{B}\epsilon^{j0} = \mathbf{B}. \qquad [175]$$

From the figure it is apparent that

$$\mathcal{R}_e[\mathrm{B}\epsilon^{j\omega_0 t}] = B\cos(\omega_0 t + \delta). \qquad [176]$$

Since its real part is the transient component of charge, $\mathrm{B}\epsilon^{j\omega_0 t}$ may well be called the *transient-charge time vector*, sometimes shortened to transient-charge vector.

This method of including the phase angle δ in the coefficient B by making B a complex number gives expressions that are more easily manipulated than those in which δ is added to $\omega_0 t$ in the argument of the exponential. That is, the form $\mathrm{B}\epsilon^{j\omega_0 t}$ is usually found more convenient to use than $B\epsilon^{j(\omega_0 t+\delta)}$.

Equations 162, 171, and 173 provide three different ways of writing the function q, namely,

$$q = \frac{E}{S} + B_1\cos\omega_0 t - B_2\sin\omega_0 t \qquad ▶[162]$$

$$q = \frac{E}{S} + B\cos(\omega_0 t + \delta) \qquad ▶[171]$$

$$q = \frac{E}{S} + \mathcal{R}_e[\mathrm{B}\epsilon^{j\omega_0 t}]. \qquad ▶[173a]$$

The choice of the one to use is purely a matter of convenience. Although at first sight the last form may appear more awkward than the other two, in reality it is usually the simplest to use.

The current i as given by Eq. 160 is considered next. Carrying through the same kind of processes that were used with q obtains a form for i similar to Eq. 173a. It is much simpler as well as more instructive, however, to take the derivative of the exponential form of q. In order to do so, one needs to know how to differentiate the real part of a complex function. For this differentiation, the complex function $\mathrm{A}(t)$ of the real variable t is written in rectangular form as

$$\mathrm{A}(t) = A_1(t) + jA_2(t), \qquad [177]$$

$A_1(t)$ and $A_2(t)$ being real functions. It can be seen by the usual process of obtaining a derivative as the limit of a quotient of increments that

$$\frac{d}{dt}\mathrm{A}(t) = \frac{d}{dt}A_1(t) + j\frac{d}{dt}A_2(t). \qquad [178]$$

In words, the real part of the derivative of a complex function is the derivative of the real part of the function, and the imaginary part of the

derivative is the derivative of the imaginary part. Another way in which this can be expressed is

$$\frac{d}{dt}\mathcal{R}_e[\mathrm{A}(t)] = \mathcal{R}_e\left[\frac{d}{dt}\mathrm{A}(t)\right], \tag{179}$$

$$\frac{d}{dt}\mathcal{I}[\mathrm{A}(t)] = \mathcal{I}\left[\frac{d}{dt}\mathrm{A}(t)\right]. \tag{180}$$

That is, the operations of differentiation and of taking the real or imaginary part are commutative.

Thus one can write for the current function in question, from Eq. 173a,

$$i = \frac{dq}{dt} = 0 + \frac{d}{dt}\mathcal{R}_e[\mathrm{B}\epsilon^{j\omega_0 t}] = \mathcal{R}_e\left[\frac{d}{dt}\mathrm{B}\epsilon^{j\omega_0 t}\right] = \mathcal{R}_e[j\omega_0\mathrm{B}\epsilon^{j\omega_0 t}]. \quad ▶[181]$$

The differentiation of q to obtain i has the effect of making the transient component of i differ from the transient component of q by the factor $j\omega_0$. In words, the transient-current vector is $\pi/2$ radians or 90 degrees ahead of the transient-charge vector and is ω_0 times as large. This 90-degree angle by which the current vector leads the charge vector is the general relation between any sinusoidally varying charge and the corresponding current. This phase relation can be seen trigonometrically also. If a charge is given by the sine function, the current is given by the derivative, or cosine function, which is merely a sine function advanced in phase by 90 degrees. This phase relation will be encountered frequently in alternating-current theory.

In Eqs. 173a and 181 the complex coefficient B can be determined by Eq. 174 from B_1 and B_2 as given in Eqs. 164 and 165. It is instructive and illustrative of the simpler method generally used, however, to determine its real and imaginary components B_1 and B_2, respectively, directly from the initial conditions. Thus it is quite unnecessary in an actual problem to go through all the steps used in proceeding from the initial solutions Eqs. 96 and 97 to Eq. 173a or Eq. 181, once the nature of the result has been found. Therefore Eqs. 173a and 181 with t equal to zero are set equal to the given initial conditions:

$$q(0-) = \frac{E}{S} + \mathcal{R}_e[\mathrm{B}\epsilon^{j0}] = \frac{E}{S} + B_1, \tag{182}$$

$$i(0-) = \mathcal{R}_e[j\omega_0\mathrm{B}\epsilon^{j0}] = \mathcal{R}_e[j\omega_0(B_1 + jB_2)] = -\omega_0 B_2; \tag{183}$$

whence

$$B_1 = q(0-) - \frac{E}{S}, \tag{184}$$

$$B_2 = -\frac{i(0-)}{\omega_0}. \tag{185}$$

Putting these values in Eqs. 173a and 181 gives for q and i,

$$q = \frac{E}{S} + \mathcal{R}_e\left[\left\{q(0-) - \frac{E}{S} - j\frac{i(0-)}{\omega_0}\right\}\epsilon^{j\omega_0 t}\right], \qquad [186]$$

$$\left.\begin{aligned} i &= \mathcal{R}_e\left[j\omega_0\left\{q(0-) - \frac{E}{S} - j\frac{i(0-)}{\omega_0}\right\}\epsilon^{j\omega_0 t}\right] \\ &= \mathcal{R}_e\left[\left\{i(0-) + j\omega_0\left(q(0-) - \frac{E}{S}\right)\right\}\epsilon^{j\omega_0 t}\right] \end{aligned}\right\}. \qquad [187]$$

17. Illustrative example of series LS circuit

It was stated in introducing the series LS circuit in Art. 12 that such a circuit is physically unrealizable because self-inductance is unavoidably accompanied by resistance. However, this circuit is sufficiently important on two grounds to justify considering a numerical example. The first is that the effects of resistance may sometimes be neglected with fair approximation in a practical case. The second is that this circuit offers a simple and rigorous introduction to complex numbers as applied to circuit analysis.

A comment regarding the term *steady state* as used in this example is necessary. In general, the term as applied to a component of current or charge means that component remaining after the transient has subsided completely. Because there is no resistance in this particular theoretical circuit, the transients never subside and the foregoing definition of steady state has no meaning. However, if the circuit contains even the smallest amount of resistance, the transient does subside. Furthermore, the remaining steady-state components are independent of the value of the resistance in this series circuit. These same steady-state values are therefore arbitrarily taken as the correct ones in the limiting case of zero resistance.

The diagram of the illustrative circuit is shown in Fig. 19. Switch K is operated in the following sequence: With K in position 1 the circuit is assumed to have reached the steady state, as defined, with i equal to zero. Switch K is then moved suddenly to position 2, where it remains for 0.00110 second, and is then suddenly moved back to position 1. The problem is:

(a) To calculate and plot the current and the charge on the elastance as functions of time;

(b) To find the maximum positive value of i which follows the last operation of K, and the time at which it occurs;

(c) To find the instant following the last operation of K at which the elastance voltage is maximum, with plate a positive with respect to plate b, and the value of this voltage.

In making the solution the foregoing theory is used only as a guide in the reasoning; numbers are not merely substituted in the expressions there obtained. This procedure is usually found to be quite as direct as mere substitution, and it avoids the frequent difficulty of interpretation of a formula in terms of the particular circuit under consideration.

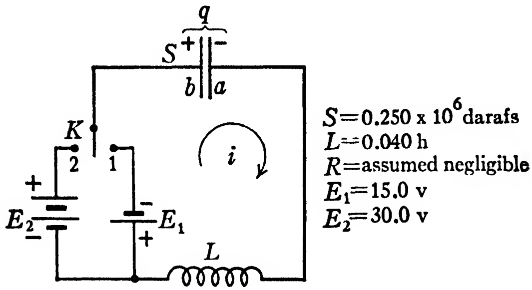


FIG. 19 Diagram for series LS circuit having initial charge.

Solution: The first step is the determination of the steady-state charge on S with K at 1 and i equal to 0. Since positive charge displaced around the arbitrarily assumed arrow direction is described by a positive number, a positive value for q means that b is at a positive potential with respect to a. Summing the potential drops in the arrow direction and equating to zero, with I_{s1} equal to 0, give

$$E_1 + Sq = 0; \tag{188}$$

whence

$$q = Q_{s1} = -\frac{E_1}{S} = -\frac{15.0}{0.250} \times 10^{-6} = -60.0 \times 10^{-6} \text{ coulomb.} \tag{189}$$

The symbols Q_{s1} and I_{s1} are used to designate the steady-state components of q and i, respectively, with K in position 1. The negative value of q agrees with the obvious physical fact that a is at a higher potential than b.

Next to be considered are conditions following the movement of K to position 2 at an instant from which time t is measured. The first step is to find the steady-state component Q_{s2} of charge for the new condition, using the term steady state as explained before in this example. Since the driving force is a constant voltage and there is a series condenser in the circuit, the final steady-state component I_{s2} of the current is known to be zero, and the entire battery voltage appears across the condenser. Since the voltage E_2 has such polarity as to displace charge in the positive direction, it is seen by inspection that

$$Q_{s2} = \frac{30.0}{0.25 \times 10^6} = 120 \times 10^{-6} \text{ coulomb.} \tag{190}$$

This result is also easily obtained by summing the potential drops in the arrow direction, with I_s equal to zero, giving

$$-E_2 + SQ_{s2} = 0, \tag{191}$$

from which

$$Q_{s2} = 120 \times 10^{-6} \text{ coulomb.} \tag{192}$$

The initial conditions are next analyzed. Before K was moved to position 2, the condenser charge $q(0-)$ was -60.0×10^{-6} coulomb. The new steady-state value is 120×10^{-6} coulomb, and because of the inductance the charge cannot be displaced instantaneously. Hence, the actual charge on the condenser is the same immediately after K is moved to position 2 as it was before; that is,

$$q(0+) = q(0-) = Q_{s1} = -60.0 \times 10^{-6} \text{ coulomb.} \tag{193}$$

Just after switching there evidently must be such a transient component q_{t2} that

$$q(0+) = -60.0 \times 10^{-6} \text{ coulomb} = Q_{s2} + q_{t2}(0+) = 120 \times 10^{-6} + q_{t2}(0+) \tag{194}$$

or

$$q_{t2}(0+) = (-60 - 120)10^{-6} = -180. \times 10^{-6} \text{ coulomb.} \tag{194a}$$

Since

$$i(0-) = i(0+) = I_{s2} = 0, \tag{195}$$

there is no initial value of transient current; that is,

$$i_{t2}(0+) = 0. \tag{196}$$

The steady-state components and the initial values of the transient components of charge and current having been found, the roots of the characteristic equation are evaluated:

$$p = \pm\sqrt{-\frac{S}{L}} = \pm j\sqrt{\frac{0.250 \times 10^6}{0.040}} = \pm j\sqrt{6.25 \times 10^6} = \pm j2{,}500 \text{ sec}^{-1}. \tag{197}$$

That is,

$$\omega_0 = 2{,}500 \text{ radians/sec.} \tag{197a}$$

The results of Art. 16 show that the transient component of charge can be expressed

$$q_t(t) = \mathcal{R}_e[\mathrm{B}\epsilon^{j\omega_0 t}], \tag{198}$$

in which B is a complex number,

$$\mathrm{B} = B_1 + jB_2. \tag{174a}$$

Also differentiating Eq. 198 gives the transient component of current

$$i_t(t) = \mathcal{R}_e[j\omega_0\mathrm{B}\epsilon^{j\omega_0 t}]. \tag{199}$$

The two components of B are readily found from the initial values $q_t(0+)$ and $i_t(0+)$. Thus

$$q_t(0+) = -180. \times 10^{-6} = \mathcal{R}_e[\mathrm{B}] = B_1, \tag{200}$$

$$i_t(0+) = 0 = \mathcal{R}_e[j\omega_0(B_1 + jB_2)] = -\omega_0 B_2, \tag{201}$$

$$\mathrm{B} = -180 \times 10^{-6} + j0 = 180 \times 10^{-6}\underline{/180^\circ} \text{ vector coulomb.} \tag{202}$$

The unit for B is conveniently described as a *vector coulomb* because it has the physical dimensions of charge and is at the same time a complex number, or plane vector. To B itself no simple physical interpretation can be attached. All that can be said is that when it is multiplied by the exponential time function $\epsilon^{j\omega_0 t}$, the real part of the product is the transient component of the actual charge, which is a measurable quantity.

Putting the numerical value of B in Eqs. 198 and 199 gives for the transient components of charge and current

$$q_t(t) = \mathcal{R}_e[180 \times 10^{-6}\underline{/180^\circ}\, \epsilon^{j2,500t}] \text{ coulomb,} \qquad [198a]$$

$$\left.\begin{aligned} i_t(t) &= \mathcal{R}_e[j2{,}500 \times 180 \times 10^{-6}\underline{/180^\circ}\, \epsilon^{j2,500t}] \\ &= \mathcal{R}_e[0.450\underline{/-90^\circ}\, \epsilon^{j2,500t}] \text{ amp.} \end{aligned}\right\} \qquad [199a]$$

The graphical interpretation of these expressions is a great aid in visualizing their meaning. The time vectors of which $q_t(t)$ and $i_t(t)$ are the real parts are plotted in the complex plane in Fig. 20. The charge time vector $180 \times 10^{-6}\underline{/180^\circ}\, \epsilon^{j2,500t}$ is shown for two instants. It is at $0m$ when t is 0, and at $0n$ when t is 0.00110 sec, or $\omega_0 t$ is 2.75 radians, or 157.4°. The current time vector $0.450\underline{/-90^\circ}\, \epsilon^{j2,500t}$ is also shown for the same two instants at $0p$ and $0r$, respectively. The length scales for the

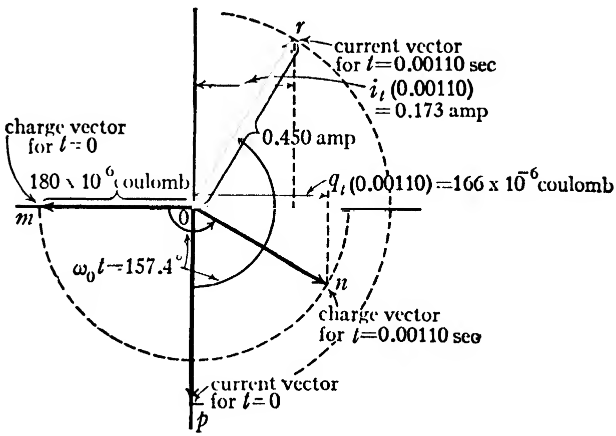


FIG. 20. Vector interpretation of transient components of charge and current $q_t(t)$ and $i_t(t)$ for circuit of Fig. 19.

current and charge time vectors are of course quite different. At any given instant the transient component of charge or current is the projection of the appropriate vector on the axis of reals. The current vector in each case is 90° ahead of the charge vector, as pointed out in the preceding section.

When the expressions for the transient components of charge and current have thus been found, the expressions for the actual charge and current are readily obtained as the sums of the transient and steady-state components. These are

$$q(t) = Q_{s2} + q_t(t) = 120 \times 10^{-6} + \mathcal{R}_e[180 \times 10^{-6}\underline{/180^\circ}\, \epsilon^{j2,500t}] \text{ coulomb,} \qquad [203]$$

$$i(t) = I_{s2} + i_t(t) = 0 + \mathcal{R}_e[0.450\underline{/-90^\circ}\, \epsilon^{j2,500t}] \text{ amp,} \qquad [204]$$

for t between 0 and 0.00110 sec. The conditions immediately prior to the second switching operation, which occurs when t is 0.00110 sec, are considered next, in order to find the charge and current that serve as initial values for the period subsequent

to the second switching. At this value of t,

$$\left.\begin{aligned} q(0.00110) &= 120 \times 10^{-6} + \mathcal{R}_e[180 \times 10^{-6} \underline{/180^\circ}\, \epsilon^{j2.75}], \\ &= 120 \times 10^{-6} - 180 \times 10^{-6} \cos 2.75 \end{aligned}\right\} \quad [205]$$

$$\left.\begin{aligned} i(0.00110) &= \mathcal{R}_e[-j0.450\epsilon^{j2.75}] = \mathcal{R}_e[-j0.450(\cos 2.75 + j \sin 2.75)] \\ &= 0.450 \sin 2.75. \end{aligned}\right\} \quad [206]$$

Evaluating these expressions,

$$\cos 2.75 = \cos 157.4^\circ = -\cos 22.6^\circ = -0.923, \quad [207]$$

$$\sin 2.75 = \sin 157.4^\circ = \sin 22.6^\circ = 0.385, \quad [208]$$

$$\left.\begin{aligned} q(0.00110) &= (120 + 180 \times 0.923)10^{-6} \\ &= (120 + 166)10^{-6} = 286 \times 10^{-6} \text{ coulomb}, \end{aligned}\right\} \quad [209]$$

$$i(0.00110) = 0.450 \times 0.385 = 0.173 \text{ amp}. \quad [210]$$

The transient components of $q(0.00110)$ and $i(0.00110)$ are indicated in Fig. 20.

For expressing the current and charge subsequent to 0.00110 sec, it is convenient to measure the time t' from this instant. That is, a new time variable

$$t' = t - 0.00110 \text{ sec} \quad [211]$$

is taken whose value is zero at the instant that K is moved from 2 back to 1.

For t' greater than zero, it is evident that the steady-state components Q_{s1} and I_{s1} are those found at the beginning of the solution, namely, -60.0×10^{-6} coulomb and 0, respectively, while the actual values of charge and current at t' equal to zero are those just found for t equal to 0.00110 sec:

$$q(t' = 0) = q(t = 0.00110) = 286 \times 10^{-6} \text{ coulomb}, \quad [209a]$$

$$i(t' = 0) = i(t = 0.00110) = 0.173 \text{ amp}. \quad [210a]$$

Thus there must be transient components q_{t1} and i_{t1} such that at the initial instant, t' equal to zero,

$$q(t' = 0) = Q_{s1} + q_{t1}(t' = 0) \quad [212]$$

and

$$i(t' = 0) = I_{s1} + i_{t1}(t' = 0). \quad [213]$$

Consequently the initial values of the transient components for t' greater than zero are

$$\left.\begin{aligned} q_{t1}(t' = 0) = q(t' = 0) - Q_{s1} &= [286 - (-60)]10^{-6} \\ &= 346 \times 10^{-6} \text{ coulomb}, \end{aligned}\right\} \quad [212a]$$

$$i_{t1}(t' = 0) = i(t' = 0) - I_{s1} = 0.173 - 0 = 0.173 \text{ amp}. \quad [213a]$$

By considering next the form of the transient, one sees that the characteristic equation is the same for both the 1 and 2 positions of K and that one can write

$$q_{t1}(t') = \mathcal{R}_e[\mathrm{D}\epsilon^{j\omega_0 t'}], \quad [214]$$

$$i_{t1}(t') = \mathcal{R}_e[j\omega_0 \mathrm{D}\epsilon^{j\omega_0 t'}], \quad [215]$$

$$\mathrm{D} = D_1 + jD_2. \quad [216]$$

The symbol D is used to indicate the complex coefficient instead of B to avoid confusion with the value for the preceding interval. Putting in values for t' equal to zero gives

$$346 \times 10^{-6} = \mathcal{R}_e[\mathrm{D}] = D_1, \tag{217}$$

$$0.173 = \mathcal{R}_e[j\omega_0\mathrm{D}] = \mathcal{R}_e[j\omega_0(D_1 + jD_2)] = -\omega_0 D_2, \tag{218}$$

$$D_2 = -\frac{0.173}{2{,}500} = -69.2 \times 10^{-6}, \tag{219}$$

$$\mathrm{D} = (346 - j69.2)10^{-6} = 353 \times 10^{-6}\underline{/-11.3^\circ} \text{ vector coulomb.} \tag{220}$$

Using the numerical value of D, one can write for the transient components,

$$q_{t1}(t') = \mathcal{R}_e[353 \times 10^{-6}\underline{/-11.3^\circ}\, \epsilon^{j2{,}500t'}] \text{ coulomb,} \tag{214a}$$

$$\left.\begin{aligned} i_{t1}(t') &= \mathcal{R}_e[j2{,}500 \times 353 \times 10^{-6}\underline{/-11.3^\circ}\, \epsilon^{j2{,}500t'}] \\ &= \mathcal{R}_e[0.883\underline{/78.7^\circ}\, \epsilon^{j2{,}500t'}] \text{ amp.} \end{aligned}\right\} \tag{215a}$$

The interpretation of j as $1\underline{/90^\circ}$ is used in obtaining the last expression. Plots of the transient time vectors are shown in Fig. 22.

The actual charge and current are given by

$$\left.\begin{aligned} q(t') &= Q_{s1} + q_{t1}(t') \\ &= -60 \times 10^{-6} + \mathcal{R}_e[353 \times 10^{-6}\underline{/-11.3^\circ}\, \epsilon^{j2{,}500t'}] \text{ coulomb} \end{aligned}\right\} \tag{221}$$

$$i(t') = I_{s1} + i_{t1}(t') = 0 + \mathcal{R}_e[0.883\underline{/78.7^\circ}\, \epsilon^{j2{,}500t'}] \text{ amp} \tag{222}$$

for $t' > 0$, or $t > 0.00110$ sec.

The calculations for part (a) of the problem are complete, the results being given by Eqs. 203 and 204 for the time interval during which K is in position 2 and by Eqs. 221 and 222 for the succeeding time interval, during which K is in position 1. These equations are plotted as functions of time in Fig. 21.

The maxima of parts (b) and (c) are readily determined by inspection and are particularly well seen by plotting the transient-charge and current time vectors in the complex plane, as shown in Fig. 22. From this figure it is evident that i_t has its maximum positive value when its vector has rotated to the positive axis of reals, as occurs at $78.7^\circ + \omega_0 t'$, or 360°; whence

$$\omega_0 t' = 281.3^\circ = \frac{281.3\pi}{180} = 4.90 \text{ radians} \tag{223}$$

and

$$t' = \frac{4.90}{2{,}500} = 1.960 \times 10^{-3} \text{ sec.} \tag{224}$$

At this instant i has the value 0.883 amp, its maximum value.

Part (c) shows that terminal a of the condenser is most positive with respect to b when q has its most negative value, because a positive displacement of charge around the circuit makes b positive with respect to a. From Eq. 221 it can be seen by inspection that $q(t')$ has its greatest negative value when the transient component $q_t(t')$ is most negative. From Fig. 22 it is seen that this situation occurs when the transient-charge vector has rotated to the negative axis of reals or when

$$\omega_0 t' = 180^\circ + 11.3^\circ = 3.34 \text{ radians.} \tag{225}$$

At this instant

$$t' = \frac{3.34}{2{,}500} = 1\ 335 \times 10^{-3}\ \text{sec}, \tag{226}$$

$$q(t' = 1.335 \times 10^{-3}) = (-60 - 353)10^{-6} = -413 \times 10^{-6}\ \text{coulomb}, \tag{227}$$

and the potential drop from a to b is

$$v_{ab} = 413 \times 10^{-6} \times 0.250 \times 10^{6} = 103\ \text{v}. \tag{228}$$

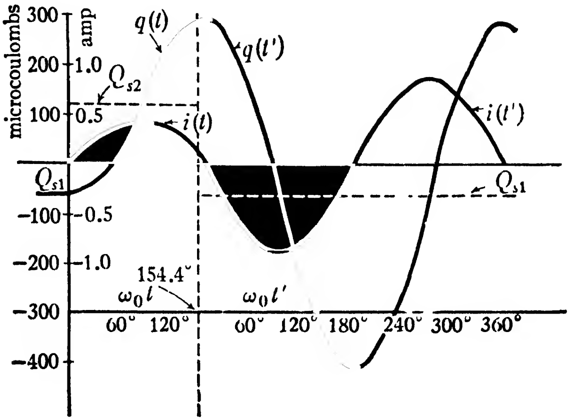


FIG. 21. Plots of charge and current in circuit of Fig. 19.

These maxima can also be located readily from a time plot such as Fig. 21, but it is found that the vector diagrams are usually simpler to draw and easier to use. This problem illustrates how switching may cause the voltage across the condenser in such

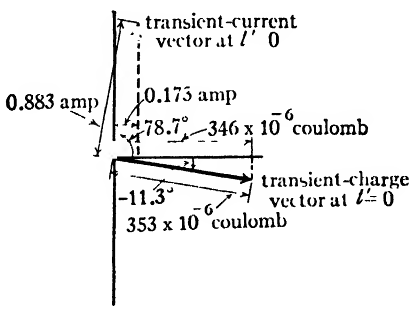


FIG. 22. Vector interpretation of transient components of charge and current $q_t(t')$ and $i_t(t')$ for circuit of Fig. 19.

a circuit to build up to high values. Thus if the switch is operated numerous times in succession at such instants as to make the transient amplitudes as large as possible, arbitrarily large voltages theoretically can be obtained.

18. Series *RLS* circuit

The few preceding articles analyze the series *LS* circuit, which has limited practical applicability but which furnishes a relatively simple means of gaining familiarity with the application of complex numbers to circuits. Adding series resistance to this circuit provides a circuit that is physically realizable but one that is slightly more complicated to handle analytically. The analysis applicable to this series *RLS* circuit is of particular interest, however, because of the importance of this circuit and because of the large class of mechanical and other physical systems that are described by the same form of differential equation. That is, by merely redefining the symbols used for the coefficients and variables, the differential equation next to be considered, and its solution, can be applied without change to any of a wide variety of important practical equipment. Examples that may be mentioned are the moving systems of many varieties of indicating instruments, automatic control devices, and many mechanical systems containing inertia and elastance. Because of its importance, the development of this analysis is considered in detail as applied to the *RLS* circuit to which a constant driving force can be suddenly applied.

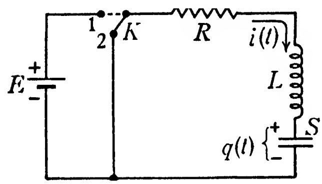


FIG. 23. Series *RLS* circuit.

In Fig. 23 is shown a series *RLS* circuit which can be short-circuited or connected to the constant-voltage source E by putting switch K in position 2 or position 1, respectively. As in the preceding case the analysis is somewhat more conveniently carried out in terms of charge rather than current as the dependent variable. Writing the Kirchhoff voltage equation for potential drops in the arbitrarily selected arrow direction, with K in position 1, gives

$$L\frac{di}{dt} + Ri + S\int idt = E, \tag{229}$$

which can be written in terms of charge as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + Sq = E. \tag{230}$$

In Eqs. 229 and 230, q and $\int idt$ mean the actual charge on the elastance. Equation 230 is a linear differential equation with constant coefficients. It is of second order because the highest order of derivative is the second.

As in the preceding cases of linear differential equations that are considered, the solution of Eq. 230 consists of two parts. The first is the steady-state solution; the second, the transient solution. In mathematical

nomenclature these are respectively the particular integral and the complementary function.

The steady-state solution satisfies Eq. 230 identically. It represents conditions in the circuit at sufficient time after any change in the circuit for a steady state to be reached — a state that remains unchanged until the circuit or forces are changed. By intuition one expects that ultimately, with K in position 1, the current becomes zero and the elastance is charged to the battery potential. That is, one expects the steady-state component Q_s of charge and the steady-state component I_s of current to be, respectively,

$$Q_s = \frac{E}{S} \tag{83}$$

and

$$I_s = 0. \tag{84}$$

The correctness of this result is easily verified by noting that this value of Q_s satisfies Eq. 230.

By the same reasoning that was used in the preceding cases, it is observed that, while the values of Eqs. 83 and 84 satisfy the differential equation 230, they do not necessarily satisfy the initial conditions. The current in an inductance cannot be changed instantaneously; neither can the charge on an elastance that is in series with either a resistance or an inductance. If at the instant before K is moved to position 1 there is current in the circuit or charge on the elastance, this current and this charge are unaltered by the switching operation, assuming, as before, that K is so made that its movement does not open the circuit. In other words, the actual current and charge have the same values immediately after K is moved as they had immediately before. It is easily seen that these values generally are not those called for by the steady-state solution Eqs. 83 and 84. Thus arises the need for the transient, or temporary, components of current and charge having such values immediately after K is moved that when they are added to the steady-state values the sums are respectively equal to the then existing values or initial values. As these transient components gradually die out, the charge and current undergo a continuous or smooth transition from their initial to their steady-state values. These ideas have been stated in the preceding articles but are repeated here for emphasis, for they are fundamental to the understanding of circuit transient behavior.

In order to express the foregoing ideas quantitatively, time is measured from the instant when K is moved; then whatever the values $i(0-)$ and $q(0-)$ are at an instant immediately before K is moved, the values

$i(0+)$ and $q(0+)$ immediately thereafter are

$$i(0+) = i(0-) \tag{231}$$

and

$$q(0+) = q(0-). \tag{232}$$

But it is known that

$$q(0+) = Q_s + q_t(0+) = q(0-) \tag{233}$$

and

$$i(0+) = I_s + i_t(0+) = i(0-), \tag{234}$$

in which $q_t(0+)$ and $i_t(0+)$ are the initial values of the transient components of charge and current, respectively — their values, that is, immediately after K is moved. Thus

$$q_t(0+) = q(0-) - Q_s \tag{233a}$$

and

$$i_t(0+) = i(0-) - I_s. \tag{234a}$$

But q must satisfy Eq. 230. The component Q_s does. However, if q_t satisfies the reduced or force-free equation

$$L\frac{d^2q_t}{dt^2} + \frac{R\,dq_t}{dt} + Sq_t = 0, \tag{235}$$

then q satisfies Eq. 230.

To find the q_t that satisfies Eq. 235 is not an explicit operation, but it is well known that an exponential function in general satisfies such an equation. It is therefore assumed — subject to verification by direct substitution into Eq. 235 — that

$$q_t = A\epsilon^{pt}, \tag{236}$$

in which A and p are constants. This gives the conditional equation

$$Lp^2A\epsilon^{pt} + RpA\epsilon^{pt} + SA\epsilon^{pt} = 0 = (Lp^2 + Rp + S)A\epsilon^{pt}, \tag{237}$$

which is satisfied if A, ϵ^{pt}, or the expression in parentheses is zero. The first possibility is trivial because it merely makes q_t equal to zero. The second possibility also is trivial because, in order to have Eq. 237 satisfied for all values of t, p must equal minus infinity. Eliminating the first two possibilities shows that Eq. 236 is a solution of Eq. 235 provided

$$Lp^2 + Rp + S = 0 \tag{237a}$$

or

$$p = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{S}{L}}. \tag{238}$$

Thus there are two values of p,

$$p_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{S}{L}} \qquad [238a]$$

and

$$p_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{S}{L}} \qquad [238b]$$

that satisfy Eq. 237a, that is, for which Eq. 236 becomes a solution of the force-free equation 235.[3]

The values of p determine the nature or form of the transient, that is, the way in which it varies with time. The characteristic equation, 237a, from which they are determined contains the circuit parameters L, R, and S but in no way involves the impressed force E. It is emphasized again that the *form of the transient depends only on the circuit parameters* (and the circuit connections) *and in no way upon the driving forces applied to the circuit,* a statement found to be applicable to all linear circuits. As a consequence of the foregoing analysis, the characteristic equation can be written immediately by inspection from the reduced equation, 235.

From Eqs. 238a and 238b it can be seen that the expression under the radical may be positive, negative, or zero. Since these three possibilities lead to quite different forms of transient, they are treated in the following articles separately and in the order mentioned.

19. Case I: Overdamped circuit

A series RLS circuit is said to be *overdamped*, for reasons that become apparent subsequently, if

$$\frac{R^2}{4L^2} > \frac{S}{L}. \qquad [239]$$

The radical is then real, so that both roots are real. However, since the value of the radical is always less than $R/(2L)$, both roots are negative. This is to be expected, since if these roots were positive, the corresponding terms of the form Eq. 236 would grow exponentially. This behavior is not possible in a network containing no source of energy, that is, in a passive network.

[3] This general method is further considered in an outstanding paper by G. A. Campbell, "Cisoidal Oscillations," *A.I.E.E. Trans.*, XXX, Part 2 (1911), 873–913. Other treatments are given by V. Bush, "Oscillating Current Circuits by the Method of Generalized Angular Velocities," *A.I.E.E. Trans.*, XXXVI (1917), 207–234; and A. E. Kennelly, "The Impedances, Angular Velocities, and Frequencies of Oscillating Current Circuits," *I.R.E. Proc.*, IV (1916), 47–94. The discussions immediately following this last paper are particularly interesting. They present a number of additional points of view and, by reference to original sources, shed valuable light on the history and development of the methods.

The following discussion is simplified somewhat if one uses the abbreviations

$$\alpha \equiv \frac{R}{2L}, \tag{240}$$

$$\omega_0 \equiv \sqrt{\frac{S}{L}}, \tag{94}$$

the second of which is introduced in the nondissipative case, Art. 12. The condition Eq. 239 then corresponds to α greater than ω_0; that is, half the resistance to inductance ratio of the coil is larger than the corresponding nondissipative characteristic angular frequency of the circuit.

If one uses the further abbreviation

$$\beta \equiv \sqrt{\alpha^2 - \omega_0^2} = \sqrt{\frac{R^2}{4L^2} - \frac{S}{L}}, \tag{241}$$

then the roots for this case are given by

$$p_1 = -\alpha + \beta, \tag{238c}$$

and

$$p_2 = -\alpha - \beta. \tag{238d}$$

The transient solution is composed of the sum of two terms of the form of Eq. 236, one for each of these roots. Thus the complete charge solution becomes

$$q = Q_s + q_t = \frac{E}{S} + A_1\epsilon^{(-\alpha+\beta)t} + A_2\epsilon^{(-\alpha-\beta)t}, \tag*{▶[242]}$$

in which A_1 and A_2 are constants. The corresponding current solution is

$$i = \frac{dq}{dt} = 0 + (-\alpha + \beta)A_1\epsilon^{(-\alpha+\beta)t} + (-\alpha - \beta)A_2\epsilon^{(-\alpha-\beta)t}. \tag*{▶[243]}$$

The integration constants A_1 and A_2 are evaluated by using the initial current $i(0+)$ and the initial charge $q(0+)$. Writing Eqs. 242 and 243 for t equal to zero gives

$$q(0+) = \frac{E}{S} + A_1 + A_2, \tag{244}$$

$$i(0+) = 0 + (-\alpha + \beta)A_1 + (-\alpha - \beta)A_2. \tag{245}$$

In any given problem Eqs. 244 and 245 are readily solved simultaneously for A_1 and A_2.

As in the preceding cases, the number of arbitrary constants provided by the mathematical solution — in this case two — is just sufficient to

suit the physical nature of the problem. Thus A_1 and A_2 can be any real numbers and Eq. 242 satisfies Eq. 230. Their values are uniquely determined, however, when Eqs. 242 and 243 are required to satisfy not only the differential equation but the two initial conditions, Eqs. 231 and 232, as well.

Equations 242 and 243 are sometimes expressed in terms of hyperbolic functions, but the simple exponential forms given are fully as convenient for calculation and interpretation. Their only deficiency lies in a certain lack of mathematical elegance which appears when comparisons are made among the three cases of this circuit.

This case of the overdamped RLS circuit is illustrated numerically in the example of Art. 23, in which the characteristic behavior of such a circuit as contrasted with that of an underdamped circuit becomes very apparent. The reasons for the terms *underdamped* and *overdamped* become evident at that point.

20. Case II: Underdamped, or oscillatory, circuit

If in a series RLS circuit

$$\frac{R^2}{4L^2} < \frac{S}{L}, \tag{246}$$

the circuit is said to be *underdamped*. When this condition obtains, the expression under the radical of Eq. 238 becomes negative, and it is found convenient to rewrite Eqs. 238a and 238b as

$$p_1 = -\frac{R}{2L} + j\sqrt{\frac{S}{L} - \left(\frac{R}{2L}\right)^2} = -\alpha + j\omega_d, \tag{238e}$$

$$p_2 = -\frac{R}{2L} - j\sqrt{\frac{S}{L} - \left(\frac{R}{2L}\right)^2} = -\alpha - j\omega_d, \tag{238f}$$

in which

$$\omega_d \equiv \sqrt{\frac{S}{L} - \left(\frac{R}{2L}\right)^2} = \sqrt{\omega_0^2 - \alpha^2} \tag{241a}$$

and, as before,

$$\omega_0 \equiv \sqrt{\frac{S}{L}}, \tag{94}$$

$$\alpha \equiv \frac{R}{2L}. \tag{240}$$

A word regarding the notation for the components of p may be helpful at this point. The symbol ω_0 is used to denote the angular frequency of the

force-free, or characteristic, oscillations of a circuit containing no dissipative elements, that is, no resistance, as illustrated by the series LS circuit. If a small amount of series resistance is added to such a circuit, its force-free behavior is still oscillatory but the oscillations have a slightly lower angular frequency denoted by ω_d, and they gradually die out, as is shown in the following analysis. Angular frequencies ω_0 and ω_d are characteristic of the circuit parameters and connections and in no way depend upon the driving forces. Chapter IV and following chapters discuss the behavior of circuits subjected to sinusoidal driving forces whose angular velocity is denoted by ω without a subscript. It may be mentioned that α is often called the *damping constant.*

Before proceeding, the status of the problem under consideration should be recapitulated. The steady-state solution is given by Eqs. 83 and 84. The initial values of the transient charge and current are given by Eqs. 233a and 234a, while the form of the transient is given by Eq. 236, for which the two values of p are those of Eqs. 238e and 238f. For the transient component of charge one may assume an expression containing two terms, one for each value of p, which provides two arbitrary constants. Thus

$$q_t = \frac{\mathrm{B}}{2}\,\epsilon^{(-\alpha+j\omega_d)t} + \frac{\mathrm{D}}{2}\,\epsilon^{(-\alpha-j\omega_d)t}, \qquad [247]$$

in which B/2 and D/2 are arbitrary constants to be evaluated from the initial conditions. Here B/2 and D/2 are written, rather than B and D, merely in order that the 2's shall disappear from forms which are used more often subsequently. That this expression satisfies Eq. 235 can be verified readily by direct substitution. From Eq. 247 differentiation gives

$$i_t = \frac{dq_t}{dt} = (-\alpha + j\omega_d)\,\frac{\mathrm{B}}{2}\,\epsilon^{(-\alpha+j\omega_d)t} + (-\alpha - j\omega_d)\,\frac{\mathrm{D}}{2}\,\epsilon^{(-\alpha-j\omega_d)t}. \qquad [248]$$

Equations 247 and 248 are the general forms of the transient charge and current.

The constants B and D can be evaluated from the known values of $q_t(0+)$ and $i_t(0+)$ as given by Eqs. 233a and 234a. Thus putting t equal to zero in Eqs. 247 and 248 gives

$$q_t(0+) = \frac{\mathrm{B}}{2} + \frac{\mathrm{D}}{2}, \qquad [249]$$

$$i_t(0+) = (-\alpha + j\omega_d)\,\frac{\mathrm{B}}{2} + (-\alpha - j\omega_d)\,\frac{\mathrm{D}}{2}, \qquad [250]$$

in which B and D are the only unknown quantities. These are easily solved for B and D. It is interesting that B and D are conjugate complex

numbers; that is, their real parts are equal, while their imaginary parts are equal in magnitude but opposite in sign. This latter is required by the fact that both $q_t(0)$ and $i_t(0)$ must be real. In rectangular and polar form,

$$\mathrm{B} = B_1 + jB_2 = B\underline{/\delta}, \tag{251}$$

$$\mathrm{D} = B_1 - jB_2 = B\underline{/-\delta}, \tag{252}$$

in which

$$B = \sqrt{B_1^2 + B_2^2}, \tag{253}$$

$$\delta = \tan^{-1}\frac{B_2}{B_1}. \tag{254}$$

If Eq. 247 is written using Eqs. 251 and 252 and $\epsilon^{-\alpha t}$ is factored out, the transient component of charge appears as

$$\left.\begin{aligned} q_t &= \tfrac{1}{2}\epsilon^{-\alpha t}[(B_1 + jB_2)\epsilon^{j\omega_d t} + (B_1 - jB_2)\epsilon^{-j\omega_d t}] \\ &= \tfrac{1}{2}\epsilon^{-\alpha t}[B\underline{/\delta}\epsilon^{j\omega_d t} + B\underline{/-\delta}\epsilon^{-j\omega_d t}]. \end{aligned}\right\} \tag{255}$$

Inspection of these expressions shows the significant fact that the two terms in the bracket are conjugate complex functions; that is, the first term has a magnitude B and makes an angle $\delta + \omega_d t$ with the axis of reals, while the second term has the same magnitude but makes an angle $-(\delta + \omega_d t)$ with the axis of reals. Since these two terms are conjugates, their sum is real and has a value of twice the real part of either. This fact allows the expression for q_t in even simpler form. Thus one can write

$$q_t = \epsilon^{-\alpha t}\mathcal{R}_e[(B_1 + jB_2)\epsilon^{j\omega_d t}] = \epsilon^{-\alpha t}\mathcal{R}_e[\mathrm{B}\epsilon^{j\omega_d t}]. \tag{247a}$$

The transient current can be obtained directly from Eq. 247a by differentiation, remembering that differentiation and taking the real part of a complex function are commutative operations. First the real function $\epsilon^{-\alpha t}$ is put inside the brackets, as can be done since $\epsilon^{-\alpha t}$ is merely a real coefficient. Taking the real part and multiplying by a real coefficient are also commutative operations when applied to a complex function. Thus

$$\left.\begin{aligned} i_t &= \frac{dq_t}{dt} = \frac{d}{dt}\mathcal{R}_e[\mathrm{B}\epsilon^{(-\alpha + j\omega_d)t}] = \mathcal{R}_e[(-\alpha + j\omega_d)\mathrm{B}\epsilon^{(-\alpha+j\omega_d)t}] \\ &= \epsilon^{-\alpha t}\mathcal{R}_e[(-\alpha + j\omega_d)\mathrm{B}\epsilon^{j\omega_d t}]. \end{aligned}\right\} \tag{248a}$$

Equations 247a and 248a are good forms for numerical calculation of the transient components of charge and of current. A simple procedure for finding B_1 and B_2 in any given case is to recall that q_t can be written in the form Eq. 247a, considering B as a complex constant still to be determined. Then by setting t equal to zero in Eqs. 247a and 248a and equating the resulting expressions to the known initial values Eqs. 233a

and 234a, the two components of B are readily found. Thus

$$q_t(0) = q(0-) - Q_s = \mathcal{R}_e[\mathrm{B}] = B_1, \tag{256}$$

$$\left.\begin{aligned} i_t(0) &= i(0-) - I_s = \mathcal{R}_e[(-\alpha + j\omega_d)(B_1 + jB_2)] \\ &= -\alpha B_1 - \omega_d B_2, \end{aligned}\right\} \tag{257}$$

from which B_1 and B_2 are readily calculated.

The complete expressions for charge and current can now be written as

$$q = Q_s + q_t(t) = \frac{E}{S} + \epsilon^{-\alpha t}\mathcal{R}_e[\mathrm{B}\epsilon^{j\omega_d t}], \tag{258}$$

$$i = I_s + i_t(t) = 0 + \epsilon^{-\alpha t}\mathcal{R}_e[(-\alpha + j\omega_d)\mathrm{B}\epsilon^{j\omega_d t}]. \tag{259}$$

Equation 247a for the transient component of charge is often written in the trigonometric form obtained as follows:

$$\left.\begin{aligned} q_t &= \epsilon^{-\alpha t}\mathcal{R}_e[B\underline{/\delta}\,\epsilon^{j\omega_d t}] = \epsilon^{-\alpha t}\mathcal{R}_e[B\epsilon^{j(\omega_d t+\delta)}] \\ &= B\epsilon^{-\alpha t}\cos(\omega_d t + \delta). \end{aligned}\right\} \tag{247b}$$

The similar change for the transient component of current is

$$\left.\begin{aligned} i_t &= \epsilon^{-\alpha t}\mathcal{R}_e[\omega_0 B\underline{/\delta + \gamma}\,\epsilon^{j\omega_d t}] = \epsilon^{-\alpha t}\mathcal{R}_e[\omega_0 B\epsilon^{j(\omega_d t+\delta+\gamma)}] \\ &= \omega_0 B\epsilon^{-\alpha t}\cos(\omega_d t + \delta + \gamma), \end{aligned}\right\} \tag{248b}$$

in which γ is defined by

$$\gamma \equiv \tan^{-1}\frac{\omega_d}{-\alpha}\,. \tag{260}$$

Generally it is found that the exponential forms for the transient components as given in Eqs. 258 and 259 are the simplest forms to use for the determination of the constants and for all analytical work. For numerical calculation the exponential forms are readily changed to the trigonometric forms as shown.

The angle γ is that of the complex number $-\alpha + j\omega_d$. This number lies always in the second quadrant because its real part is negative for any passive network, and ω_d is a positive number. If the real part $(-\alpha)$ of $-\alpha + j\omega_d$ were positive, the exponential term $\epsilon^{-\alpha t}$ would increase without limit. This fact, in turn, implies a current or charge that grows without limit, an obvious absurdity for passive networks. It may be remarked that the analysis of networks containing vacuum tubes, which may serve as concealed energy sources, sometimes leads to roots of the characteristic equation that have positive real parts.

21. Vector interpretation of solution

The solutions for series circuits containing inductance and elastance without resistance and with resistance — make possible some very interesting and instructive comparisons. For this purpose the exponential, or vector, form of the results is most effective. Furthermore, it is convenient to divide the solutions into their steady-state and transient components, since, for this circuit, resistance has no effect on the steady-state component, whereas it changes the transient component in a significant way.

For making the comparisons Eqs. 173, 181, 258, and 259 are used primarily for the cases without and with resistance. The steady-state components Q_s and I_s are identical in the two circumstances, a point that is further discussed subsequently. This being so, attention is turned primarily to the transient components. These are repeated here, those for zero resistance and finite resistance being indicated by added subscripts 0 and R, respectively.

$$\left.\begin{aligned} q_{t0} &= \mathcal{R}_e[\mathbf{B}_0\epsilon^{j\omega_0 t}] && [261]\\ i_{t0} &= \mathcal{R}_e[j\omega_0\mathbf{B}_0\epsilon^{j\omega_0 t}] && [262]\end{aligned}\right\}\quad \text{for } LS \text{ circuit,}$$

$$\left.\begin{aligned} q_{tR} &= \mathcal{R}_e[\epsilon^{-\alpha t}\mathbf{B}_R\epsilon^{j\omega_d t}] && [263]\\ i_{tR} &= \mathcal{R}_e[\epsilon^{-\alpha t}(-\alpha + j\omega_d)\mathbf{B}_R\epsilon^{j\omega_d t}] && [264]\end{aligned}\right\}\quad \text{for } RLS \text{ circuit.}$$

As the first point in the comparison it should be observed that as α approaches zero in Eqs. 247a and 248a, these equations reduce immediately to Eqs. 261 and 262. Thus $\mathbf{B}_R$ approaches $\mathbf{B}_0$, ω_d approaches ω_0, and $\epsilon^{-\alpha t}$ approaches one as α approaches zero. This is to be expected, for there is no essential or sudden change in the circuit as its resistance is made smaller and smaller and reaches zero. A more detailed study of how the introduction of a small amount of resistance into the LS circuit alters its behavior shows three effects: First — and most important — appears the *damping factor* $\epsilon^{-\alpha t}$, which makes the entire transient shrink in size exponentially with time. Thus the RLS-circuit transients decrease with time and eventually disappear because of this damping factor $\epsilon^{-\alpha t}$, whereas the LS-circuit "transients" persist indefinitely with undiminished amplitude. The second effect of the introduction of resistance is to reduce the angular frequency of oscillation from ω_0 to ω_d, a very minor change unless the circuit is rather highly damped. A third and rather minor effect is the change in the coefficients from $\mathbf{B}_0$ and $j\omega_0\mathbf{B}_0$ to $\mathbf{B}_R$ and $(-\alpha + j\omega_0)\mathbf{B}_R$, respectively. This alters somewhat the initial magnitudes and angles of the time vectors, of which the real parts are the transient charge and current, respectively. Since this third aspect is rela-

tively obvious and also unimportant as far as the nature of the transient is concerned, it is not discussed in further detail. The first two, however, merit further attention.

The first effect of resistance, namely, the introduction of the damping factor $\epsilon^{-\alpha t}$, can be seen graphically by comparing the charge time vectors for the *LS* and *RLS* circuits. Since for each circuit the current time vector differs from the charge time vector only by containing an additional constant complex factor, for simplicity only the charge time vectors are considered.

For concreteness two very simple cases are compared quantitatively, in both of which $q_t(0)$, $i_t(0)$, and ω_0 are each unity but in one of which α is zero, while in the other α is 0.2. Plots of the charge time vectors in the

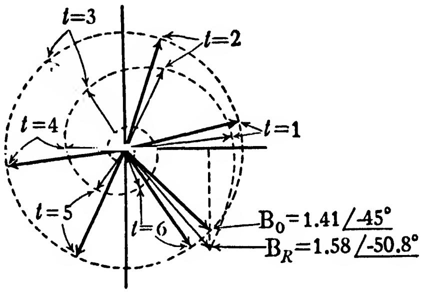


FIG. 24. Interpretation of transient component of charge in underdamped series *RLS* circuit in terms of shrinking vector.

complex plane, as calculated by Eqs. 261 and 263, using these values, are shown in Fig. 24. Here the locus of the tip of the charge time vector is seen to be a circle for α equal to zero, that is, for the case of zero circuit resistance. This vector rotates at the rate of ω_0, or one radian per second, or 57.3 degrees per second. It continues to do so indefinitely or until some change is made in the circuit. Its length meanwhile remains unaltered. In contrast, the tip of the charge time vector for the case of α equal to 0.2 traces a spiral locus. That is, this vector as it rotates also decreases in length. In unit time it rotates through an angle of ω_d or 0.98 radian, or 56.1 degrees, which corresponds to a slightly lower speed than that of the first vector. It also decreases in length by the end of each unit time interval to $\epsilon^{-0.2}$, or 0.819 of its value at the beginning of the interval.

The initial vectors B_0 and B_R, while having different magnitudes and angles, have the same projection on the axis of reals. That they must follows from the fact that the initial value of the transient component of charge is the same in both cases.

The contrast between the charge time vectors for the circuits with and

without resistance can also be seen by comparing the velocities of the tips of the two charge time vectors.

For the LS case, the current time vector, $j\omega_0 B_0 \epsilon^{j\omega_0 t}$, is the time derivative of the charge time vector, $B_0 \epsilon^{j\omega_0 t}$. The current time vector therefore may be interpreted as the tip velocity of the charge time vector. As illustrated in Fig. 25a, the tip of the charge vector moves in a direction 90 degrees ahead of the direction of the charge time vector, with a linear speed ω_0 times the length of the time charge vector.

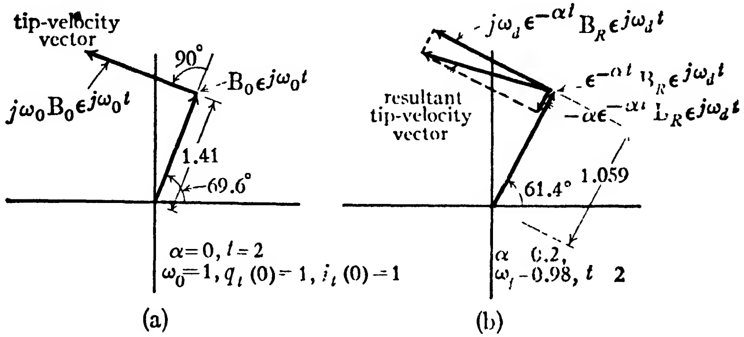


FIG. 25. Vector interpretation of transient component of charge in LS and RLS circuits.

For the RLS case, the current time vector $\epsilon^{-\alpha t}(-\alpha + j\omega_d)B_R\epsilon^{j\omega_d t}$ likewise is the time derivative of the charge time vector, $\epsilon^{-\alpha t}B_R\epsilon^{j\omega_d t}$, and may be interpreted as its tip velocity. Here the tip velocity may be conceived as made up of two components, $-\alpha\epsilon^{-\alpha t}B_R\epsilon^{j\omega_d t}$, which is α times the length of the charge time vector and displaced 180 degrees from it, and $j\omega_d\epsilon^{-\alpha t}B_R\epsilon^{j\omega_d t}$, which is ω_d times the length of the charge time vector and is in a direction 90 degrees ahead of it. These are illustrated in Fig. 25b. The first represents a component which causes the charge time vector to shrink in length at a rate that is always α times as large as the vector. The significance of $1/\alpha$ as a time constant is discussed subsequently in this article. The second is essentially the same as the actual tip velocity of the charge time vector for the LS case and may be called the tangential component. This component differs from the tip velocity of the charge time vector for the LS case because ω_d is slightly smaller than ω_0 and also because the charge time vector for the RLS case is itself changing in length, and its tip tangential velocity is proportional to its length as well as to ω_d. Since

$$\omega_0 = \sqrt{\omega_d^2 + \alpha^2}, \qquad [241b]$$

the magnitude of the actual tip velocity of the charge time vector for the

RLS case is ω_0 times its instantaneous magnitude, as in the *LS* case. For the *RLS* case the tip has an inwardly directed radial component, however, whereas in the *LS* case the tip moves tangentially only.

The effect of adding resistance on the behavior of the charge functions themselves is now considered. As shown by Eqs. 261 to 264 and by the preceding developments of the vector representation of functions, the charge and current as functions of time are merely the real parts of the corresponding time vectors. For the two examples under consideration, therefore, the charge functions are

$$q_{t0} = \mathcal{R}_e[\mathrm{B}_0\epsilon^{j\omega_0 t}] = 1.41 \cos (1.00t - 45°), \tag{261a}$$

$$q_{tR} = \mathcal{R}_e[\epsilon^{-\alpha t}\mathrm{B}_R\epsilon^{j\omega_d t}] = 1.58\epsilon^{-0.2t} \cos (0.98t - 50.8°). \tag{263a}$$

In these expressions the arguments of the cosine functions are given in mixed units, $1.00t$ and $0.98t$ being in radians and the phase angles -45

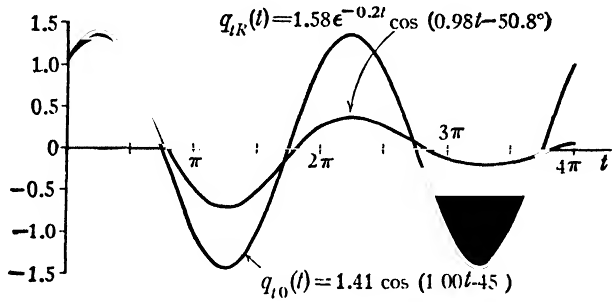


FIG. 26 Charge time functions for *LS* and *RLS* circuits.

and -50.8 being in degrees. While this form is inconsistent mathematically, it may be justified as a matter of convenience provided no ambiguity arises regarding the meaning. These functions are plotted in Fig. 26.

Another useful comparison showing the effect of resistance in this circuit is made in Fig. 27. Here ω_0 and $q_t(0)$ are each unity, $i_t(0)$ is zero, and α takes the various values indicated. In addition, a curve is shown for the critically damped case (discussed in the next article) together with curves for overdamped cases. For the values of α that are less than ω_0, that is, for the underdamped cases, the charge curve is a sinusoidal function modified at every instant by the factor $\epsilon^{-\alpha t}$. Thus the curves $\pm\epsilon^{-\alpha t}$ may be said to be an envelope inside which the sinusoid is confined. Such a sinusoid with exponentially decreasing amplitude is called a *damped oscillation*. If the amplitude decreases relatively little per cycle, the oscillations are said to be *slightly damped*, or the circuit in which they occur is said to be *highly oscillatory*. This condition obtains when α is small compared to ω_0, or when ω_0 and ω_d differ by a negligible amount.

On the other hand, if the oscillations substantially disappear after a few cycles, the oscillations and the circuit are said to be *highly damped*.

The coefficient α may well be thought of as the reciprocal of a time constant. Thus $1/\alpha$, or $2L/R$, which has the dimensions of time, is the time required for the charge time vector to decrease its length to $1/\epsilon$, or 0.368, of its initial value. From the nature of the exponential function it is evident that the vector length decreases by this same factor during any time interval of the same length. This is illustrated by the shrinking vector of Fig. 24. Thus when t is six its length is $1/\epsilon$ of its length when

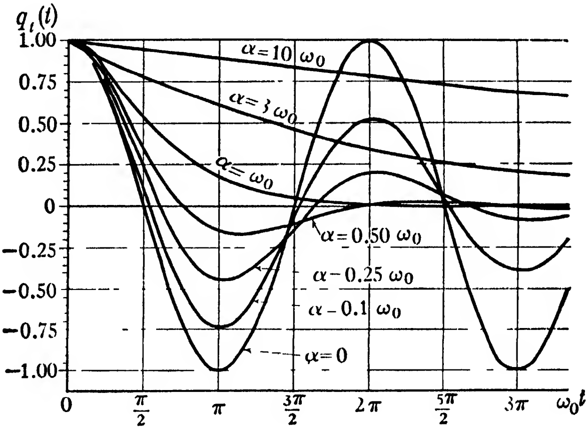


FIG. 27. Plot showing the effect of various values of R on $q_t(t)$ in a series RLS circuit having constant L and S.

t is one, or when t is ten its length is $1/\epsilon$ of its length when t is five. The same effect of α is seen also in Fig. 27. Here the factor $\epsilon^{-\alpha t}$ causes the envelope inside which the charge function oscillates to decrease in height as time increases. The statements previously made concerning the effect of the $\epsilon^{-\alpha t}$ factor on the length of the charge time vector apply without change to its effect on the height of the envelope. Thus $1/\alpha$ is the time during which the envelope decreases in height by the factor $1/\epsilon$ equal to 0.368. This condition can be seen more readily perhaps if q_{tR} is rewritten with the $\epsilon^{-\alpha t}$ term outside the bracket as

$$q_{tR} = \epsilon^{-\alpha t}\mathcal{R}_e[\mathrm{B}_R\epsilon^{j\omega_d t}]. \qquad [263b]$$

Thus the real part of the rotating vector $\mathrm{B}\epsilon^{j\omega_d t}$, which is a sinusoidal function of time, is multiplied by the shrinking function $\epsilon^{-\alpha t}$.

Another useful interpretation of the time constant $1/\alpha$ is obtained as

follows: At any given time t_1, the function $\epsilon^{-\alpha t}$ is changing at the rate

$$\frac{d}{dt}\epsilon^{-\alpha t}\Big|_{t=t_1} = -\alpha\epsilon^{-\alpha t_1} \qquad [265]$$

units per second. If this rate of change, which is a decrease, were to remain constant, the value of the function $\epsilon^{-\alpha t}$ would be reduced to zero in a time Δt such that

$$\epsilon^{-\alpha t_1} - \alpha\epsilon^{-\alpha t_1}\Delta t = 0 \qquad [266]$$

or

$$\Delta t = \frac{1}{\alpha}. \qquad [266a]$$

Stated in words, $1/\alpha$ is the time that would be required for the transient to disappear if its rate of decrease at any given instant were to continue unchanged. Thus the envelope for α equal to 0.2 in Fig. 26 has a slope at every point such that if this slope remained constant the height of the envelope would be reduced to zero in a time $1/\alpha$, or five seconds. For the *RLS* circuit $1/\alpha$ equal to $2L/R$ is exactly twice the time constant for an *RL* circuit having the same L and R.

As stated near the beginning of this article, the principal effect of resistance in an oscillatory *RLS* circuit is the reduction in the amplitude of the oscillations by the factor $\epsilon^{-\alpha t}$. The second effect mentioned, which is the reduction in the characteristic angular frequency from ω_0 to ω_d radians per second, is discussed next.

In an *RLS* circuit that is to any appreciable extent oscillatory — that is, one in which oscillations of appreciable magnitude persist for at least a few cycles — ω_d is found to differ but little from ω_0. This statement can be verified by considering a particular case. If α and ω_d are such that during one cycle, that is, the interval $\omega_d t$ equal to 2π, the amplitude decreases to $1/\epsilon$ of its value, then

$$t = 1/\alpha = 2\pi/\omega_d \qquad [266b]$$

or

$$\alpha = \omega_d/2\pi, \qquad [266c]$$

and

$$\omega_0 = \sqrt{\omega_d^2 + \alpha^2} = \omega_d\sqrt{1 + \left(\frac{1}{2\pi}\right)^2} = 1.013\omega_d. \qquad [241c]$$

Thus an oscillation so highly damped that it decreases by nearly two-thirds in each cycle has an angular frequency differing from the undamped frequency by only slightly over one per cent. It may be concluded therefore that the effect of resistance on the characteristic angular velocity ω_d is small as long as the circuit is distinctly oscillatory, the ratio of ω_0 and ω_d in this case being very nearly unity.

22. Case III: Critically damped circuit

While the critically damped case of the RLS circuit might seem to be a rather special one which merely separates the underdamped from the overdamped, it is in reality of practical importance, especially in numerous analogous mechanical systems such as those of indicating instruments. The results for this case are readily obtained by taking a limit in either of the other two cases.

An RLS circuit is said to be critically damped if

$$\left(\frac{R}{2L}\right)^2 = \frac{S}{L}. \tag{267}$$

Since a single exponential term for the transient component of charge provides only one arbitrary constant instead of the necessary two, it appears that for this particular case a modified procedure must be used. The simplest way is that suggested in the preceding paragraph; namely, to take the limit of the underdamped solution as ω_d approaches zero, or as α approaches ω_0.

By solving Eqs. 256 and 257 for B_1 and B_2, one sees that the imaginary term in Eq. 268 apparently approaches infinity as α approaches ω_0 because ω_d approaches zero in the denominator of B_R:

$$\mathrm{B}_R\epsilon^{j\omega_d t} = \left\{q_t(0) - \frac{j}{\omega_d}\left[i_t(0) + \alpha q_t(0)\right]\right\}\epsilon^{j\omega_d t} \tag{268}$$

Further consideration shows that the angle of B_R in this equation at the same time approaches $\pm\pi/2$ [depending on the signs of the numerical values of $i_t(0)$ and $q_t(0)$]. Since the form looks somewhat formidable to evaluate, recourse is had to Eq. 247; the imaginary parts of B and D are opposite in sign and may lead to a form more simply evaluated than Eq. 268. Putting the value of B_R from Eq. 268 in Eq. 247 gives

$$\left.\begin{aligned} 2q_t &= \epsilon^{-\alpha t}\left[\left\{q_t(0) - \frac{j}{\omega_d}\left[i_t(0) + \alpha q_t(0)\right]\right\}\epsilon^{j\omega_d t}\right. \\ &\quad \left. + \left\{q_t(0) + \frac{j}{\omega_d}\left[i_t(0) + \alpha q_t(0)\right]\right\}\epsilon^{-j\omega_d t}\right] \\ &= \epsilon^{-\alpha t}\left\{q_t(0)(\epsilon^{j\omega_d t} + \epsilon^{-j\omega_d t})\right. \\ &\quad \left. - \frac{j}{\omega_d}\left[i_t(0) + \alpha q_t(0)\right](\epsilon^{j\omega_d t} - \epsilon^{-j\omega_d t})\right\}, \end{aligned}\right\} \tag{247c}$$

$$\left.\begin{aligned} q_t &= \epsilon^{-\alpha t}\left\{q_t(0)\cos\omega_d t - \frac{j}{\omega_d}[i_t(0) + \alpha q_t(0)]j\sin\omega_d t\right\} \\ &= \epsilon^{-\alpha t}\left\{q_t(0)\cos\omega_d t + [i_t(0) + \alpha q_t(0)]\frac{\sin\omega_d t}{\omega_d}\right\}. \end{aligned}\right\} \qquad [247d]$$

In the last step but one the trigonometric-exponential relations Eqs. 152 and 158 were used. Equation 247d is readily evaluated as ω_d approaches zero since

$$\lim_{\omega_d \to 0} \cos\omega_d t = 1 \qquad [269]$$

and

$$\lim_{\omega_d \to 0} \frac{\sin\omega_d t}{\omega_d} = t, \qquad [269a]$$

which gives

$$q_t = \epsilon^{-\alpha t}\{q_t(0) + [i_t(0) + \alpha q_t(0)]t\}. \qquad [247e]$$

The limit of $(\sin\omega_d t)/\omega_d$ is readily obtained by l'Hôpital's rule or by expanding the sine function into a series and noting that higher-order terms become negligible for all finite values of t as ω_d approaches zero.

While Eq. 247e may be used directly in solving a problem, it is usually more convenient to express the transient solutions in the form

$$q_t = \epsilon^{-\alpha t}(a_1 + a_2 t), \qquad ▶[270]$$

$$\left.\begin{aligned} i_t &= \frac{d}{dt}q_t = \epsilon^{-\alpha t}a_2 + (a_1 + a_2 t)(-\alpha\epsilon^{-\alpha t}) \\ &= (a_2 - \alpha a_1 - \alpha a_2 t)\epsilon^{-\alpha t} \end{aligned}\right\} \qquad ▶[271]$$

and to evaluate the two unknown real constants a_1 and a_2 from the known initial values of transient charge and current.

The behavior in the above critically damped circuit is in general like that for the overdamped circuit, except that the surge is briefer; in fact, the critical case is the one for which the surge in a nonoscillatory *RLS* circuit is substantially over in the shortest time. Thus, if a condenser is to be discharged through an inductance and a resistance in the minimum time, the resistance should be adjusted to the critical value. There are many electrical, as well as mechanical and electrochemical, devices in which this principle is employed. Thus an instrument (voltmeter, ammeter, or galvanometer, for instance), the needle of which has a tendency to swing back and forth (oscillate) when suddenly excited, may be made to assume its final position most rapidly by introducing mechanical friction or electrical resistance as a dissipative element of sufficient magnitude to

establish the critical condition discussed here. This case is considered in the treatment of devices of similar kind in Ch. XII.

It is well to bear in mind the mechanical analog — a pendulum or a weight on a spring — in connection with the physical interpretation of the circuit here treated. If charge and mechanical displacement are made analogous, the overdamped discharge of an *initially* charged condenser corresponds to the motion of an *initially* displaced pendulum immersed in a liquid of high viscosity. When such a pendulum is raised from its equilibrium position and released, it settles slowly back to its equilibrium position. Similarly, when a condenser is charged and then connected to the *RL* elements, the charge on the condenser slowly passes around the circuit until the condenser is discharged. Thinning the liquid surrounding the pendulum so that the bob descends more swiftly corresponds to reducing the circuit resistance so that the discharge occurs more rapidly. If the liquid is made very thin, the bob oscillates back and forth with decreasing amplitude about the equilibrium point in a way analogous to the way in which the electric charge oscillates if the circuit resistance is made very small. For some critical value of the liquid viscosity the bob just fails to overswing on the first descent. Similarly, for some critical value of circuit resistance, the charge just fails to reverse its motion during the discharge. These two cases in which oscillation just fails to take place are the critically damped cases for the mechanical and electrical systems, respectively.

23. Illustrative example of series *RLS* circuit

In Fig. 28a is shown the circuit diagram of a device known as a Marx surge generator, which is used to apply a very rapidly rising voltage similar to that resulting naturally from lightning or from switching to apparatus for test purposes. Its development and use have greatly facilitated the design of electric-power apparatus with suitable insulation to withstand lightning voltages without injury.

The voltage surge is obtained from numerous condensers that are charged in parallel at a moderate voltage and suddenly and simultaneously discharged in series to yield the high voltage. The charging process is relatively slow as the charging resistors R_C have a relatively high resistance. When the condensers are charged to a predetermined voltage slightly under that at which the spark gaps g break down if undisturbed, a spark is started from the auxiliary electrode a, which immediately causes a spark across the main gap g_1. This immediately results in additional voltage across the second gap which breaks down, and so on. In an extremely short time interval all the spark gaps have become excellent conductors and the combined voltage of all the condensers in series is applied to the remainder of the surge-generator circuit.

The additional part of the circuit consists primarily of the two resistors R_1 and R_2, the apparatus to be tested being connected to the terminals of R_2. Often the capacitance C_2 of the tested apparatus significantly affects the behavior of the surge-generator circuit. In this particular case, however, the specimen is assumed to have negligible capacitance and, prior to breakdown, a negligible conductance. The circuit also unavoidably has a small inductance, which is sometimes increased and which plays a significant part in the circuit operation.

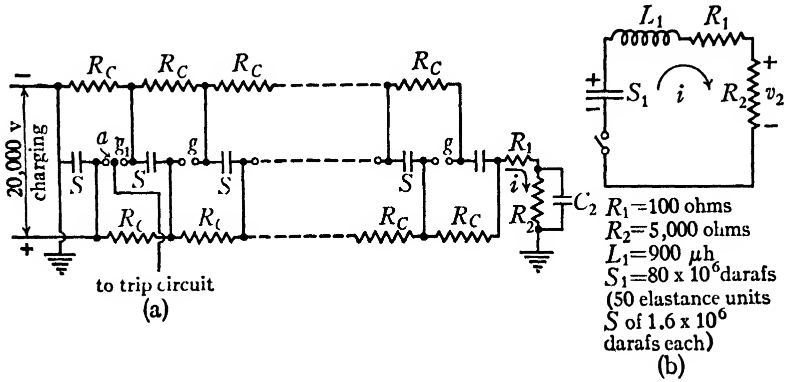


Fig. 28. Circuit diagram for a Marx surge generator.

When the spark gaps are broken down the voltage applied to the test specimen rises very rapidly within a few microseconds at most — to its maximum value and then usually decreases much more slowly. However, in case the test specimen "breaks down" – that is, becomes highly conducting or an arc forms around it in air, the voltage across the specimen decreases rapidly, and the current may become large.

For the purpose of calculating the performance of the surge generator the equivalent circuit of Fig. 28b is found to represent the actual circuit rather accurately. In case the specimen breaks down electrically, R_2 is, in effect, temporarily short-circuited. Ordinarily, the voltage drop in the generator spark gaps can be neglected during the condenser-discharge period, as can the current in the charging resistors. These two effects are therefore not represented in the equivalent circuit.

In this problem the condensers are each charged to 20,000 volts prior to initiating the surge. The specimen is assumed to break down 3.0 microseconds (3.0×10^{-6} second) after the surge is started, and to have negligible resistance during the discharge period following breakdown. The problem is to determine:

(a) the curve of voltage applied to the specimen prior to its breakdown;
(b) the maximum value of this voltage and the time at which it occurs;
(c) the current in the specimen during the test.

While certain of the foregoing assumptions might need to be reconsidered for accurate results in actual practice, the salient features of surge-generator behavior are illustrated by this problem.

Solution: The surge generator described in the problem evidently reduces for analysis to a simple *RLS* circuit, to which the treatment of the preceding articles applies. It is therefore permissible to omit the usual first step, which is to write the differential equation of the circuit, and to proceed at once to the solution of this equation. Following the methods previously developed, one finds, first, the *steady-state component; second, the initial value of the transient component; and, third, the complete* expression for the transient component.

After the tripping of the generator, the circuit is assumed to contain no sources and therefore reaches a steady state with zero condenser charge and zero current. The justification of this assumption of no sources when the 20,000-v charging circuit remains connected is left to the reader. Thus

$$Q_s = 0, \tag{83a}$$

and

$$I_s = 0. \tag{84}$$

Prior to tripping of the generator the resultant elastance S_1 is charged to 50(−20,000), or -1.00×10^6 v, or with a charge

$$q(0-) = \frac{1.00 \times 10^6}{80 \times 10^6} = -0.0125 \text{ coulomb}, \tag{272}$$

while

$$i(0-) = 0. \tag{273}$$

The condenser is charged negatively in order that the discharge current shall be in the positive direction. Therefore the initial values of the transient components are

$$q_t(0+) = q(0-) - Q_s = -0.0125 \text{ coulomb}, \tag{233b}$$

and

$$i_t(0+) = i(0-) - I_s = 0 \text{ amp}. \tag{234b}$$

Next the nature of the transient for the time interval prior to the breakdown of the specimen is determined; that is, whether it is under or over or critically damped is ascertained by calculating α and ω_0:

$$\alpha = \frac{R_1 + R_2}{2L} = \frac{5100}{2 \times 900 \times 10^{-6}} = 2.83 \times 10^6 \text{ sec}^{-1}, \tag{240a}$$

$$\omega_0 = \sqrt{\frac{S}{L}} = \sqrt{\frac{80 \times 10^6}{900 \times 10^{-6}}} = 10^6\sqrt{0.0890} = 2.98 \times 10^5 \text{ radians/sec.} \tag{94a}$$

Hence α is greater than ω_0, the circuit is overdamped, and the solution has the general form

$$q(t) = q_t(t) = A_1\epsilon^{(-\alpha+\beta)t} + A_2\epsilon^{(-\alpha-\beta)t}, \tag{242}$$

$$i(t) = i_t(t) = (-\alpha + \beta)A_1\epsilon^{(-\alpha+\beta)t} + (-\alpha - \beta)A_2\epsilon^{(-\alpha-\beta)t}. \tag{243}$$

Proceeding with the evaluation of p_1 and p_2,

$$\beta = \sqrt{\alpha^2 - \omega_0^2} = 10^6\sqrt{2.83^2 - 0.298^2} = 2.82 \times 10^6, \tag{241d}$$

$$p_1 = -\alpha + \beta = -\alpha + (\alpha^2 - \omega_0^2)^{1/2} \approx -\alpha + (\alpha - \tfrac{1}{2}\alpha^{-1}\omega_0^2 + \cdots), \tag{238g}$$

$$p_1 \approx -\frac{\omega_0^2}{2\alpha} = -\frac{2.98^2 \times 10^{10}}{2 \times 2.83 \times 10^6} = -1.58 \times 10^4, \tag{238h}$$

$$p_2 = -\alpha - \beta = -5.65 \times 10^6. \tag{238i}$$

In evaluating p_1 the first two terms in the binomial expansion of $\sqrt{\alpha^2 - \omega_0^2}$ are used to avoid taking the difference of two nearly equal numbers.

Just after tripping the generator,

$$q(0+) = q(0-) = -0.0125 = A_1 + A_2, \tag{274}$$

$$i(0+) = 0 = -1.58 \times 10^4 A_1 - 5.65 \times 10^6 A_2. \tag{275}$$

Solving simultaneously,

$$A_1 = -1.25 \times 10^{-2}, \tag{276}$$

$$A_2 = +3.51 \times 10^{-5}, \tag{277}$$

and

$$q(t) = -0.0125\epsilon^{-1.58\times10^4 t} + 3.51 \times 10^{-5}\epsilon^{-5.65\times10^6 t} \text{ coulomb}, \tag{278}$$

$$i(t) = 198\epsilon^{-1.58\times10^4 t} - 198\epsilon^{-5.65\times10^6 t} \text{ amp}. \tag{279}$$

The voltage drop v_2 across the specimen is $5{,}000i$,

$$v_2 = 5{,}000i = +0.991 \times 10^6\epsilon^{-1.58\times10^4 t} - 0.991 \times 10^6\epsilon^{-5.65\times10^6 t} \text{ v}. \tag{280}$$

To determine the time at which v_2 is maximum,

$$\frac{di}{dt} = 0 = (-\alpha + \beta)^2 A_1\epsilon^{(-\alpha+\beta)t} + (-\alpha - \beta)^2 A_2\epsilon^{(-\alpha-\beta)t}, \tag{281}$$

$$\frac{\epsilon^{(-\alpha+\beta)t}}{\epsilon^{(-\alpha-\beta)t}} = \epsilon^{2\beta t} = -\frac{(-\alpha - \beta)^2 A_2}{(-\alpha + \beta)^2 A_1}, \tag{282}$$

$$t = \frac{1}{2\beta}\ln\frac{-(-\alpha - \beta)^2 A_2}{(-\alpha + \beta)^2 A_1}. \tag{283}$$

For this particular case, from $i(0+)$ equal to zero,

$$\frac{A_1}{A_2} = -\frac{(-\alpha - \beta)}{(-\alpha + \beta)}; \tag{284}$$

hence

$$t = \frac{1}{2\beta}\ln\frac{-\alpha - \beta}{-\alpha + \beta} = \frac{1}{2 \times 2.82 \times 10^6}\ln\frac{-5.65 \times 10^6}{-1.58 \times 10^4} = 1.042 \times 10^{-6} \text{ sec}. \tag{285}$$

Putting this value of t in the expression for v_2 gives for the maximum value of v_2,

$$\left.\begin{aligned} v_{2max} &= +0.991 \times 10^6\epsilon^{-1.58\times10^4\times1.042\times10^{-6}} - 0.991 \times 10^6\epsilon^{-5.65\times10^6\times1.042\times10^{-6}} \\ &= 0.991 \times 10^6(+\epsilon^{-0.0165} - \epsilon^{-5.89}) \\ &= 0.991 \times 10^6 \times 0.980 = 0.971 \times 10^6 \text{ v}. \end{aligned}\right\} \tag{286}$$

When t equals 3×10^{-6} sec, R_2 becomes zero because of the breakdown of the specimen; hence for t greater then 3×10^{-6} sec a new solution must be worked out:

$$\omega_0 = 2.98 \times 10^5 \text{ as before,} \tag{94b}$$

$$\alpha = \frac{R_1}{2L} = \frac{100}{2 \times 900 \times 10^{-6}} = 5.56 \times 10^4. \tag{240b}$$

Hence ω_0 is greater than α, and the circuit is oscillatory.

For convenience the time

$$t' = t - 3 \times 10^{-6} \tag{287}$$

is measured from the instant of breakdown for this new solution. Again the steady-state components of charge and current are zero, so that

$$q(t') = q_t(t') \tag{288}$$

and

$$i(t') = i_t(t'). \tag{289}$$

The initial values $q_t(t' = 0)$ and $i_t(t' = 0)$ are found as follows, it being remembered that neither condenser charge nor current can change instantaneously in this circuit:

$$q_t(t' = 0) = q(t = 3 \times 10^{-6}) \tag{290}$$

and

$$i_t(t' = 0) = i(t = 3 \times 10^{-6}). \tag{291}$$

Evaluating these initial values,

$$\left.\begin{aligned} q(t = 3 \times 10^{-6}) &= -1.25 \times 10^{-2}\epsilon^{-1.58\times10^4\times3\times10^{-6}} \\ &\quad + 3.51 \times 10^{-5}\epsilon^{-5.65\times10^6\times3\times10^{-6}} \\ &= -1.20 \times 10^{-2} \text{ coulomb,} \end{aligned}\right\} \tag{290a}$$

$$\left.\begin{aligned} i(t = 3 \times 10^{-6}) &= 198\,(+\epsilon^{-1.58\times10^4\times3\times10^{-6}} - \epsilon^{-5.65\times10^6\times3\times10^{-6}}) \\ &= +189 \text{ amp.} \end{aligned}\right\} \tag{291a}$$

Since the transient is underdamped or oscillatory, the charge and current can be assumed to have the form

$$q(t') = q_t(t') = \mathcal{R}_e[\mathrm{B}\epsilon^{(-\alpha+j\omega_d)t'}], \tag{247f}$$

$$i(t') = i_t(t') = \mathcal{R}_e[(-\alpha + j\omega_d)\mathrm{B}\epsilon^{(-\alpha+j\omega_d)t'}], \tag{248c}$$

$$\mathrm{B} = B_1 + jB_2, \tag{251a}$$

B_1 and B_2 being real constants. These can be evaluated after calculating ω_d.

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 10^4\sqrt{29.8^2 - 5.56^2} = 2.93 \times 10^5. \tag{241c}$$

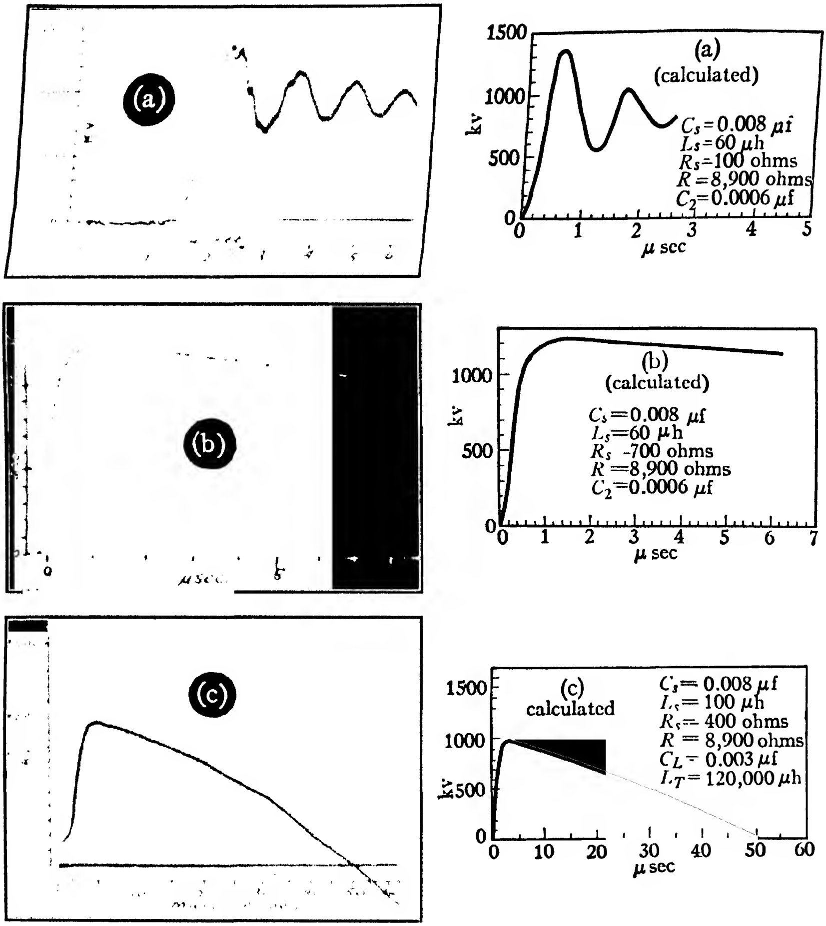
When the specimen breaks down,

$$q_t(t' = 0) = -0.0120 = B_1, \tag{292}$$

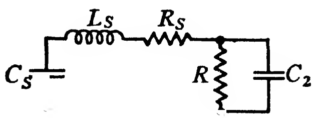
$$\left.\begin{aligned} i_t(t' = 0) &= 189 = \mathcal{R}_e[(-\alpha + j\omega_d)(B_1 + jB_2)] = -\alpha B_1 - \omega_d B_2 \\ &= 5.56 \times 10^4 B_1 - 29.3 \times 10^4 B_2. \end{aligned}\right\} \tag{293}$$

From these equations,

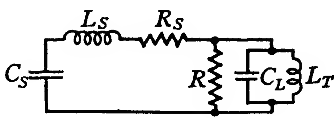
$$B_1 = -0.0120 \tag{294}$$



Oscillograms of output voltage of surge generator used to study the effect of change in parameter values and to justify the prediction of performance of the actual complicated circuits from simplified approximate equivalents: (a) no load on generator except stray capacitance to ground, no external series resistance added to condenser bank C_s; (b) same as (a) but with external series resistance of 600 ohms added in R_s in order to render discharge nonoscillatory; (c) 42,000 kilovolt-ampere transformer load, 300 ohms external series resistance added in R_s. [Taken from P. L. Bellaschi, " Characteristics of Surge Generators for Transformer Testing," *A.I.E.E. Trans.*, LI(1932), 936–951, with permission of author and publisher.]



Equivalent circuit for conditions of oscillograms (a) and (b)



Equivalent circuit for conditions of oscillogram (c)

and

$$B_2 = 1.63 \times 10^{-3}, \tag{295}$$

$$\mathrm{B} = -0.0120 + j1.63 \times 10^{-3} = 0.0121\underline{/172.3^\circ}. \tag{296}$$

For plotting, the trigonometric form is useful. Evaluating the complex coefficient in $i(t')$ gives

$$\left.\begin{aligned}(-\alpha + j\omega_d)\mathrm{B} &= \left(\omega_0 \underline{/\tan^{-1}\frac{\omega_d}{-\alpha}}\right)\mathrm{B}\\ &= \left(29.8 \times 10^4 \underline{/\tan^{-1}\frac{29.3}{-5.56}}\right)(0.0121\underline{/172.3^\circ})\\ &= 3{,}600\underline{/273^\circ}.\end{aligned}\right\} \tag{297}$$

Hence

$$\left.\begin{aligned}q(t') &= \mathcal{R}_e[0.0121\underline{/172.3^\circ}\epsilon^{-5.56\times10^4t'}\epsilon^{j2.93\times10^5t'}]\\ &= 0.0121\epsilon^{-5.56\times10^4t'}\cos(2.93\times10^5t' + 172.3^\circ)\text{ coulomb,}\end{aligned}\right\} \tag{298}$$

$$\left.\begin{aligned}i(t') &= \mathcal{R}_e[3{,}600\underline{/273.0^\circ}\epsilon^{-5.56\times10^4t'}\epsilon^{j2.93\times10^5t'}]\\ &= 3{,}600\epsilon^{-5.56\times10^4t'}\cos(2.93\times10^5t' + 273.0^\circ)\text{ amp.}\end{aligned}\right\} \tag{299}$$

These results together with those for the preceding interval are shown in Fig. 29. The voltage v_2 shown in the enlarged portion of the figure is usually of principal interest.

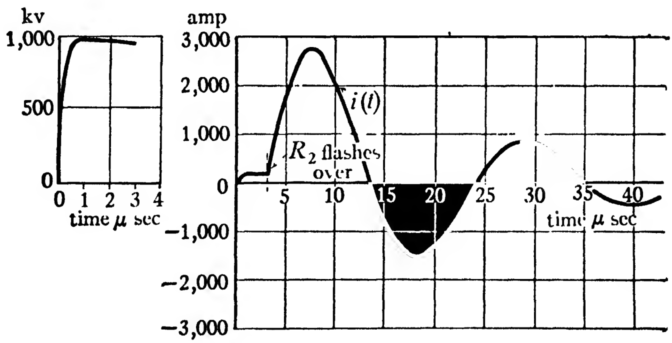


FIG. 29. Plots of current i and voltage v_2 in circuit of Fig. 28b.

24. SIMPLE PARALLEL CIRCUITS, DUALS

Thus far in this chapter only simple series combinations of the circuit elements — resistance, inductance, and elastance — are considered. For certain purposes it proves convenient to deal with a simple parallel combination of three single elements, as in the equivalent electric circuits of electromechanical systems of Ch. XII and in certain aspects of advanced circuit theory. The parallel circuit consisting of three different

single elements has a much less practical utility than the corresponding series circuit because of the unavoidable and usually significant series resistance that accompanies the inductance element. For this reason the circuit about to be considered cannot be realized physically in any but a roughly approximate way with simple apparatus. Because of the aforementioned uses and also because it broadens the circuit concepts, the three-element parallel combination is briefly treated. The two-element conductance-capacitance circuit, it may be mentioned, can be realized physically with good accuracy and is frequently of practical importance.

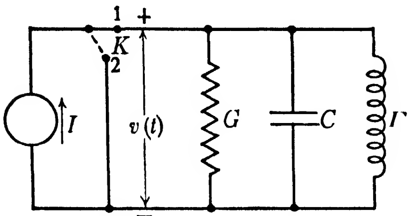


FIG. 30. Parallel $G C \Gamma$ circuit, dual of series RLS circuit

The three-element parallel circuit is shown in Fig. 30. It consists of a conductance G, a capacitance C, and a reciprocal inductance Γ, all connected in parallel. This combination can be connected in series with a source of constant current I or left open-circuited by switch K. Conductance and resistance are merely alternative ways of describing the same element; sometimes one proves more convenient, sometimes the other. In the same way capacitance and elastance are alternative descriptions, and inductance and reciprocal inductance are alternative descriptions.

Because of the simpler form that they give to the equations, constant-strength current sources prove convenient for the analysis of parallel circuits. A constant-strength current source is one that delivers a given current independent of the potential difference across its terminals, in contrast to a constant-strength voltage source which maintains a given difference of potential between its terminals which is independent of the current. Physical sources usually have more nearly the attributes of a constant-strength voltage than of a constant-strength current ideal source.

In the circuit of Fig. 30, switch K is moved from position 2 to position 1, thus suddenly impressing the current I on the parallel combination. From Ch. I it is known that if $v(t)$, which is commonly written v, is the potential drop across these elements, the currents in the elements in the direction of the potential drop are, respectively,

$$i_\Gamma = \Gamma \int v\,dt, \text{ the current in } \Gamma, \qquad [300]$$

$$i_G = Gv, \text{ the current in } G, \qquad [301]$$

$$i_C = C\frac{dv}{dt}, \text{ the current in } C, \qquad [302]$$

$$\int v\,dt \equiv \int_0^t v\,dt + \lambda(0) = \text{actual flux linkages in } \Gamma \text{ at time } t. \qquad [303]$$

The symbol λ is used to indicate flux linkages which are equivalent to a time integral of voltage. By the Kirchhoff current law, the sum of these currents with K in position 1 is I, hence

$$C\frac{dv}{dt} + Gv + \Gamma\int v\,dt = I. \qquad [304]$$

Comparison of Eq. 304 with Eq. 229 shows that they are identical in form but with symbols interchanged as follows:

v for i, λ for q, I for E, C for L, G for R, Γ for S.

The parallelism between these two cases is striking and illustrates a broad conception in circuit theory called *duality*, which can be merely mentioned here. Broadly, two circuits are said to be *duals* or *dual to each other* if their differential equations have the same form but with the coefficients and variables interchanged as indicated above, if series connections are replaced by parallel, voltages by currents, charges by flux linkages, voltage sources by current sources, open circuits by short circuits, loops by nodes, inductance by capacitance, resistance by conductance, and elastance by reciprocal inductance; or vice versa.

The principle of duality is not restricted to electric circuits. For example, in geometry, certain theorems concerning lines and points have dual theorems concerning points and lines; in electrical machinery, electromagnetic and electrostatic devices are often duals. This principle is likely to be useful in at least two ways. The first is that if the analysis of one member of the dual pair is known, it can be applied to the other directly by merely interchanging symbols. This is the way it is used here. The second is that a knowledge of duality may suggest alternative methods of accomplishing a given result at less cost or with more effectiveness, or on the theoretical side it may suggest new theorems or relations. For example, in the design of electric circuits to discriminate for or against certain frequencies of alternating current, the dual of a given circuit may be more economical or have better characteristics.[4]

[4] Further discussion of duality is found in: A. Russell, *A Treatise on the Theory of Alternating Currents* (2d ed.; Cambridge: at the University Press, 1914), Vol. I, Ch. xxi; H. Sire de Vilar, "La Dualité en Électrotechnique," *L'Éclairage Élec.*, XXVII (1901), 252-260; E. A. Guillemin, *Communications Networks* (New York: John Wiley & Sons, 1935), II, 246-254.

Because of the similarity of the differential equations of two circuits that are duals of each other, a solution of one is applicable to the other by merely interchanging symbols. This is seen from what follows:

By returning to Eq. 304 it can be verified by direct substitution that

$$\Gamma \int v\,dt = \Gamma \lambda_s = I \qquad [305]$$

and

$$V_S = 0, \qquad [306]$$

in which λ_s is a constant, are solutions of Eq. 304. Equation 306 is, in fact, the particular integral of Eq. 304 or the steady-state condition of the circuit with I impressed on it.

Next the initial conditions are considered. From the inherent nature of functions it is known that if dv/dt is always finite, v cannot change instantaneously; likewise, if v is always finite, $\int_0^t v\,dt$ cannot change instantaneously. But from Eq. 304 dv/dt is equal to a linear combination of v, $\int v\,dt$, and the constant I; hence dv/dt is itself always finite. It therefore follows that neither v nor $\int v\,dt$ can change instantaneously; or, stated another way, they can change only by an infinitesimal amount during any infinitesimal time dt. If, then, v and $\int v\,dt$ have given values immediately prior to any switching, they have these same values immediately after the switching.

From the physical point of view, v cannot change instantaneously because an instantaneous change of charge on the condenser would thus be involved. But this in turn requires an infinite condenser current, which can be produced only by an infinite voltage, which is not available. Similarly, by Faraday's induction law, the flux linkages $\int v\,dt$ cannot change instantaneously, except with the appearance of an infinite voltage, an impossibility in this circuit. Thus physical reasoning leads to the same conclusions as the mathematical reasoning.

These initial conditions correspond exactly with those imposed upon i and q, respectively, in the series *RLS* circuit. Mathematically, therefore, the solution of the parallel $GC\Gamma$ circuit is identical with that of the series *RLS* circuit as given on p. 222 with the variables interchanged.

With the foregoing ideas in mind the solutions for the $GC\Gamma$ circuit can be written at once by analogy. There are three forms of the result depending on whether the circuit is over, under, or critically damped. As an example, the solution for v in the underdamped case is written. For K

suddenly moved to position 1 when λ is equal to $\lambda(0)$ and v is equal to $v(0)$, analogy with Eq. 248a, gives

$$v = \epsilon^{-\alpha t}\mathcal{R}_e[(-\alpha + j\omega_d)\mathrm{B}\epsilon^{j\omega_d t}], \qquad [307]$$

in which

$$\alpha = \frac{G}{2C}, \qquad [240c]$$

$$\omega_0 = \sqrt{\frac{\Gamma}{C}}, \qquad [94c]$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}, \qquad [241e]$$

$$\mathrm{B} = \lambda_t(0) - \frac{j}{\omega_d}[v_t(0) + \alpha\lambda_t(0)], \qquad [268a]$$

$$\lambda(0+) = \lambda(0-) = \lambda_t(0) + \lambda_s, \qquad [233c]$$

$$v(0+) = v(0-) = v_t(0) + V_s. \qquad [234c]$$

The subscripts have the same significance as they have in the series *RLS* case.

The student may find it instructive to carry out in detail the solution of either this or the simpler two-element parallel circuits, paralleling the analysis given for the series two-element cases.

While the three-element parallel circuit has relatively limited utility, and that principally in more advanced work, the parallel circuit concepts are sometimes useful as shown in the illustrative problems of the following article.

25. Illustrative Example of the Use of Reciprocal Parameters

In order to show how the use of reciprocal parameters often simplifies the solution of a problem, the following illustrative examples are given.

The first problem is that of finding the behavior of the circuit of Fig. 31 after K is suddenly closed.

Solution: Writing Kirchhoff's current law for the node above the capacitance gives

$$(E - v)G_1 = C\frac{dv}{dt} + G_2 v \qquad [308]$$

or

$$C\frac{dv}{dt} + Gv = G_1 E, \qquad [308a]$$

in which

$$G = G_1 + G_2. \tag{309}$$

The form of this differential equation is well known from the solution of the series RL circuit for current and the series RC circuit for charge. From the parallelism, the solution for v is immediately written

$$v = \frac{G_1E}{G_1 + G_2}\left(1 - \epsilon^{-(G/C)t}\right). \tag{310}$$

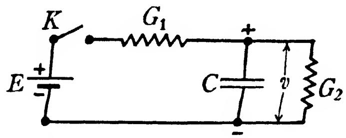


FIG. 31. Circuit for illustration of use of reciprocal parameters.

The second problem is that of finding the behavior of the circuit of Fig. 32 after K is suddenly closed.

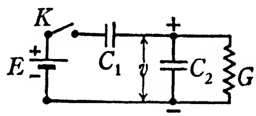


FIG. 32. Circuit for illustration of use of reciprocal parameters.

Solution: Writing Kirchhoff's current law for the node above the capacitance C_2 gives

$$C_1\frac{d}{dt}(E - v) = C_2\frac{dv}{dt} + Gv, \tag{311}$$

or

$$(C_1 + C_2)v + G\int v\,dt = C_1E. \tag{311a}$$

The form of this equation is like the equation of the series RC circuit for current. The solution for v is immediately written

$$v = \frac{EC_1}{C_1 + C_2}\epsilon^{-[Gt/(C_1+C_2)]}. \tag{312}$$

The third problem is that of finding the response of the circuit shown in Fig. 33 after K is closed. This differs from the series RLS circuit only in that the condenser is shunted by a certain amount of leakage as indicated by the conductance G. If this problem is solved on the loop-current basis, it is quite evident that two loop currents must be assumed. The treatment may be simplified by considering the voltage drop v as the unknown and using a mixture of the loop and node methods.

Solution: Writing Kirchhoff's voltage law for the loop containing the source gives

$$E - v = L\frac{di}{dt} + Ri. \qquad [313]$$

But

$$i = C\frac{dv}{dt} + Gv, \qquad [314]$$

so that

$$E - v = LC\frac{d^2v}{dt^2} + LG\frac{dv}{dt} + RC\frac{dv}{dt} + RGv, \qquad [315]$$

from which

$$LC\frac{d^2v}{dt^2} + (LG + RC)\frac{dv}{dt} + (RG + 1)v = E. \qquad [316]$$

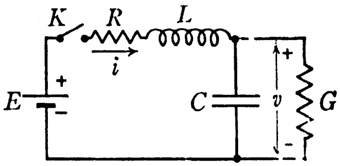


FIG. 33. Circuit for illustration of use of reciprocal parameters.

The steady-state solution for v is evidently a constant, and thus becomes

$$V_s = \frac{E}{RG + 1}. \qquad [317]$$

The transient portion is the integral of the corresponding reduced or force-free equation

$$LC\frac{d^2v}{dt^2} + (LG + RC)\frac{dv}{dt} + (RG + 1)v = 0. \qquad [318]$$

The equation

$$v_t = A\epsilon^{pt} \qquad [319]$$

being assumed, leads to the characteristic equation

$$p^2 + \left(\frac{R}{L} + \frac{G}{C}\right)p + \left(\frac{R}{L}\frac{G}{C} + \frac{1}{LC}\right) = 0, \qquad [320]$$

which gives

$$p = -\left(\frac{R}{2L} + \frac{G}{2C}\right) \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L} - \frac{G}{2C}\right)^2}. \qquad [321]$$

The dissipative effects in the coil and condenser oppose as far as their net effect upon the angular frequency is concerned, while they are additive with regard to the resulting damping constant. Thus an initially over or critically damped circuit might be made oscillatory by the addition of leakage across the condenser. In such a case, however, the total damping becomes so severe as practically to obliterate the oscil-

latory character. It is interesting, nevertheless, that if R/L equals G/C, the natural frequency of the circuit is the same as for the corresponding nondissipative case.

For example, if L is 1 h, C is 10^{-6} farad, R is 500 ohms, and G is 0.0005 mho, then

$$\frac{R}{L} = \frac{G}{C} = 500, \tag{322}$$

$$\frac{1}{\sqrt{LC}} = 1{,}000, \tag{323}$$

$$RG = 0.25, \tag{324}$$

and

$$p = -500 \pm j1{,}000. \tag{325}$$

The complete solution, therefore, has the form

$$v = \frac{E}{0.25 + 1} + (A_1\epsilon^{j1000t} + A_2\epsilon^{-j1000t})\epsilon^{-500t}. \tag{326}$$

If the condenser charge and the current in the coil are each initially zero, then $v(0)$ is zero and $[dv/dt]_0$ is zero. Thus

$$A_1 + A_2 = -0.8E, \tag{327}$$

and, since

$$\left.\frac{dv}{dt}\right|_{t=0} = (-500 + j1{,}000)A_1 + (-500 - j1{,}000)A_2 = 0, \tag{328}$$

then

$$\frac{A_2}{A_1} = \frac{-500 + j1{,}000}{500 + j1{,}000} = \epsilon^{j2\tan^{-1}1/2}, \tag{329}$$

so that

$$A_1 = -\frac{E}{\sqrt{5}}\epsilon^{-j\tan^{-1}1/2};\ A_2 = -\frac{E}{\sqrt{5}}\epsilon^{j\tan^{-1}1/2}. \tag{330}$$

The final solution thus becomes

$$v = 0.8E - \frac{2E}{\sqrt{5}}\epsilon^{-500t}\cos\left[1{,}000t - \tan^{-1}\tfrac{1}{2}\right], \tag{331}$$

or

$$v = 0.8E[1 - (\cos 1{,}000t + 0.5\sin 1{,}000t)\epsilon^{-500t}]. \tag{332}$$

PROBLEMS

1. In order to obtain a time axis in a cathode-ray oscillograph, it is desired to have a voltage which varies directly as time. Furthermore, it is desired to have the voltage reset to zero each time it reaches 200.

One way of accomplishing these results is to charge a condenser from a high-voltage source through a fixed resistor and to use the voltage across the condenser. An automatic device may be used to discharge the condenser in 10^{-6} sec each time its voltage exceeds 200 v.

(a) What values of R, C, and E cause the reset mechanism to operate 1,000 times/sec, and cause the resulting voltage-time curve to deviate no more than 25 μsec from the straight line of the initial slope (Fig. 34)?
(b) If a current source I is used in place of R and E, what values of C and I satisfy the conditions of (a)?

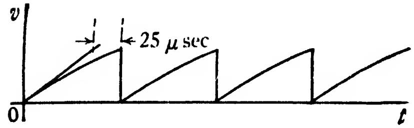


FIG. 34. Voltage-time curve of sweep circuit, Prob. 1.

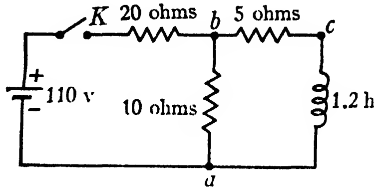


FIG. 35. Inductor with discharge resistance, Probs. 2 and 3

2. In Fig. 35 switch K is closed and steady current is established; K is then opened.
(a) What is the current in the circuit $abca$ as a function of time? What is the direction of this transient current?
(b) What voltage appears across the switch contact points at the instant the contact is broken?

3. In Fig. 35, what is the current in each branch as a function of time after closure of the switch? All the initial currents are zero.

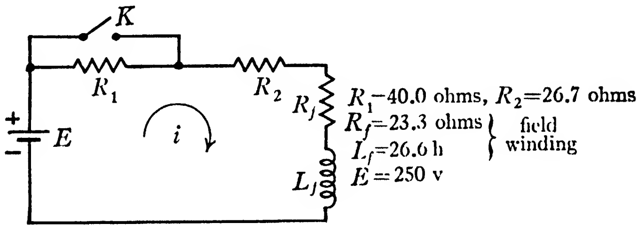


FIG. 36. Generator field circuit, Prob. 4.

4. A voltage regulator operating on the field circuit of a generator alternately closes and opens contacts which short-circuit a portion of the resistance in series with the field winding, as indicated in Fig. 36. As set, the regulator contact K is vibrating at a frequency of 5.0 ~. The ratio of time closed to time open is

$$\frac{t_c}{t_0} = 4.0.$$

(a) Between what limits is the field current varying?
(b) What is the average value of the field current?

5. In Fig. 37, the galvanometer is assumed to be absent.
(a) What is the time expression for current through the battery upon the closure of K? Initially, there is no current in the inductance and no charge on the elastance.
(b) What are the forms of current-time curves for branches acb and adb (as indicated by sketches)?
(c) At what instant are the two currents equal?

6. In Fig. 37, if the galvanometer is absent an oscillograph element placed in series with the battery indicates that when K is suddenly closed the battery current rises as though the circuit contained resistance only.

(a) What must be the relations among the parameters? (The numerical values on Fig. 37 should be ignored.)
(b) What is the form (as indicated by a sketch) of the current-time curves for each branch of the circuit?

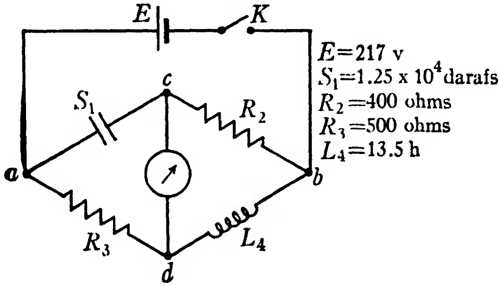


FIG. 37. Bridge circuit, Probs. 5, 6, and 7.

7. Figure 37 represents a bridge which might be used to measure an inductance or a capacitance by a ballistic method if three of the four bridge elements are known. The bridge is balanced if no net charge passes through the galvanometer after the switch K is closed. (The numerical values on the figures are to be ignored.)

(a) What is the equation for balance?
(b) At balance, what are the currents in the arms during the transient period?
(c) What is the voltage across each of the four arms immediately after the switch is closed? (These answers should be obtained by inspection.)
(d) At balance what is the voltage between c and d as a function of time?
(e) What physical considerations affect the actual use of such a bridge?

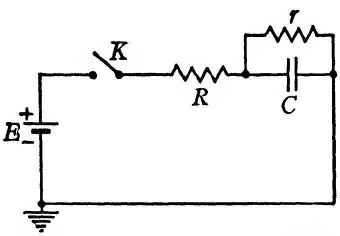


FIG. 38. For charging of condenser, Prob. 8.

8. In Fig. 38, r represents the leakage resistance of the condenser of capacitance C. What are the equations for the charge on the condenser as functions of time after the switch K is closed:

(a) For a perfect condenser (r absent)?
(b) For a leaky condenser?

9. One method of obtaining a high direct voltage when very little current is needed is as follows: 20 condensers, each of 1 μf capacitance, are connected in parallel to a 230-v source. As soon as they are charged, they are disconnected from the source and from each other. Then they are connected in series aiding, and the whole combination is placed in parallel with a high-voltage condenser, which is also 1 μf capacitance. Then the original 20 condensers are immediately disconnected from the high-voltage one, and the cycle is repeated until the high-voltage condenser reaches a voltage above 4,000 v. Then the cycle is repeated by automatic means once whenever the voltage falls to 4,000 v. The leakage current is 0.4 μa at 4,000 v, and may be assumed to be proportional to the voltage.

(a) How often does the automatic charging device operate?
(b) If the automatic charging device becomes inoperative, how long does it take for the voltage to drop to 100 v?
(c) What is the answer to (b) in terms of time constant?

10. By using the complex numbers below,

$$A = 3 + j5, \qquad [333]$$

$$B = 7\,(\cos 135^\circ + j \sin 135^\circ), \qquad [334]$$

$$C = 4\underline{/50^\circ}, \qquad [335]$$

$$D = 6\underline{/-130^\circ}, \qquad [336]$$

$$E = 8\epsilon^{j2}, \qquad [337]$$

$$F = 9\epsilon^{j3}, \qquad [338]$$

the following are to be obtained:

(a) A table expressing each number in rectangular, polar, and exponential form.

(b) Sums:

(1) $A + B$
(2) $C + D$
(3) $E - F$
(4) $A - C + F$

(c) Products:

(1) CD
(2) EF
(3) CF
(4) AB
(5) BDE

(d) Quotients:

(1) $\dfrac{A}{B}$
(2) $\dfrac{C}{F}$
(3) $\dfrac{E}{F}$
(4) $\dfrac{1}{A - C + F}$
(5) $\dfrac{BDE}{ACF}$

(e) Graphs of the quantities:

(1) A
(2) C
(3) F
(4) $A - C + F$
(5) $\dfrac{1}{A - C + F}$
(6) CF
(7) $\dfrac{C}{F}$

(f) Evaluations of expressions:

(1) $\mathcal{R}_e(CD)$
(2) $[\mathcal{R}_e(C)\,\mathcal{R}_e(D)]$
(3) $\sqrt{F}$
(4) $\ln E$
(5) ϵ^{A}

(g) Solutions of the equations for x:

(1) $Fx + C = 0,$ [339]
(2) $Fx^2 + Cx = E.$ [340]

11. The expression below is to be simplified and stated (a) in rectangular form, (b) in polar form, (c) in exponential form. Each step in the solution is to be illustrated on a plot.

$$\mathrm{A} = \frac{[(141 + j141)\underline{/-15^\circ} + 100\epsilon^{j(5\pi/6)}]\underline{/30^\circ} + \mathcal{R}_e[141\epsilon^{-j(3\pi/4)}]}{5.00 + j8.66}. \qquad [341]$$

12. What is the value of V obtained from

$$\mathrm{E} = \mathrm{V} + \mathrm{IZ}, \qquad [342]$$

wherein

$$E = 270, \qquad [343]$$

$$\mathrm{I} = 80 - j60, \qquad [344]$$

$$\mathrm{Z} = 0.10 + j0.5, \qquad [345]$$

$$\mathrm{V} = V\underline{/0^\circ}\ ? \qquad [346]$$

13. The following equalities are to be demonstrated:

(a) $$j \tan \theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{\epsilon^{j\theta} + \epsilon^{-j\theta}}. \qquad [347]$$

(b) $$\epsilon^{x+jy} = \epsilon^x \cos y + j\epsilon^x \sin y. \qquad [348]$$

(c) $$\sin (x + jy) = \frac{\epsilon^y + \epsilon^{-y}}{2} \sin x + j\frac{\epsilon^y - \epsilon^{-y}}{2} \cos x. \qquad [349]$$

(d) $$\mathcal{R}_e[-j\mathrm{X}] = \mathcal{I}[\mathrm{X}]. \qquad [350]$$

14. In Prob. 5 the switch has been closed long enough for the steady state to be reached. Then the switch is opened.

(a) What is the voltage across the switch as a function of time?
(b) What is the maximum voltage across the condenser?
(c) What is the maximum voltage across the inductance?

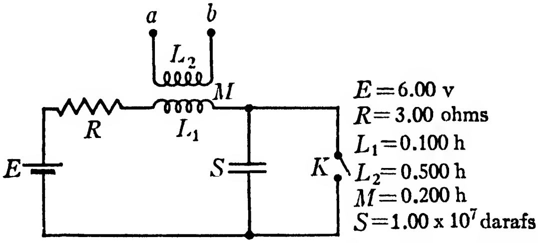


FIG. 39. Arrangement for minimizing sparking at contacts, Prob. 16.

15. A condenser of 0.4 μf capacitance, initially uncharged, is charged from a 2,000-v direct-current source. The effective inductance of the charging circuit is 1 mh. What are the maximum voltage across and the maximum current to the condenser, if the resistance of the circuit is

(a) zero, (b) 60 ohms, (c) 120 ohms?

16 In order to minimize sparking which occurs when an inductive circuit is opened, a condenser is sometimes connected across the switch as shown in Fig. 39.

Switch K is closed long enough for the steady state to be reached, then is opened.

(a) What is the initial voltage across the terminals a–b?

(b) What is the initial rate of voltage build-up across the open switch terminals?

(c) What is the time expression for voltage across the switch?

(d) What is the maximum voltage across the switch, and at what time does it occur?

All the results desired are numerical values based on the constants given for the circuit.

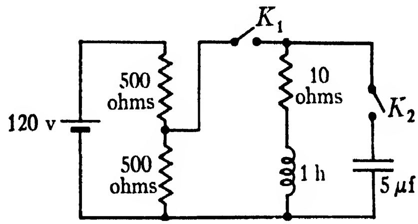


FIG. 40. Circuit supplied from dropwire, Prob. 17.

17. In the circuit of Fig. 40 with both switches open there is no initial charge on the condenser. What is the current in the inductance as a function of time after

(a) K_1 is closed?

(b) K_2 is closed a long time after K_1?

CHAPTER IV

Elementary Alternating-Current Theory: Steady State

1. Introductory comments

The preceding chapter considers the current response of circuits consisting of various combinations of constant resistance, inductance, and capacitance, when a constant voltage is applied. The voltage response of these circuits to a source of constant current is also discussed. Neither the value of the current nor the voltage of the source varies after being applied to the circuit.

This chapter is concerned with the response of the same constant-parameter circuits to energy sources whose voltages or currents vary sinusoidally with time. These are called sinusoidal or simple harmonic currents and voltages.

As with the constant applied forces considered in the preceding chapter, the solutions here are found to consist of two components, the steady-state component and the transient component. Since the theory of the alternating steady state as here developed is fundamental to all alternating-current circuit analysis, its importance to the electrical engineer cannot be overemphasized. It is probably the most-used part of electrical theory in all subsequent work and should therefore be thoroughly mastered in all of its details.

Though the resultant circuit behavior during the transient interval may be materially different from that for constant applied forces, the transient components differ in no way from those already considered and consequently present no new problems to the student. They do furnish additional practice in applying the principles relating to the transient behavior of linear circuits.

2. Sinusoidal voltages and currents

Before the theory of alternating-current circuits is attacked, the fundamental ideas pertaining to currents and voltages whose instantaneous values vary with time in a sinusoidal or simple harmonic manner must be reviewed. Such a simple harmonic function can be described mathematically by either the sine or cosine function. As seen in the preceding chapter, it can also be described in terms of the complex exponential function, a form that often proves more convenient in subsequent mathematical work than the sine or cosine representation.

If $e(t)$ is the instantaneous value at a time t of an electromotive force

that is a sinusoidal function of time, it can be expressed as

$$e(t) = E_m \sin(\omega t + \psi'). \tag{1}$$

The maximum value of $e(t)$ is E_m and is called the *amplitude* of the electromotive force. The argument $(\omega t + \psi')$ of the sine is composed of the constant term or *phase angle* ψ' and the term ωt that varies linearly with the time t. The constant ω is called the *angular frequency* of the function. A plot of $e(t)$ as given by Eq. 1 is shown in Fig. 1. The curve is located along the time axis by noting that sin (0) occurs when t is $-\psi'/\omega$ or ωt is $-\psi'$. Geometrically, the ordinates of the sine curve can be thought

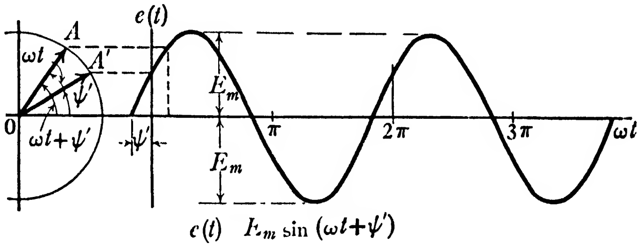


FIG. 1. Generation of sine wave by rotating vector.

of as being generated by the vector $0A$ of length E_m rotating at a speed of ω radians per second, the position of this vector being $0A'$ at the instant that t is equal to zero.

In Fig. 1, $e(t)$ is plotted in terms of ωt in order that the curve shall apply for any value of ω. In mathematical work such as differentiation and integration, t rather than ωt is usually taken as the independent variable.

The sine curve of Fig. 1 consists of positive and negative loops. Two complete adjacent loops, one positive and one negative, or the function over any other interval 2π of ωt constitutes one *cycle*. The time corresponding to one cycle is the *period* of the function. The number of periods in one second is the *frequency* f of the function and is measured in cycles per second. Frequency f should be distinguished from the angular frequency ω which is measured in radians per second.

The function $e(t)$ as given by Eq. 1 can be expressed in several other forms which are used in subsequent work:

$$\begin{aligned} e(t) &= E_m \sin(\omega t + \psi') = E_m \frac{\epsilon^{j(\omega t+\psi')} - \epsilon^{-j(\omega t+\psi')}}{j2} \\ &= \mathcal{I}[E_m \epsilon^{j(\omega t+\psi')}] = \mathcal{I}[\mathrm{E}'_m \epsilon^{j\omega t}] \\ &= \mathcal{I}[-E_m \epsilon^{-j(\omega t+\psi')}] = \mathcal{I}[-\bar{\mathrm{E}}'_m \epsilon^{-j\omega t}] \end{aligned} \tag{2}$$

in which $\mathbf{E}'_m$ and $\overline{\mathbf{E}}'_m$ are the complex constants

$$\mathbf{E}'_m \equiv E_m \epsilon^{j\psi'} \tag{3}$$

and

$$\overline{\mathbf{E}}'_m \equiv E_m \epsilon^{-j\psi'}, \tag{4}$$

$\overline{\mathbf{E}}'_m$ being the conjugate of $\mathbf{E}'_m$.

The complex exponential form $\mathscr{I}[\mathbf{E}'_m \epsilon^{j\omega t}]$ is merely a very convenient *way of describing mathematically the ordinate of the rotating vector 0A* of Fig. 1. Thus if $0A$ is put in the complex plane, its position $0A'$ when t is zero is described by the complex number $\mathbf{E}'_m$. The factor $\epsilon^{j\omega t}$ produces rotation at the speed of ω radians per second but does not change the length of the vector. Hence $\mathbf{E}'_m \epsilon^{j\omega t}$ describes the vector $0A$. Taking the imaginary part $\mathscr{I}[\mathbf{E}'_m \epsilon^{j\omega t}]$ of $\mathbf{E}'_m \epsilon^{j\omega t}$ is the mathematical way of expressing the ordinate of $0A$. It is seen, therefore, that the equation

$$e(t) = E_m \sin (\omega t + \psi') = \mathscr{I}[\mathbf{E}'_m \epsilon^{j\omega t}] \tag{2a}$$

is a mathematical expression of the geometrical concept of the sine curve shown in Fig. 1.

Complex quantities such as $\mathbf{E}'_m$ and $\mathbf{E}'_m \epsilon^{j\omega t}$ are much used in the subsequent work. Ordinarily $\mathbf{E}'_m$ is a complex constant, although if a range of frequencies is considered it may be a function of the angular frequency ω. It is called a *complex* or *vector voltage*. The complex quantity $\mathbf{E}'_m \epsilon^{j\omega t}$ is a function of time and for that reason may be distinguished from $\mathbf{E}'_m$ by calling it a *time-vector voltage*.

The function $e(t)$ may also be expressed in terms of the cosine function as

$$e(t) = E_m \cos (\omega t + \psi), \tag{5}$$

in which Eqs. 1 and 5 describe the same function if

$$\psi' = \psi + \frac{\pi}{2}. \tag{6}$$

The function $e(t)$ as given by Eq. 5 can be expressed in terms of complex exponential functions of the same general form as in Eq. 2:

$$\begin{aligned} e(t) &= E_m \cos (\omega t + \psi) = E_m \frac{\epsilon^{j(\omega t+\psi)} + \epsilon^{-j(\omega t+\psi)}}{2} \\ &= \mathscr{R}_e[E_m \epsilon^{j(\omega t+\psi)}] = \mathscr{R}_e[\mathbf{E}_m \epsilon^{j\omega t}] \qquad [7] \\ &= \mathscr{R}_e[E_m \epsilon^{-j(\omega t+\psi)}] = \mathscr{R}_e[\overline{\mathbf{E}}_m \epsilon^{-j\omega t}], \end{aligned}$$

$$\mathbf{E}_m \equiv E_m \epsilon^{j\psi}, \tag{8}$$

and

$$\overline{\mathbf{E}}_m \equiv E_m \epsilon^{-j\psi}. \tag{9}$$

From Eq. 6 it can be seen that

$$\mathbf{E}_m = \mathbf{E}'_m \underline{/-90^\circ}. \qquad [10]$$

and

$$\overline{\mathbf{E}}_m = \overline{\mathbf{E}}'_m \underline{/90^\circ} \qquad [11]$$

Although degree units are properly not used as the argument of an exponential, their use in numerical work is often convenient. The symbols $\underline{/\quad\quad}$ and $\overline{/\quad\quad}$ mean "at a positive angle of" and "at a negative angle of," respectively. The symbol $\underline{/\quad\quad}$ with the negative sign prefixed to the angle is recommended as preferable to the symbol $\overline{/\quad\quad}$ for negative angles, because in complicated expressions involving both multiplication

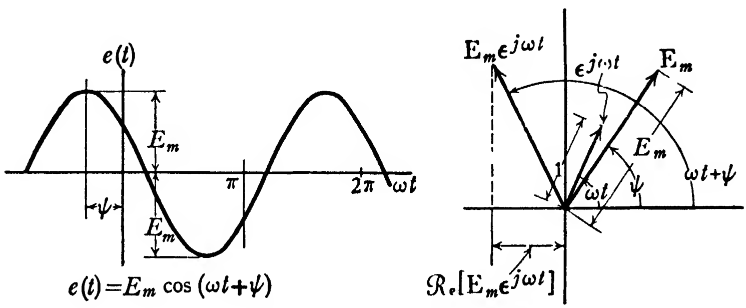


FIG. 2. Cosine wave and associated vectors.

and division there is less opportunity for confusion if there is only one convention for designation of algebraic sign. Representation of some of the forms given by Eqs. 7 to 11 are shown in Fig. 2. The geometrical construction for the cosine form, similar to Fig. 1 for the sine curve, is suggested as an exercise for the student.

So far the cosine function appears to be representable only by the real part and the sine function by the imaginary part. This apparent restriction is removed by the following consideration of Fig. 3 in which

$$\mathcal{I}[\mathbf{E}'_m] = 0A', \qquad [12]$$

$$\mathcal{R}_e[\mathbf{E}_m] = 0A. \qquad [13]$$

If

$$E_m = E'_m, \qquad [14]$$

then

$$0A = 0A'. \qquad [15]$$

Hence

$$\mathcal{I}[\mathbf{E}'_m] = \mathcal{R}_e[\mathbf{E}_m], \qquad [16]$$

so $$E_m \sin(\omega t + \psi') = \mathcal{I}[\mathrm{E}'_m \epsilon^{j\omega t}] = \mathcal{R}_e[\mathrm{E}_m \epsilon^{j\omega t}] \qquad [17]$$

and $$E_m \cos(\omega t + \psi) = \mathcal{R}_e[\mathrm{E}_m \epsilon^{j\omega t}] = \mathcal{I}[\mathrm{E}'_m \epsilon^{j\omega t}]. \qquad [18]$$

The sine function may alternatively be written as the real part of a complex function whose complex amplitude E_m *lags* behind the amplitude E'_m by 90 degrees, or the cosine function may alternatively be written as the imaginary part of a complex function whose amplitude E'_m *leads* the amplitude E_m by 90 degrees.

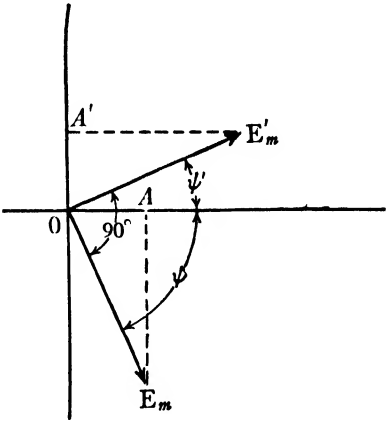


FIG. 3. Relations between real and imaginary parts of complex numbers.

A current that is a sinusoidal function of time is described by mathematical functions of exactly the same form as Eq. 2 or Eq. 7, E being replaced by I and ψ by ϕ. The same geometrical and analytical considerations apply to both equally.

In what follows, the cosine forms (Eq. 7) are used more often than the sine forms (Eq. 2), although either may be used. The various ways of expressing the simple harmonic function $e(t)$ as given in Eqs. 2 and 7, including the graphical representations both of the entire expressions in the real plane of $e(t)$ and t and of the complex or vector expressions in the complex plane, should be clearly understood and mastered before the mathematical work of the following articles is undertaken.

In commercial power practice, the alternating voltage and currents encountered usually have very nearly simple harmonic form, and circuits often may be analyzed on the assumption that the voltage and current are actually simple harmonic time functions. Departures from this form are, however, occasionally important; telephone apparatus is especially sensitive to them, for instance, and they are magnified by the action of certain other types of apparatus, such as power rectifiers. In either situation it is usually necessary to consider functions of the general type illustrated by the voltage

$$e(t) = E_{m1} \cos \omega t + E_{m3} \cos(3\omega t + \theta_3) + E_{m5} \cos(5\omega t - \theta_5). \qquad [19]$$

This equation means that the voltage wave has a *fundamental* component, whose maximum value is E_{m1} and whose angular velocity is ω radians per second; a *third harmonic* component of E_{m3} maximum value with an angular velocity of 3ω radians per second, which is θ_3 radians ahead of the fundamental when t is zero; and a *fifth harmonic* component

of maximum value E_{m5} with an angular velocity of 5ω radians per second, which is θ_5 radians behind the fundamental when t is zero. In this chapter attention is confined to voltages and currents that are simple harmonic functions of time. Complex wave forms, such as that of Eq. 19, are considered in all subsequent volumes of this series in a variety of situations which produce nonsinusoidal voltages and currents.

3. Steady-state response of the series *RLS* circuit with an alternating-voltage source

The character of transient components in alternating-current circuits and the manner of obtaining the solutions are essentially no different from those presented in Ch. III for direct-current circuits, because the character and magnitude of the transients are determined entirely by the circuit parameters and the initial conditions, and are independent of the nature of the source. Therefore this portion of the problem is postponed for review subsequent to the treatment of the steady state.

Instead of solving separately the series circuits having various combinations of parameters, the more direct method is to go at once to the general *RLS* case and obtain from it the cases involving fewer parameters. Therefore the circuit of Fig. 4 is used, with the assumption that K has been in position 1 for an indefinitely long time. If the source voltage has the form

$$\left.\begin{aligned} e(t) &= E_m \cos(\omega t + \psi) = E_m \frac{\epsilon^{j(\omega t+\psi)} + \epsilon^{-j(\omega t+\psi)}}{2} \\ &= \mathcal{R}_e[\mathrm{E}_m \epsilon^{j\omega t}] = \mathcal{R}_e[\bar{\mathrm{E}}_m \epsilon^{-j\omega t}], \end{aligned}\right\} \qquad [7a]$$

it is reasonable to assume that the steady-state charge has a similar form,*

$$\left.\begin{aligned} q_s(t) &= Q_m \cos(\omega t + \phi_q) = Q_m \frac{\epsilon^{j(\omega t+\phi_q)} + \epsilon^{-j(\omega t+\phi_q)}}{2} \\ &= \mathcal{R}_e[\mathrm{Q}_m \epsilon^{j\omega t}] = \mathcal{R}_e[\bar{\mathrm{Q}}_m \epsilon^{-j\omega t}], \end{aligned}\right\} \qquad [20]$$

in which case the steady-state current also has a similar form,

$$\left.\begin{aligned} i_s(t) &= \frac{dq_s(t)}{dt} = \omega Q_m \cos\left(\omega t + \phi_q + \frac{\pi}{2}\right) \\ &= \omega Q_m \frac{\epsilon^{j(\omega t+\phi_q+\pi/2)} + \epsilon^{-j(\omega t+\phi_q+\pi/2)}}{2} \\ &= I_m \cos(\omega t + \phi) = I_m \frac{\epsilon^{j(\omega t+\phi)} + \epsilon^{-j(\omega t+\phi)}}{2} \\ &= \mathcal{R}_e[\mathrm{I}_m \epsilon^{j\omega t}] = \mathcal{R}_e[\bar{\mathrm{I}}_m \epsilon^{-j\omega t}]. \end{aligned}\right\} \qquad [21]$$

* This is discussed in Art. 5, Ch. III. Sine instead of cosine functions could of course be used, as pointed out in Art. 2 of this chapter, but are less convenient in exponential form because of j in the denominator.

In the foregoing equations,

$$\mathrm{E}_m \equiv E_m \epsilon^{j\psi}, \tag{22}$$

$$\mathrm{Q}_m \equiv Q_m \epsilon^{j\phi_q}, \tag{23}$$

$$I_m \equiv \omega Q_m, \tag{24}$$

$$\mathrm{I}_m \equiv j\omega \mathrm{Q}_m = I_m \epsilon^{j\phi}, \tag{24a}$$

$$\phi = \phi_q + \frac{\pi}{2}. \tag{25}$$

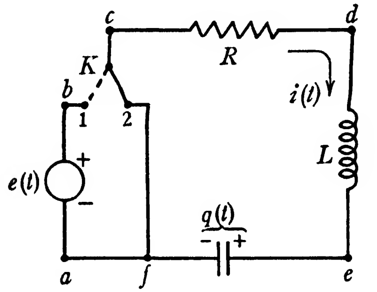


FIG. 4. Series RLS circuit with alternating-voltage source.

The only requirement on the steady-state component is that it reduce to an identity

$$e = Ri + L\frac{di}{dt} + S\int i\,dt, \tag{26}$$

(wherein functional notation has been omitted). Any of the forms of Eqs. 7a, 20, and 21 can be substituted into Eq. 26 in order to determine the conditions which must be satisfied to obtain an identity. However, the use of

$$e = \mathcal{R}_e[\mathrm{E}_m \epsilon^{j\omega t}] \tag{7b}$$

and

$$i_s = \mathcal{R}_e[\mathrm{I}_m \epsilon^{j\omega t}] \tag{21a}$$

leads most directly to the form of result in most common engineering use. Substitution of these forms gives

$$\left.\begin{aligned} \mathcal{R}_e[\mathrm{E}_m \epsilon^{j\omega t}] &= \mathcal{R}_e[R\mathrm{I}_m \epsilon^{j\omega t}] + \mathcal{R}_e[j\omega L\mathrm{I}_m \epsilon^{j\omega t}] + \mathcal{R}_e\left[\frac{S}{j\omega}\mathrm{I}_m \epsilon^{j\omega t}\right] \\ &= \mathcal{R}_e\left\{\left[R + j\left(\omega L - \frac{S}{\omega}\right)\right]\mathrm{I}_m \epsilon^{j\omega t}\right\}, \end{aligned}\right\} \tag{27}$$

which reduces to an identity for all values of time only if

$$\mathbf{E}_m = \left[R + j\left(\omega L - \frac{S}{\omega}\right)\right]\mathbf{I}_m = \mathbf{I}_m \mathbf{Z} \qquad \blacktriangleright[27a]$$

or

$$E_m\underline{/\psi} = (I_m\underline{/\phi})(Z\underline{/\theta_z}) = I_m Z\underline{/\phi + \theta_z} \qquad \blacktriangleright[27b]$$

and

$$\theta_z = \psi - \phi = \tan^{-1}\frac{\omega L - \dfrac{S}{\omega}}{R}. \qquad \blacktriangleright[28]$$

The quantity

$$\mathbf{Z} \equiv R + j\left(\omega L - \frac{S}{\omega}\right) = R + j(X_L + X_C) \equiv R + jX, \qquad \blacktriangleright[29]$$

$$Z = \sqrt{R^2 + \left(\omega L - \frac{S}{\omega}\right)^2} = \sqrt{R^2 + X^2} \qquad \blacktriangleright[29a]$$

is called the *impedance** of the circuit, and Eq. 27a is sometimes called

* Instead of substituting for i in Eq 26, it is just as logical to substitute $\mathcal{R}_e[\mathbf{Q}_m\epsilon^{j\omega t}]$ in the equation

$$e = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + Sq \qquad [26a]$$

and obtain

$$\left.\begin{aligned}\mathcal{R}_e[\mathbf{E}_m\epsilon^{j\omega t}] &= \mathcal{R}_e[-\omega^2 L\mathbf{Q}_m\epsilon^{j\omega t}] + \mathcal{R}_e[j\omega R\mathbf{Q}_m\epsilon^{j\omega t}] + \mathcal{R}_e[S\mathbf{Q}_m\epsilon^{j\omega t}]\\ &= \mathcal{R}_e\{[-\omega^2 L + S + j\omega R]\mathbf{Q}_m\epsilon^{j\omega t}\},\end{aligned}\right\} \qquad [27c]$$

which reduces to an identity for all values of time only if

$$\mathbf{E}_m = [S - \omega^2 L + j\omega R]\mathbf{Q}_m = \mathbf{Q}_m\mathbf{Z}_q \qquad [27d]$$

or

$$E_m\underline{/\psi} = (Q_m\underline{/\phi_q})(Z_q\underline{/\theta_q}) = Q_m Z_q\underline{/\phi_q + \theta_q} \qquad [27e]$$

and

$$\theta_q = \psi - \phi_q = \tan^{-1}\frac{\omega R}{S - \omega^2 L}. \qquad [28a]$$

The quantity

$$\mathbf{Z}_q = S - \omega^2 L + j\omega R, \qquad [29b]$$

$$Z_q = \sqrt{(S - \omega^2 L)^2 + \omega^2 R^2} \qquad [29c]$$

is called the *charge impedance* By using it, the steady-state charge can be computed directly from the impressed voltage. However, this method is not commonly used. The use of charge impedance merely makes one more set of relations which are rather superfluous to the engi-

the generalized Ohm's law for alternating-current circuits. Voltage, current, and impedance of Eq. 27a are, of course, complex numbers and are sometimes designated in engineering usage as *vector volts, vector amperes,* and *vector ohms.* The quantity

$$X \equiv X_L + X_C \equiv \omega L - \frac{S}{\omega} \qquad ▶[30]$$

is called the *reactance* of the circuit, and its components

$$X_L \equiv \omega L, \qquad ▶[31]$$

$$X_C \equiv -\frac{S}{\omega} = -\frac{1}{\omega C} \qquad ▶[32]$$

are called, respectively, the *inductive reactance* and the *elastive* or *capacitive reactance* of the circuit.

If there is no elastance in the circuit, it is a series RL circuit, and

$$Z = R + j\omega L = R + jX_L, \qquad [33]$$

$$Z = \sqrt{R^2 + \omega^2 L^2} = \sqrt{R^2 + X_L^2}, \qquad [33a]$$

and

$$\theta_z = \tan^{-1}\frac{\omega L}{R} = \tan^{-1}\frac{X_L}{R}. \qquad [34]$$

If there is no inductance in the circuit, it is a series RS circuit, and

$$Z = R - j\frac{S}{\omega} = R + jX_C, \qquad [35]$$

$$Z = \sqrt{R^2 + \frac{S^2}{\omega^2}} = \sqrt{R^2 + X_C^2}, \qquad [35a]$$

and

$$\theta_z = \tan^{-1}\left(\frac{-S}{\omega R}\right) = \tan^{-1}\frac{X_C}{R}. \qquad [36]$$

---

neer's stock, since charge can be obtained so readily from current merely by use of Eq. 24a. It is interesting that

$$E_m = [S - \omega^2 L + j\omega R]Q_m = \frac{1}{j\omega}[S - \omega^2 L + j\omega R]j\omega Q_m = \left[R + j\left(\omega L - \frac{S}{\omega}\right)\right]I_m \qquad [27f]$$

or

$$Q_m Z_q = j\omega Q_m \frac{Z_q}{j\omega} = I_m Z \qquad [27g]$$

or

$$Z_q = j\omega Z. \qquad [29d]$$

For the more artificial condition where there is no resistance, that is, for a series LC circuit,

$$Z = j\left(\omega L - \frac{S}{\omega}\right) = j(X_L + X_C), \tag{37}$$

$$Z = X_L + X_C = X, \tag{37a}$$

and

$$\theta_z = +\frac{\pi}{2}. \tag{38}$$

Hence for any series circuit, the steady-state solution can be made completely in terms of complex numbers, independent of time,[1] provided that any two of the quantities in the equation

$$\mathbf{E}_m = \mathbf{I}_m Z \tag{27a}$$

are known or can be computed. For example, the amplitude and phase angle of the electromotive force may be known. The impedance and its angle can be computed from the values of the parameters of the circuit and the angular frequency of the source. Then the amplitude and phase angle of the current can be computed. If the amplitude and phase angle of the current instead of the electromotive force are known, the amplitude and phase angle of the electromotive force can, of course, be computed. Finally, if the amplitude and phase angle of both electromotive force and current are known, the impedance and its angle can be computed, but the reactance cannot be separated into its components without additional information.

The actual phase angle of the electromotive force or current depends upon the instant of operation of a switch (or some other circuit change) and usually is of little interest when only the steady-state solution is desired. Ordinarily, in steady-state computations, the relative phase angle, or the angle between voltage and current, is adequate, a fact which

[1] The use of complex algebra in the solution of electric circuit and other physical problems appears to have been introduced between 1877 and 1887 largely by Rayleigh and Heaviside, the latter being responsible for the invention of the terms "impedance" and "admittance."

John William Strutt Rayleigh, *The Theory of Sound* (2d ed.; London: Macmillan & Co., Ltd., 1894), Vol. I, Chs. iv and v, "The Reaction upon the Driving-Point of a System Executing Forced Harmonic Oscillations of Various Periods, with Applications to Electricity," *Phil. Mag.*, fifth series, XXI (1886), 369–381, O. Heaviside, "On Resistance and Conductance Operators, and Their Derivatives, Inductance and Permittance, Especially in Connection with Electric and Magnetic Energy," *Phil. Mag.*, fifth series, XXVI (1878), 479–502.

These references contain in them other references on the subject. The use of the complex method was introduced to engineers largely by Kennelly and Steinmetz: A. E. Kennelly, "Impedance," *A.I.E.E. Trans.*, X (1893), 175–216; C. P. Steinmetz, "Reactance," *idem*, XI (1894), 640–648.

means that one of the phase angles ψ or ϕ can be assumed to be zero or any convenient value. Furthermore, instead of dealing with amplitude, engineers find it more convenient to deal with *root-mean-square* or *effective* values of voltage and current as explained presently. If, however, true instantaneous steady-state values are desired, it is necessary to have the *true phase angles and the amplitudes, and to express the results as time* functions in accordance with Eqs. 7a, 20, and 21.

4. GRAPHICAL REPRESENTATION IN THE COMPLEX PLANE: VECTOR DIAGRAMS*

Graphical representation of complex-quantity relations such as those of the preceding article is of great assistance in the solution of engineering problems. For example, the expressions for $e(t)$ and $i(t)$ of Eqs. 7a and 21 are so represented in Fig. 5. However, if the expressions $\mathcal{R}_e[\mathrm{E}_m\epsilon^{j\omega t}]$ and $\mathcal{R}_e[\mathrm{I}_m\epsilon^{j\omega t}]$ are used, as is done in the derivations of Art. 3, only the positively rotating arrows on the top half of the picture need be drawn. Since those in the bottom half of the plane are merely the images or conjugates of those in the top half of the plane, they give no additional information. It is of course just as logical to use the complex numbers in the bottom half of the plane, the negatively rotating set, and to throw away the others. This procedure results in the relation

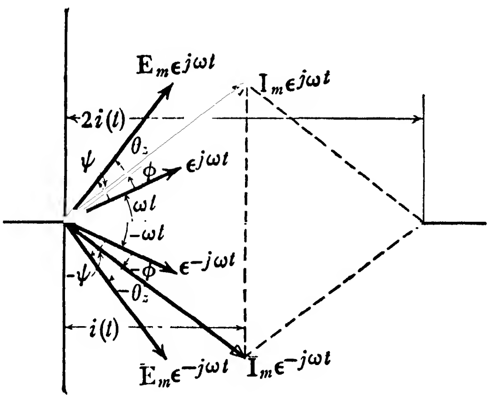


FIG. 5. Representation of real quantities by conjugates.

$$\overline{\mathrm{E}}_m = \overline{\mathrm{I}}_m\overline{\mathrm{Z}} \qquad [39]$$

wherein $\overline{\mathrm{E}}_m$, $\overline{\mathrm{I}}_m$, and $\overline{\mathrm{Z}}$ are, respectively, the conjugates of E_m, I_m, and Z. This system, in fact, is used in the early American literature and still is used in some contemporary European literature. All contemporary American literature, however, uses the positively rotating set.

The next step in the use of diagrams is to free them of time. Since in the steady state the instantaneous values are commonly not desired, but rather the complex quantities of Eq. 27a, the interrelations of which are

* This is another common engineering use of the term "vector" interchangeably with "complex quantity."

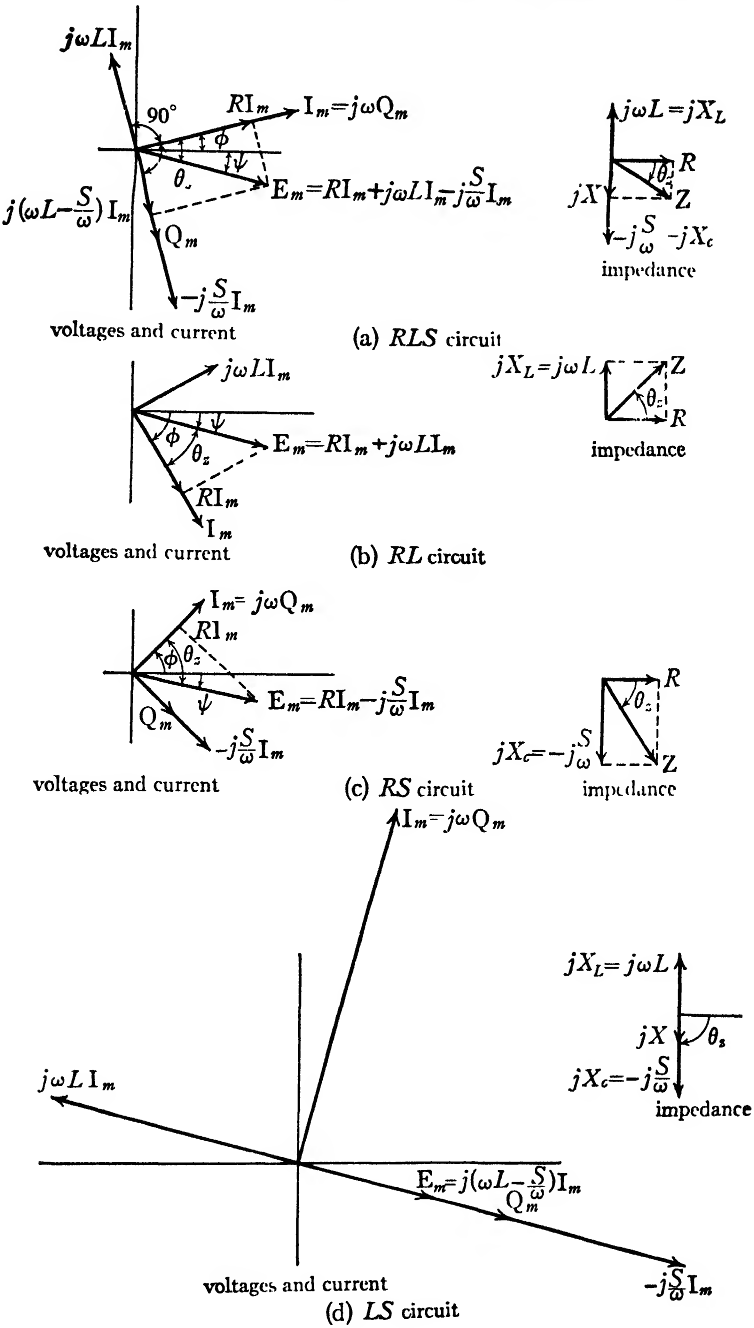


FIG. 6. Vector diagrams of $\mathbf{E}_m$, $\mathbf{I}_m$, $\mathbf{Q}_m$, and Z for series circuits

independent of time, the diagram representing such an equation can utilize any reference axis which may be convenient. Drawing this diagram corresponds frequently to drawing that for the instant when t equals zero in Eq. 21, a fact which means that ψ and ϕ are retained from the time expressions. However, if instantaneous values are not desired, ψ and ϕ are not important, θ_z alone being of consequence. In fact, in most steady-state problems, the engineer does not know the exact time expressions, but deals only in amplitudes,* frequencies, parameters, and angles. Hence ψ or ϕ can be assumed at will in order to obtain the most convenient reference axis.† Ordinarily one or the other is taken as zero. If ψ is taken as zero, E_m lies along the reference axis; if ϕ is taken as zero, I_m lies along the reference axis. The latter usually is more convenient for a series circuit, because the same current exists in all branches. Similarly, the former usually is more convenient for a parallel circuit, because the same voltage exists across all branches.

The diagrams for the series RLS, RL, RS, and LS circuits, and the associated impedance triangles, by means of which the impedance Z is computed in each case, are shown in Fig. 6.

In the addition of vectors it is not necessary to draw them all from a common origin. For example, if in Fig. 6a

$$E_m = RI_m + j\omega LI_m - j\frac{S}{\omega}I_m = V_{m(cd)} + V_{m(de)} + V_{m(ea)}, \qquad [27h]$$

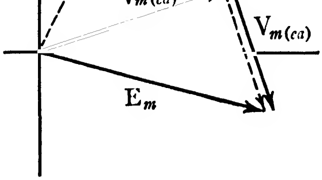


FIG. 6e. Addition of vectors by drawing them end to end.

corresponding to the circuit of Fig. 4, with K in position 1, the vector diagram can be drawn by placing the vectors end to end in their proper sequence around the circuit, as shown in Fig. 6e. The voltages between other points on the diagram can then be scaled from the diagram, as shown by the dotted lines. If phase angles are desired, however, care must be exercised to insure obtaining in each case the angle desired and not its complement, its supplement, or its negative. An advisable procedure is to transfer the vectors in question to a common origin mentally when angles are desired. Furthermore, though it is sometimes possible and perhaps convenient to have the

* Or effective values, defined presently.

† Hence the question of whether sine or cosine functions are used becomes superfluous. If time is reintroduced, however, it is important to remember that ψ and ϕ are defined on the basis of cosine functions, and that the sine relations are in Art. 2.

geometric configuration of the vector diagram correspond to the outline of the circuit diagram, the student should recognize that such correspondence has no physical significance. In fact, unless great care is exercised in drawing the vectors in the proper sequence and direction and unless there is a full knowledge of existence or nonexistence of common grounds, some voltages scaled from this type of diagram may be entirely meaningless. Not only for this situation alone but for the entire process of manipulating alternating-current circuit quantities by means of complex numbers, an understanding of the conventions of direction is essential.

5. Directions of voltages and currents when expressed in complex notation

When dealing with instantaneous values, it is not particularly difficult to visualize polarities for voltages and directions for currents in accordance with the conventions established in previous chapters. These conceptions, when carried into the realm of complex numbers, however, cease to have physical meaning but are nevertheless just as essential to consistent procedure in order to avoid errors of sign. As stated in Art. 2, Ch. III, a time expression for electromotive force, such as Eq. 7a applying to Fig. 4, $e(t)$ or $e_{ab}(t)$ represents the amount by which point b is higher in potential than point a *algebraically* as a function of time. Obviously, from the nature of the sinusoidal function, the amount by which point b is higher in potential than point a must be represented by a negative number half the time; in other words, point b is actually lower in potential than point a half the time. Similarly, a time expression for current, such as Eq. 21 applying to Fig. 4, $i(t)$ or $i_{cd}(t)$ represents the amount of current in the arrow or c–d direction *algebraically* as a function of time. Again, it is obvious from the nature of the time function that the current in the arrow direction must be represented by a negative number half the time; in other words, the current is actually in the opposite direction half the time. Finally, a time expression for charge, such as Eq. 20 applying to Fig. 4, $q(t)$ or $q_c(t)$ represents the amount of positive charge on the c plate *algebraically* as a function of time. The other plate always has a charge of equal magnitude but of opposite sign to that on the c plate. Here also it is obvious from the nature of the time function that the amount of positive charge on the c plate must be represented by a negative number half the time; in other words, the c plate actually carries a negative charge half the time. Another way of viewing the entire matter is to see that the polarity marks or arrows indicate the actual polarities or directions when the corresponding *time vectors* are in the fourth or first quadrants if cosine functions are used, or are in the first or second quadrants if sine functions are used.

In associating the ideas of polarity and direction with complex numbers, it is merely necessary to bear in mind that

$$\mathrm{E}_m = \mathrm{E}_{m(ab)} = -\mathrm{E}_{m(ba)} \qquad [40]$$

is the complex number associated with

$$\left.\begin{aligned} e = e_{ab} = -e_{ba} &= \mathcal{R}_e[\mathrm{E}_m\epsilon^{j\omega t}] = \mathcal{R}_e[\mathrm{E}_{m(ab)}\epsilon^{j\omega t}] \\ &= \mathcal{R}_e[-\mathrm{E}_{m(ba)}\epsilon^{j\omega t}], \end{aligned}\right\} \qquad [41]$$

and that

$$\mathrm{I}_m = \mathrm{I}_{m(cd)} = -\mathrm{I}_{m(dc)} \qquad [42]$$

is the complex number associated with

$$\left.\begin{aligned} i = i_{cd} = -i_{dc} &= \mathcal{R}_e[\mathrm{I}_m\epsilon^{j\omega t}] = \mathcal{R}_e[\mathrm{I}_{m(cd)}\epsilon^{j\omega t}] \\ &= \mathcal{R}_e[-\mathrm{I}_{m(dc)}\epsilon^{j\omega t}], \end{aligned}\right\} \qquad [43]$$

and that

$$\mathrm{Q}_m = \mathrm{Q}_{m(e)} = -\mathrm{Q}_{m(f)} \qquad [44]$$

is the complex number associated with

$$\left.\begin{aligned} q = q_e = -q_f &= \mathcal{R}_e[Q_m\epsilon^{j\omega t}] = \mathcal{R}_e[Q_{m(e)}\epsilon^{j\omega t}] \\ &= \mathcal{R}_e[-Q_{m(f)}\epsilon^{j\omega t}] \end{aligned}\right\} \qquad [45]$$

as derived in Art. 3, and to pay the same attention to polarities and directions in dealing with complex numbers as in dealing with the associated time functions themselves. From the nature of the derivation of the use of complex numbers in alternating-current representation, evidently the quantity

$$\mathrm{I}_m Z = \mathrm{I}_m R + j\mathrm{I}_m X_L + j\mathrm{I}_m X_C \qquad [46]$$

must be considered as a complex voltage drop in the arrow direction of the complex current I_m, as must each of the components of $\mathrm{I}_m Z$. An assembly of vector diagrams for each type of element may be helpful.

Figure 7 illustrates the position of the complex drop $\mathrm{V}_{m(ab)}$ with respect to the complex current $\mathrm{I}_{m(ab)}$. For the resistance the voltage drop $\mathrm{V}_{m(ab)}$ is said to be in phase with the current $\mathrm{I}_{m(ab)}$. For the inductance, the voltage drop $\mathrm{V}_{m(ab)}$ is said to lead the current $\mathrm{I}_{m(ab)}$ (or the current is said to lag the voltage drop) by 90 degrees. For the capacitance the voltage drop $\mathrm{V}_{m(ab)}$ is said to lag the current $\mathrm{I}_{m(ab)}$ (or the current is said to lead the voltage drop) by 90 degrees. For the general network, the voltage drop $\mathrm{V}_{m(ab)}$ may be anywhere in the fourth or first quadrants. The voltage and current vectors are distinguished by different arrowheads.

For all passive networks, therefore, $\frac{1}{2}V_{m(ab)}I_{m(ab)}\cos\theta_z$, or electric power absorbed, is a positive quantity. If the load is directly connected

to a source, the electromotive force of which $\mathbf{E}_{m(ab)}$ is equal to $\mathbf{V}_{m(ab)}$, the power $\frac{1}{2}E_{m(ba)}I_{m(ab)}\cos\theta_z$ delivered by the source is also a positive quantity, equal to that absorbed by the load.*

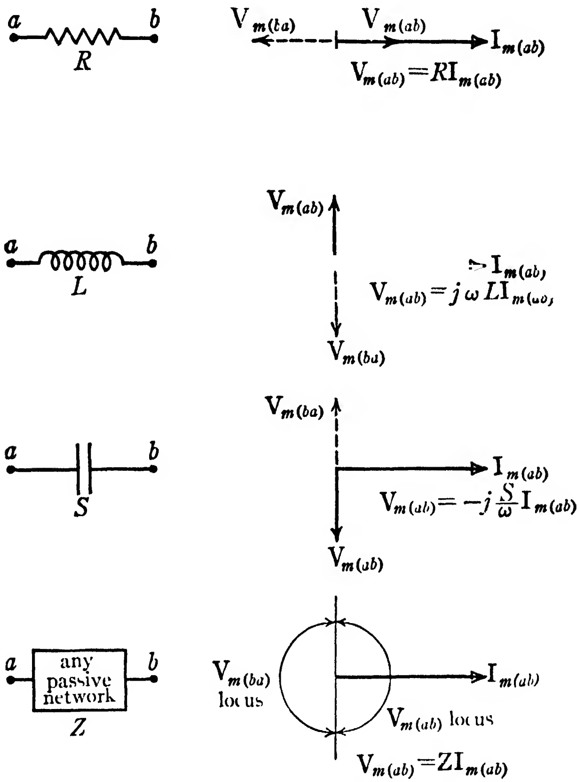


FIG. 7. Vector relations of $\mathbf{V}_{m(ab)}$ and $\mathbf{I}_{m(ab)}$ for passive elements.

In dotted lines is shown $\mathbf{V}_{m(ba)}$ or $-\mathbf{V}_{m(ab)}$, which can be interpreted as the voltage drop from b to a, or as the voltage rise from a to b. This fact illustrates the folly of talking loosely about the voltage leading the current, and so on, and emphasizes the necessity for maintaining a systematic terminology and notation.

6. NOTATION

In order to continue conventions previously established, lower-case letters are in general used to represent time functions, and capital letters are in general used to represent constants. Functional notation is

* These expressions for power are developed in Art. 10. As evident from study of Art 10, $\cos\theta_z$ always is positive for a passive network because R is always positive.

used only where it seems to lend clarity in the solution of problems involving transients. The symbols e, E, or $\mathbf{E}$ always stand for the electromotive force of an active element (a source), and the symbols v, V, or $\mathbf{V}$ always stand for the voltage drop through passive elements (resistance, inductance, capacitance) *in the direction of the appended subscripts* if double-subscript notation is used.

7. Illustrative example of series *RL* circuit

This numerical example is given to illustrate the use of complex quantities in obtaining the steady-state solution of a series *RL* circuit. A 500-cycle-per-second voltage of 155.2 volts amplitude is applied to 95.6 ohms resistance and 0.128 henry inductance in series. The questions are:

(a) *What is the amplitude of the current?*
(b) What is the instantaneous value of the current when the voltage is half its maximum negative value and decreasing in size, and what is the time at which this occurs?

Time is measured from an instant at which the voltage is $\sqrt{3}/2$ times its positive maximum and increasing. This example might be encountered in practice as a load connected to an alternating-current generator. Figure 4 may be used if the condenser is omitted.

Solution: [Part (a)]

$$\mathbf{Z} = 95.6 + j2\pi 500 \times 0.128 = 95.6 + j402 \text{ vector ohms,} \qquad [33b]$$

$$Z = \sqrt{95.6^2 + 402^2} = 413 \text{ ohms.} \qquad [33c]$$

$$\theta_z = \tan^{-1}\frac{402}{95.6} = \tan^{-1} 4.21 = 76.6° \text{ (not needed for Part a),} \qquad [34a]$$

$$I_m = \frac{E_m}{Z} = \frac{155.2}{413} = 0.376 \text{ amp.} \qquad [27i]$$

Solution: [Part (b)]

From the conditions of the problem the voltage is described by the expression

$$e = 155.2 \cos (2\pi 500t - 30°). \qquad [5a]$$

This can be expressed in exponential form as

$$e = \mathcal{R}_e[\mathbf{E}_m \epsilon^{j\omega t}], \qquad [7c]$$

in which

$$\mathbf{E}_m = 155.2\epsilon^{j(-\pi/6)} = 155.2\underline{/-30°}, \qquad [47]$$

since

$$\psi = -\frac{\pi}{6} \text{ radian} = -30°. \qquad [48]$$

The problem is to find the current when e is $-(\frac{1}{2})(155.2)$ and is decreasing in size, that is, when $\omega t + \psi$ is 240°. Then

$$\omega t = 240° + 30° = 270° = \frac{3\pi}{2} \tag{49}$$

and

$$t = \frac{3\pi}{2} \times \frac{1}{2\pi 500} = \frac{3}{2{,}000} = 1.5 \times 10^{-3} \text{ sec.} \tag{50}$$

Merely one value of t is found. Of course any value corresponding to

$$\omega t = \frac{3\pi}{2} \pm k2\pi, \tag{51}$$

where k is an integer, is equally correct.

When t is 1.5×10^{-3} sec, the current is

$$\left.\begin{aligned} i &= \mathcal{R}_e[\mathbf{I}_m\epsilon^{j(3\pi/2)}] = \mathcal{R}_e[0.376\epsilon^{j(-30°-76.6°)}\epsilon^{j(3\pi/2)}] \\ &= \mathcal{R}_e[0.376\underline{/-30° - 76.6° + 270°}] = \mathcal{R}_e[0.376\underline{/163.4°}] \\ &= 0.376 \cos 163.4° = -0.376 \cos 16.6° \\ &= -0.376 \times 0.958 = -0.360 \text{ amp.} \end{aligned}\right\} \tag{52}$$

A vector diagram for the instant when t is 1.5×10^{-3} sec is shown in Fig. 8. The lengths of the vectors are not drawn to any particular scale.

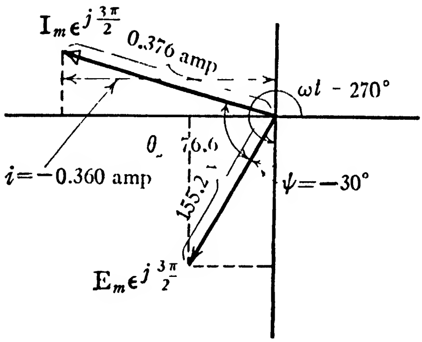


FIG. 8. Vector diagrams for example of Art. 7 when t is 1.5×10^{-3} sec.

The type of vector diagram more commonly used is one in which the complex quantities $\mathbf{E}_m$ and $\mathbf{I}_m$ and their components are shown. This does not involve time but merely shows the relative angular positions of these vectors. On such a diagram is shown how the voltage drops across the resistance and inductance add vectorially to give their combined voltage drop. In the RL circuit

$$\mathbf{I}_m(R + j\omega L) = \mathbf{E}_m \tag{53}$$

or

$$R\mathbf{I}_m + j\omega L\mathbf{I}_m = \mathbf{E}_m. \tag{53a}$$

Computing these quantities for the foregoing example gives

$$\mathbf{E}_m = 155.2\epsilon^{j(-\pi/6)} = 155.2\underline{/-30}, \tag{47a}$$

$$\mathbf{I}_m = 0.376\epsilon^{j[-(\pi/6)-76.6°]} = 0.376\underline{/-106.6°}, \tag{27j}$$

$$R\mathbf{I}_m = 95.6 \times 0.376\underline{/-106.6°} = 36.0\underline{/-106.6°}; \tag{54}$$

$$\left.\begin{aligned} j\omega L\mathbf{I}_m &= j402 \times 0.376\underline{/-106.6°} \\ &= 151.2\underline{/-106.6° + 90°} \\ &= 151.2\underline{/-16.6°}. \end{aligned}\right\} \tag{55}$$

These quantities are plotted in the vector diagram of Fig. 9. In this diagram all quantities are independent of time. Figure 8, on the other hand, is properly thought of as a snapshot at a particular instant, of the time vectors that are moving at an angular velocity of ω radians per

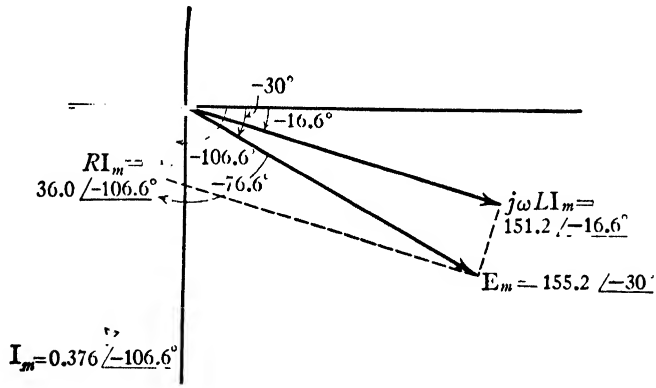


FIG. 9. Vector diagram for example of Art. 7 when t is 0

second. The advantage of the exponential form of solution is that it gives these vector quantities, such as $\mathbf{E}_m$ and $\mathbf{I}_m$, which are quite independent of time but which yield instantaneous values very readily when the rotating factor $\epsilon^{j\omega t}$ is inserted and the real part of the product is taken.

8. POWER

Thus far in this analysis of simple electric circuits only currents and voltages are considered. Another important quantity, the rate of energy flow, or power, is now discussed. This is important in both the communications and the power aspects of electrical engineering. In both these fields a significant feature of the problem is one of delivering a specified amount of power at a given point. The communications engineer must provide at the terminals of a loud-speaker or telephone receiver sufficient power to convey intelligence. The power engineer must supply a given city or factory with the power desired to operate motors and lamps.

When a given amount of power is to be delivered, the particular values of current and voltage are of relatively minor importance because of the ease with which they can be adjusted to the values desired by means of a transformer.

In analyzing circuits the usual procedure is the one thus far followed. Currents and voltages are determined first, and from these power is calculated. The electrical engineer may be interested in either or both of two power quantities: first, the instantaneous power and the way it varies from instant to instant; second, the average power or rate of energy flow over a more or less extended period. In the following articles both these power quantities are considered for alternating-current circuits operating in the steady state.

9. Instantaneous power

In a circuit branch through which there is a drop of potential $v(t)$ in the direction of current $i(t)$, the instantaneous power $p(t)$ received is

$$p(t) = v(t)i(t). \qquad [56]$$

If voltages and current are expressed in volts and amperes, respectively, power is expressed in watts, w.* Over any time interval Δt the average power P_{av} is given by

$$P_{av} = \frac{1}{\Delta t}\int_0^{\Delta t} v(t)i(t)dt. \qquad [57]$$

Equation 57 is an entirely general expression for average power and can be evaluated explicitly when the time functions $v(t)$ and $i(t)$ are known.

At this point only the special case of sinusoidally varying voltage and current is discussed; detailed consideration of the general case is reserved for other volumes of this series. For the special case of sinusoidal functions,

$$v(t) = V_m \cos(\omega t + \psi) \qquad [5b]$$

and

$$i(t) = I_m \cos(\omega t + \phi), \qquad [21b]$$

for which the expression for instantaneous power takes the form

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \psi)\cos(\omega t + \phi). \qquad [56a]$$

By means of the trigonometric identity

$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)], \qquad [58]$$

* The kilowatt (abbreviated kw) is commonly used for large quantities.

Eq. 56a may be converted into

$$p(t) = v(t)i(t) = \frac{V_m I_m}{2}[\cos(2\omega t + \psi + \phi) + \cos(\psi - \phi)]$$
$$= \frac{V_m I_m}{2}\cos(2\omega t + \psi + \phi) + \frac{V_m I_m}{2}\cos\theta_z. \qquad \blacktriangleright[56b]$$

The first of these terms represents a simple harmonic time function of twice the frequency of either voltage or current, whereas the second term is a constant whose magnitude depends only upon the product of the magnitudes of the voltage and the current and upon their *relative* phase angle θ_z.

A plot of the instantaneous power $p(t)$ according to Eq. 56a is shown

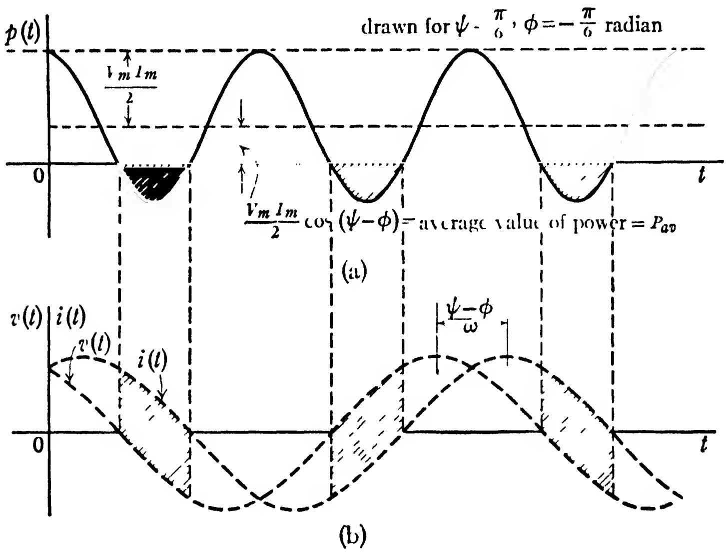


FIG. 10. Plot of instantaneous power derived from plots of instantaneous voltage and current

in Fig. 10a for the arbitrarily chosen angles $\pi/6$ radian for ψ and $-\pi/6$ radian for ϕ. In Fig. 10b are shown plots of $v(t)$ and $i(t)$. A comparison of these two figures shows graphically the relation of the instantaneous voltage, current, and power. The intervals during which the power is negative correspond to those during which the instantaneous voltage and current are opposite in sign. During these intervals, power is returned to the source from the passive elements where it was stored in either the electric or the magnetic form or in both. These intervals occur only if $v(t)$ and $i(t)$ are out of phase with each other, that is, if θ_z is not zero, which is the situation shown in these figures.

10. Average power

An inspection of Eq. 56b shows that the average rate of energy flow or average power P_{av} into the portion of the circuit considered, taken over any integral number of periods, is the constant term, or

$$P_{av} = \frac{V_m I_m}{2} \cos(\psi - \phi) = \frac{V_m I_m}{2} \cos\theta_z, \tag{56c}$$

where θ_z is the angle of the equivalent impedance.

An instructive alternative is to obtain Eq. 56c by actually calculating the average value of Eq. 56a over one complete period of the power; for example, from zero to π in terms of ωt or from zero to π/ω in terms of t. Thus,

$$\left.\begin{aligned} P_{av} &= \frac{1}{\frac{\pi}{\omega} - 0} \int_0^{\pi/\omega} p(t)dt \\ &= \frac{\omega}{\pi} \int_0^{\pi/\omega} \frac{V_m I_m}{2} [\cos(2\omega t + \psi + \phi) + \cos(\psi - \phi)]dt \\ &= \frac{V_m I_m}{2} \cos(\psi - \phi) = \frac{V_m I_m}{2} \cos\theta_z. \end{aligned}\right\} \tag{56d}$$

Over any nonintegral number of periods or cycles the average power in general is different. However, the electrical engineer associates the term "average power" only with a steady-state condition that persists for very many cycles. Consequently there can be little possibility of ambiguity in the meaning of the term.

There are several forms in which Eq. 56c for average power can be expressed when $\mathbf{V}_m$ is the vector voltage drop in the direction of the vector current $\mathbf{I}_m$ in a two-terminal impedance Z. When it is remembered that

$$\mathbf{V}_m = \mathbf{I}_m Z, \tag{27a}$$

and

$$\cos\theta_z = \frac{R}{Z}, \tag{28b}$$

the following forms of Eq. 56c are readily obtained:

$$\left.\begin{aligned} P_{av} &= \frac{V_m I_m}{2} \cos\theta_z = \frac{I_m^2}{2} R = \frac{1}{2} \frac{V_m^2}{Z} \cos\theta_z \\ &= \tfrac{1}{2} I_m^2 Z \cos\theta_z. \end{aligned}\right\} \tag{56e}$$

11. Root-mean-square or effective values of voltage and current

Because of the frequency with which expressions for average power are used, it is convenient to remove the factor 2 from Eq. 56c by defining new values of current and voltage which are $1/\sqrt{2}$ of the maximum of the sine wave. These new values, designated by I_e and V_e, are called the *effective current* and *effective voltage*, respectively; that is,

$$I_e \equiv \frac{I_m}{\sqrt{2}} \qquad [59]$$

and

$$V_e \equiv \frac{V_m}{\sqrt{2}}, \qquad [60]$$

and in terms of these effective values Eq. 56e becomes

$$\left.\begin{aligned} P_{av} &= V_e I_e \cos\theta_z \\ &= I_e^2 R = \frac{V_e^2}{Z}\cos\theta_z = I_e^2 Z \cos\theta_z. \end{aligned}\right\} \qquad \blacktriangleright[56f]$$

These are important relations. The first is quite general and gives the power input to any circuit in which the current and voltage are sinusoidal, are of the same frequency, and are displaced in phase by an angle θ_z. When the circuit consists of an impedance Z, the other expressions are also applicable.

The significance of effective value can perhaps be made clearer by considering Z to be a simple resistance R_1 so that

$$Z = Z_1 = R_1 + j0. \qquad [61]$$

If a vector voltage $\mathbf{V}_m$ is impressed on this resistance, causing a current $\mathbf{I}_m$, the average power delivered to the resistance is

$$P_{av} = \frac{V_m I_m}{2} = \frac{I_m^2}{2} R_1 = V_e I_e = I_e^2 R_1. \qquad [56g]$$

If a constant voltage V_{dc} is impressed on this same resistance, a constant current I_{dc} results, and the power delivered is

$$P_{dc} = V_{dc} I_{dc} = I_{dc}^2 R_1. \qquad [62]$$

A comparison of Eq. 56g with Eq. 62 shows that, for a resistance, a sine-wave alternating current and voltage deliver the same power as a direct current and voltage, provided the effective values of the alternating-current quantities are equal to the direct-current values.

Since the instantaneous power delivered to a resistance R carrying an instantaneous current i is i^2R, it is evident that the average power delivered is equal to the average value of i^2 times the resistance. *The average of the square of the current is quite different from the square of the average current.* In the case of a sinusoidal current, for example, the average current over any integral number of cycles, and hence the square of this average current, are zero, whereas the average of the square of the current is finite over any finite interval of time. Mathematically, the average of the square of the current is easily obtained. Thus if

$$i = I_m \cos(\omega t + \phi), \tag{21b}$$

the average of i^2 over a period is

$$\left.\begin{aligned} I_e^2 &= \frac{\omega}{2\pi}\int_0^{t=2\pi/\omega} [I_m \cos(\omega t + \phi)]^2 dt \\ &= \frac{\omega}{2\pi} I_m^2 \int_0^{t=2\pi/\omega} \cos^2(\omega t + \phi) dt \\ &= \frac{\omega}{2\pi} I_m^2 \int_0^{t=2\pi/\omega} \frac{1 + \cos 2(\omega t + \phi)}{2} dt \\ &= \frac{I_m^2}{2} \end{aligned}\right\} \tag{63}$$

where I_e is the same as defined by Eq. 59. Thus for sinusoidal currents, the effective value I_e is that whose square is equal to the average of the square of the instantaneous current. For this reason the effective value is often called the *root-mean-square* (abbreviated rms) value since it is equal to the square root of the average (or mean) of the square of the instantaneous current. Thus I_e can be defined by

$$I_e \equiv \sqrt{\text{average } i^2} = \sqrt{\frac{1}{T}\int_0^T i^2 dt}, \tag{▶64}$$

where T is for an integral number of cycles. Equation 64 is, in fact, the general definition of the effective value of a current — a definition which is always true regardless of wave form. It reduces to Eq. 59 for the special condition where the current varies sinusoidally. Similarly, the general definition of the effective value of voltage V_e is

$$V_e \equiv \sqrt{\text{average } v^2} = \sqrt{\frac{1}{T}\int_0^T v^2 dt}, \tag{▶65}$$

which likewise reduces to Eq. 60 for sinusoidal voltages. In most practical work the vector current and vector voltage are taken respectively as $\mathbf{I}_e$ and $\mathbf{V}_e$, rather than $\mathbf{I}_m$ and $\mathbf{V}_m$, in order to avoid the use of the factor 2 in

expressions for power. Measuring instruments, such as voltmeters and ammeters, for use on alternating-current circuits are nearly always calibrated in terms of effective values. Vector calculations are also usually done in terms of I_e and V_e.

Hereafter, in order to simplify symbols, the subscript e is omitted, and E, V, I, and Q are understood to represent effective values unless otherwise specified. This change makes room for double-subscript notation in representing effective values when desired. The conventions of polarity and direction apply to effective values in complex in precisely the same manner as explained in Art. 5 for complex amplitudes.*

The effective value or $1/\sqrt{2}$ of the maximum is different from, and greater than, the average value of one loop of a sine wave which is $2/\pi$ of the maximum value. This factor $2/\pi$ is readily obtained by the usual integration process for finding the average value of a function.

The application of Eqs. 64 and 65 to nonsinusoidal periodic quantities is considered in detail in subsequent volumes of this series.

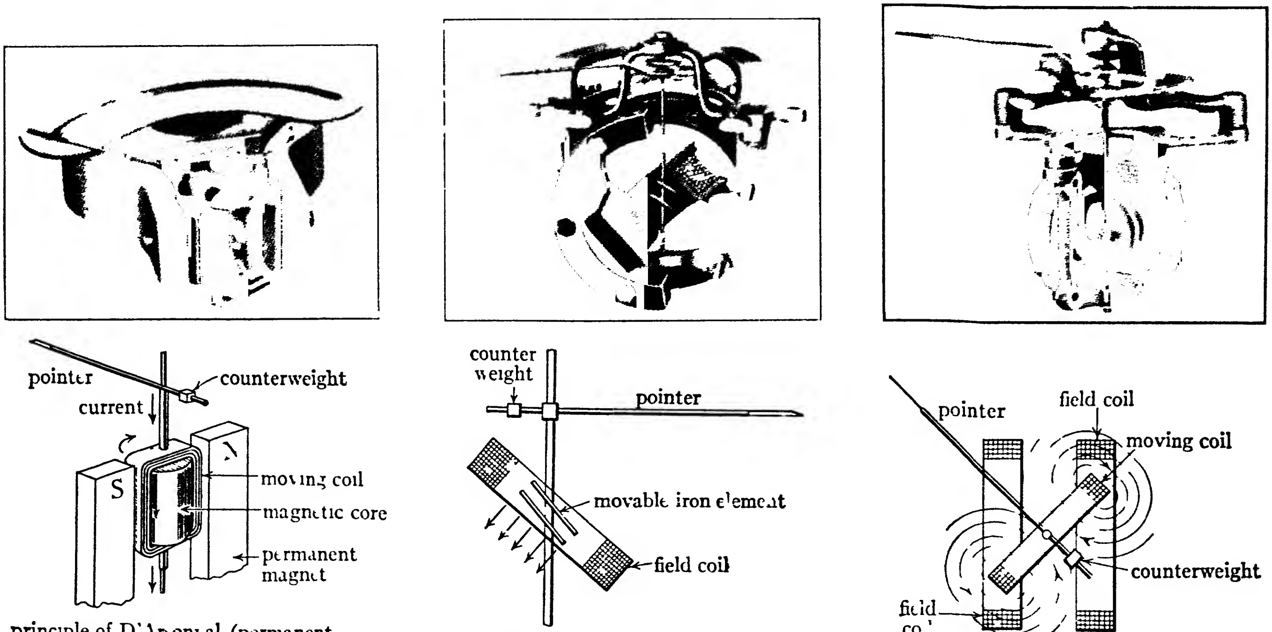
12. Measurement of effective values

The principal instruments for the measurement of alternating current and voltage are of three types: dynamometer, iron vane, and thermocouple. Each of these measures the average square of the current in the instrument and has effective, or root-mean-square, values marked on the scale.

In the dynamometer type are two coils in series, one stationary and one on the moving element. The instantaneous torque acting on the moving element is proportional to the product of the instantaneous currents in the two coils and therefore to the square of the current, since both currents are the same. From what has preceded, it is evident that this instantaneous torque varies at double the applied frequency but is always in the same direction. If the natural period of the mechanical moving system is large compared to the double-source-frequency period, the instrument is made to take a substantially steady deflection, depending on the average square of the current. Since the scale is marked in terms of the square root of the average square of the current, the instrument is made to read directly in effective values. A dynamometer voltmeter is merely a milliammeter with a large series resistance.

In the iron-vane type, a stationary coil induces poles in a soft iron armature or vane mounted on the moving element, the iron being used under such conditions that the pole strengths are proportional to the inducing field strength. The torque acting on the vane is proportional to

* Effective values in complex are also sometimes called *vector volts, vector amperes,* and *vector coulombs.*



principle of D'Arsonval (permanent magnet, moving-coil) instruments
(a)

principle of operation of Thomson inclined-coil instruments
(b)

principle of dynamometer-type instruments
(c)

Courtesy General Electric Co.

Mechanisms of most commonly used types of electrical instruments

the product of field strength and pole strength and therefore to the square of the current in the instrument. Mechanically, the instrument operates similarly to the dynamometer type.

In the thermocouple type of instrument the alternating current passes through a resistance heating element whose temperature is measured by a thermocouple operating a D'Arsonval milliammeter. The average heat input to the resistance element is proportional to the average square of the current; the temperature rise of this element and consequently the thermocouple voltage are approximately proportional to the average heat input because of the heat capacity of the heater. The D'Arsonval instrument deflection is approximately proportional, therefore, to the average square of the alternating current. As in the previous two instruments, the scale is marked in terms of the square root of the average square of the current.

All three of these instruments are calibrated on direct current, using reversed readings to eliminate any stray direct-current field effects. In general, the dynamometer and iron-vane types are limited to the lower range of audio frequencies, the dynamometer being the more accurate and expensive, though the accuracy of the iron-vane type is adequate for most measurements. The thermocouple type is more sensitive, that is, can be made to operate on much less power than the other two types, and is accurate for all frequencies, including the radio-frequency range. Its overload capacity, however, is small, the resistance element sometimes burning out at 1.2 times full-scale current. The thermocouple type is, therefore, far less rugged electrically than the other types, which may safely stand momentary overloads of many times their full-scale current.

The rectifier and vacuum-tube type of instrument, as well as other types of instruments, are also used for alternating-current measurements, their deflections usually depending on something between the half-wave average and the effective values.

13. Measurement of power

A wattmeter is an instrument for measuring the rate of flow of electrical energy (power) past a given boundary in an electric circuit. In the simplest case, as in Fig. 11, one may wish to measure the average rate of energy transfer from $a'b'$ to ab. This measurement is conveniently made in many instances by a wattmeter in which there are two coils, one fixed and one movable. The movable coil is mounted in pivots and carries a pointer that indicates on a suitable scale the angular position of the coil with respect to an arbitrary zero. A spring gives a definite angular position to the coil in the absence of any electrical torque acting on the coil.

If the two coils have currents i_c and i_p, respectively, in them at any

instant, an electromagnetic torque T that is proportional to $i_c i_p$ acts on each. This follows from the fact that the torque is proportional to the product of i_c and the magnetic flux density set up by i_p, or vice versa. This torque is also a function of the geometry of the coils, including the angular position of the moving coil. If i_c and i_p are constant, the coil takes an equilibrium angular position at which the electromagnetic and spring torques are equal in magnitude and opposite in direction. This angular position can be used as a measure of $i_c i_p$.

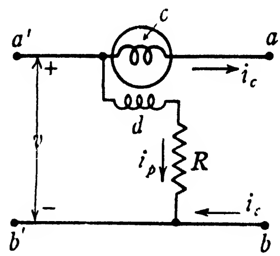


FIG. 11 Circuit diagram for wattmeter

In a wattmeter the fixed coil is usually wound with a small number of turns of relatively large wire suitable for carrying currents of a few to about a hundred amperes, depending on the rating of the instrument. The moving coil, on the other hand, is wound with many turns of fine wire suitable for a few or few tens of milliamperes. In series with the moving coil is connected a relatively large fixed resistance R so that this coil-resistance combination can be connected directly between points having a potential difference of a few tens or hundreds of volts. In Fig. 11 the fixed or *current coil* is indicated by c and carries the circuit current i_c. The moving coil is indicated by d which is in series with the fixed resistance R. This coil is called the *potential coil*, and the coil-resistance combination is called the *potential circuit* of the wattmeter. Evidently if R_d is the resistance of the potential coil, the current i_p in it is

$$i_p = \frac{v}{R_d + R}. \tag{66}$$

If v and i_c are constant, the wattmeter indication is a measure of the power p,

$$p = vi_c, \tag{56h}$$

since the indication depends on $i_c i_p$, and i_p is proportional to v. When the scale is suitably marked at the positions taken by a pointer when known values of v and i_c are applied to the instrument, the wattmeter can be used to measure power directly when v and i_c are constant.

If v and i_c are varying periodically, the equilibrium position of the moving coil varies periodically. For example, if

$$v = V_m \cos(\omega t + \psi) \tag{5b}$$

and

$$i = I_m \cos(\omega t + \phi), \tag{21b}$$

then

$$p(t) = vi = \frac{V_m I_m}{2} \cos \theta_z + \frac{V_m I_m}{2} \cos (2\omega t + \psi + \phi). \qquad [56b]$$

If the moving coil has a sufficiently small moment of inertia, as in wattmeter oscillograph elements, the deflections indicate the instantaneous values of $p(t)$ rather accurately for values of frequency as great as a few hundred cycles per second. In the usual wattmeter, however, the moving coil has sufficient moment of inertia to make the motion of the coil associated with the last term of Eq. 56b, which is a vibration at double the frequency of v or i, negligible for frequencies greater than roughly ten cycles per second. For the usual conditions of use, therefore, the wattmeter deflection depends only on

$$P_{av} = VI \cos \theta_z \qquad [56i]$$

or on the average power. A wattmeter calibrated on direct current can thus be used to measure average power in an alternating-current circuit.

Unless provided with a compensating winding, the reading of a wattmeter always must be corrected for the power loss either in its current coil or in its potential circuit, depending upon the side of the current coil to which the potential circuit is connected.[2] Whether or not this correction is important depends upon its magnitude compared to the power being measured, and upon the accuracy desired.

It should be noted that a wattmeter can be, and often is, used to measure the average of the instantaneous product of a current and voltage that are not in the same portion of a circuit. Then the wattmeter reading must be interpreted merely as this average, which may or may not be related to an actual rate of energy transfer in the circuit.

14. Power factor

For sinusoidal currents and voltages, the equation

$$P_{av} = VI \cos \theta_z \qquad [56i]$$

for average power is the same, except for the factor $\cos \theta_z$ as that for a circuit carrying a direct current I_{dc} and having across it a direct potential drop V_{dc},

$$P_{av} = V_{dc} I_{dc}. \qquad [62a]$$

This $\cos \theta_z$ factor provides for the fact that in an alternating-current circuit the current and voltage are, in general, not in phase. If they are in phase, θ_z is zero and $\cos \theta_z$ is unity, and the VI product is the average

[2] C. W. Ricker and C. E. Tucker, *Electrical Engineering Laboratory Experiments* (4th ed.; New York: McGraw-Hill Book Company, Inc., 1940), 8-14.

power. If they are displaced in phase by an angle θ_z, the factor $\cos\theta_z$ must be included with the VI product to obtain the average power. The factor $\cos\theta_z$ is therefore aptly termed the *power factor* of the circuit.

15. Illustrative example of power computations

A source of 155 volts amplitude, 60 cycles per second, supplies power to a 1,000-ohm resistor and a 1.0-microfarad condenser in series. The questions are:

(a) What is the maximum power supplied to the capacitor?
(b) At what instant does (a) occur?
(c) What is the maximum power supplied to the resistor?
(d) At what instant does (c) occur?
(e) What is the average power supplied to the capacitor?
(f) What is the average power supplied to the resistor?

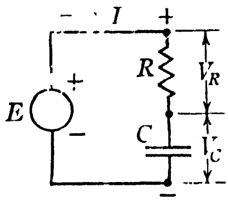


FIG. 12. Series RC circuit for example of Art. 15.

The circuit diagram is shown by Fig. 12.

Solution:

$$X_C = -\frac{10^6}{2 \times 3.14 \times 60 \times 1.0} = -2{,}650 \text{ ohms}, \qquad [67]$$

$$Z = \sqrt{(1{,}000)^2 + (2{,}650)^2} = 2{,}830 \text{ ohms}, \qquad [35b]$$

$$\sqrt{2}\, I = \frac{\sqrt{2}\, E}{Z} = \frac{155}{2{,}830} = 0.0548 \text{ amp}, \qquad [27k]$$

$$\cos\theta_z = \frac{1{,}000}{2{,}830} = 0.354; \qquad [36a]$$

$$\theta_z = -69.2°, \qquad [68]$$

$$\sqrt{2}\, V_R = 0.0548 \times 1{,}000 = 54.8 \text{ v}, \qquad [27m]$$

$$\sqrt{2}\, V_C = 0.0548 \times 2{,}650 = 145 \text{ v}. \qquad [27n]$$

If

$$e = E_m \cos \omega t, \qquad [5c]$$

$$\psi = 0, \qquad [69]$$

then

$$\phi = -\theta_z. \qquad [70]$$

$$i = 0.0548 \cos(377t + 69.2°) = 0.0548 \cos(377t + 1.21), \qquad [21c]$$

$$v_R = 54.8 \cos(377t + 69.2°) = 54.8 \cos(377t + 1.21), \qquad [5d]$$

$$v_C = 145 \cos(377t - 20.8°) = 145 \cos(377t - 0.362). \qquad [5e]$$

(a) The maximum power supplied to the capacitor is

$$p_{C(max)} = \frac{145 \times 0.0548}{2} = 3.98 \text{ w.} \qquad [71]$$

(b) This maximum occurs when

$$\cos(2 \times 377t - 0.362 + 1.21) = +1 \qquad [72]$$

or when

$$2 \times 377t - 0.362 + 1.21 = 2k\pi \qquad [72a]$$

where k is zero or an integer. Hence,

$$t = \frac{-0.85}{2 \times 377} + \frac{k\pi}{377} = -0.00113 + \frac{k\pi}{377} \text{ sec.} \qquad [73]$$

(c) The maximum power supplied to the resistor is

$$p_{R(max)} = \frac{54.8 \times 0.0548}{2} + \frac{54.8 \times 0.0548}{2} = 3.00 \text{ w.} \qquad [74]$$

(d) This maximum occurs when

$$\cos(2 \times 377t + 1.21 + 1.21) = +1 \qquad [75]$$

or when

$$2 \times 377t + 1.21 + 1.21 = 2k\pi \qquad [75a]$$

where k is zero or an integer. Hence,

$$t = \frac{-2.42}{2 \times 377} + \frac{k\pi}{377} = -0.00321 + \frac{k\pi}{377} \text{ sec.} \qquad [76]$$

(e) The average power supplied to the capacitor is

$$P_C = \frac{145 \times 0.0548}{2} \cos\frac{\pi}{2} = 0. \qquad [77]$$

(f) The average power supplied to the resistor is

$$P_R = \frac{54.8 \times 0.0548}{2} \cos 0 = \frac{(0.0548)^2}{2} \times 1{,}000 = 1.50 \text{ w,} \qquad [78]$$

which is the same as the average power supplied to the circuit,

$$P = \frac{155 \times 0.0548}{2} \times 0.354 = 1.50 \text{ w.} \qquad [79]$$

16. Steady-state response of the parallel $GC\Gamma$ circuit with an alternating-current source

The behavior of this circuit, which is the dual of the series *RLS* circuit, follows from the behavior of that circuit if current is replaced by voltage, and vice versa, resistance by conductance, inductance by capacitance, and elastance by reciprocal inductance. The circuit arrangement is shown in Fig. 13. The open-circuit condition here again replaces the closed-circuit condition in the dual case.

Current equilibrium with the switch K in position 1 is expressed by

$$i = C\frac{dv}{dt} + Gv + \Gamma\int vdt = I_m \cos(\omega t + \phi). \qquad [80]$$

The similarity of this equation to that for the series RLS circuit is seen if Eq. 26 for the latter is written in terms of current thus:

$$e = L\frac{di}{dt} + Ri + S\int idt = E_m \cos(\omega t + \psi). \qquad [26b]$$

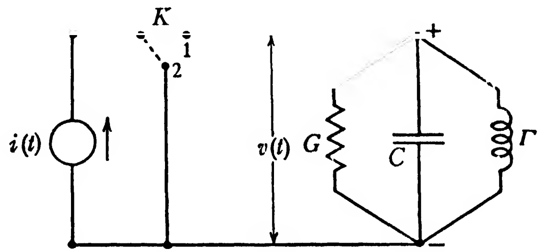


FIG. 13. Parallel $GC\Gamma$ circuit with alternating-current source.

From the complete correspondence between Eqs. 80 and 26b, term for term, the steady-state solution of Eq. 80 can immediately be written in terms of the complex functions V, I, and the circuit parameters as

$$\left(j\omega C + G + \frac{\Gamma}{j\omega}\right)\mathrm{V} = \mathrm{I}, \qquad [81]$$

or solving for V, as

$$\mathrm{V} = \frac{\mathrm{I}}{G + j\left(\omega C - \frac{\Gamma}{\omega}\right)}. \qquad [82]$$

In Eq. 82 the denominator is denoted by Y where

$$\mathrm{Y} \equiv G + j\left(\omega C - \frac{\Gamma}{\omega}\right) \equiv G + jB \equiv Y\epsilon^{j\theta_y}, \qquad [83]$$

$$Y = \sqrt{G^2 + \left(\omega C - \frac{\Gamma}{\omega}\right)^2}, \qquad [83a]$$

$$\theta_y = \tan^{-1}\left(\frac{\omega C - \frac{\Gamma}{\omega}}{G}\right) = \tan^{-1}\frac{B}{G}. \qquad [84]$$

The complex function Y is called the *admittance* of the circuit. It consists of the real part G, called the *conductance*, and the imaginary part B, called the *susceptance*.

As in the series *RLS* circuit, the relations for circuits having only two of the parameters in parallel can be obtained from the general solution for the parallel *GCΓ* case. The most important of these practically is the

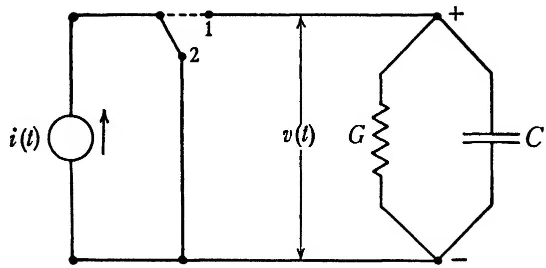


FIG. 14. Parallel *GC* circuit with alternating-current source.

parallel *GC* case, Fig. 14, which is the dual of the series *RL* circuit.

$$\mathbf{I} = \mathbf{E}\mathbf{Y}, \tag{85}$$

$$\mathbf{Y} = G + j\omega C, \tag{86}$$

$$Y = \sqrt{G^2 + \omega^2 C^2}, \tag{86a}$$

$$\theta_y = \tan^{-1}\frac{\omega C}{G} = \tan^{-1}\frac{B}{G}. \tag{87}$$

17. IMPEDANCE OF SERIES AND PARALLEL BRANCH COMBINATIONS

The alternating-current steady-state solutions developed thus far are next extended to provide means for analyzing simple series and parallel combinations of circuit branches.

The calculation of the voltage-current relations for two or more elements in series in terms of their impedances is very simple. In fact, the impedance of any number of elements in series can be readily seen to be the sum of their separate impedances. Thus a circuit consisting of several resistances, inductances, and elastances in series, behaves when viewed from its terminals, precisely like a simple series circuit whose resistance, inductance, and elastance are equal respectively to the sums of the individual resistances, inductances, and elastances. It follows, for example, that if three circuit branches whose impedances are $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$, respectively, are connected in series, their combined impedance $\mathbf{Z}_0$ is

$$\mathbf{Z}_0 = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3. \tag{▶88}$$

The addition in Eq. 88 of course must be performed according to the rules for complex numbers; that is, the real part of $\mathbf{Z}_0$ is the sum of the

real parts of Z_1, Z_2, and Z_3, while the imaginary part of Z_0 is the sum of the imaginary parts of Z_1, Z_2, and Z_3. The application of this principle appears in the numerical example which is considered in the following article.

Next is considered the case of two series branches connected in parallel to a single voltage source as shown in Fig. 15. It is desired to find the current i_0 taken by the combination.

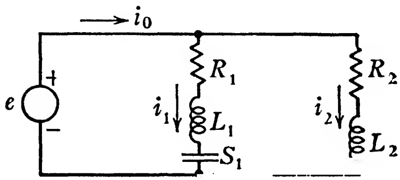


FIG. 15. Impedances in parallel.

If the voltage applied to the terminals of the combination is known, it is evident that the current in branch 1 depends in no way on the current in branch 2, and vice versa. Hence one can write from Eq. 21a

$$i_1 = \mathcal{R}_e\left[\frac{\mathrm{E}_m}{Z_1}\epsilon^{j\omega t}\right] = \mathcal{R}_e[\mathrm{I}_{1m}\epsilon^{j\omega t}], \tag{89}$$

and

$$i_2 = \mathcal{R}_e\left[\frac{\mathrm{E}_m}{Z_2}\epsilon^{j\omega t}\right] = \mathcal{R}_e[\mathrm{I}_{2m}\epsilon^{j\omega t}]. \tag{90}$$

From Kirchhoff's current law

$$\left.\begin{aligned} i_0 &= i_1 + i_2 \\ &= \mathcal{R}_e\left[\mathrm{E}_m\left(\frac{1}{Z_1}+\frac{1}{Z_2}\right)\epsilon^{j\omega t}\right] \\ &= \mathcal{R}_e[(\mathrm{I}_{1m}+\mathrm{I}_{2m})\epsilon^{j\omega t}]. \end{aligned}\right\} \tag{91}$$

Equation 91 is inconvenient in that it requires the addition of the reciprocals of the complex functions Z_1 and Z_2. This inconvenience can be somewhat lessened if two new complex functions Y_1 and Y_2 are defined by

$$\mathrm{Y}_1 \equiv \frac{1}{Z_1} = \frac{1}{R_1 + jX_1}, \tag{92}$$

$$\mathrm{Y}_2 \equiv \frac{1}{Z_2} = \frac{1}{R_2 + jX_2}, \tag{93}$$

and the expressions on the right are rationalized. This rationalization is done by multiplying both the denominator and the numerator by the

conjugate of the denominator. Thus

$$Y_1 = \frac{R_1 - jX_1}{(R_1 + jX_1)(R_1 - jX_1)} = \frac{R_1 - jX_1}{R_1^2 + X_1^2}, \qquad [92a]$$

$$Y_2 = \frac{R_2 - jX_2}{(R_2 + jX_2)(R_2 - jX_2)} = \frac{R_2 - jX_2}{R_2^2 + X_2^2}. \qquad [93a]$$

The complex expressions for such Y's are used so frequently that they are given a special notation. Considering Y_1 only, for simplicity one may write

$$Y_1 = G_1 + jB_1, \qquad [92b]$$

in which Y_1 is called a *complex admittance function,* usually shortened to *admittance function*. The real part of the admittance function, G_1, by a comparison of Eq. 92b with Eq. 92a, is

$$G_1 = \frac{R_1}{R_1^2 + X_1^2}. \qquad [94]$$

This is called the *conductance function.* The imaginary part of the admittance function is evidently given by

$$B_1 = \frac{-X_1}{R_1^2 + X_1^2} \qquad [95]$$

and is called the *susceptance function.* The distinction between G_1, which is the real part of an admittance function, and the G of the parallel $GC\Gamma$ circuit of Art. 16 should be carefully noted. In the $GC\Gamma$ circuit it turns out that G, which is the reciprocal of the resistance of one branch, is also the real part of the admittance function. In this very special case G equals $1/R$. But in general, and this should be noted very carefully, a conductance function G is not the reciprocal of a resistance or even of a resistance function R but is the real part of an admittance function. A similar caution applies to any susceptance function B, as is emphasized later.

The expression in Eq. 91 containing the Z's can now be evaluated. It becomes

$$\frac{1}{Z_1} + \frac{1}{Z_2} = Y_1 + Y_2 = G_1 + G_2 + j[B_1 + B_2]. \qquad [96]$$

The definitions of G_2 and B_2 are evident from those for G_1 and B_1. To simplify writing, the combined admittance of the two circuit branches is represented by

$$Y_0 = G_0 + jB_0, \qquad ▶[97]$$

in which

$$G_0 = G_1 + G_2, \qquad ▶[98]$$

$$B_0 = B_1 + B_2. \qquad ▶[99]$$

By use of Eq. 97, Eq. 91 can be written

$$i_0 = \mathcal{R}_e[\mathrm{E}_m \mathrm{Y}_0 \epsilon^{j\omega t}] = \mathcal{R}_e[\mathrm{I}_{0m} \epsilon^{j\omega t}]. \qquad [91a]$$

For any particular branch,

$$\theta_y = \tan^{-1} \frac{B}{G} = -\theta_z. \qquad [84a]$$

The admittance triangle is drawn in Fig. 16 for a branch such as branch 1 of Fig. 15, beside the impedance triangle for the same branch.

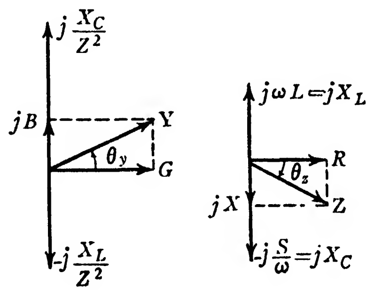


FIG. 16. Admittance and impedance triangles.

If the parallel combination is itself part of a series combination, it is desirable to write Eq. 91a in terms of an impedance, thus:

$$i_0 = \mathcal{R}_e\left[\frac{\mathrm{E}_m}{\mathrm{Z}_0} \epsilon^{j\omega t}\right], \qquad [91b]$$

where

$$\mathrm{Z}_0 = \frac{1}{\mathrm{Y}_0} = \frac{1}{G_0 + jB_0}. \qquad [100]$$

Equation 100 can be rationalized as follows:

$$\left.\begin{aligned} \mathrm{Z}_0 &= \frac{G_0 - jB_0}{(G_0 + jB_0)(G_0 - jB_0)} = \frac{G_0}{G_0^2 + B_0^2} - j\frac{B_0}{G_0^2 + B_0^2} \\ &= R_0 + jX_0, \end{aligned}\right\} \qquad [100a]$$

in which

$$R_0 = \frac{G_0}{G_0^2 + B_0^2} \qquad [101]$$

and

$$X_0 = \frac{-B_0}{G_0^2 + B_0^2}. \qquad [102]$$

Here R_0 is the real part of the impedance function Z_0 and is called the *resistance function*. In general it is not the resistance of any particular circuit element. It occupies the same position in the expression for the combined impedance of the two parallel circuits that a resistance R does *in the impedance of a simple series circuit; R_0, however, is a function of* angular frequency. The quantity X_0 is the imaginary part of Z_0 and is called the *reactance function*. As with R_0, X_0 is not the reactance of any single circuit element but is defined merely as the imaginary part of Z_0. It is very important to note carefully that

$$R_0 \neq \frac{1}{G_0} \tag{103}$$

and that

$$X_0 \neq \frac{1}{B_0}\,. \tag{104}$$

As the foregoing development is rather lengthy and involves numerous symbols, it may be well to recapitulate by outlining the method of calculation. First, the component conductance function G_1 is obtained from Eq. 94, and a similar expression is obtained for G_2. The susceptance function B_1 is obtained from Eq. 95, and B_2 is obtained similarly. These are then added as in Eqs. 98 and 99 to give the components of the resultant admittance Y_0 as shown in Eq. 97. If the impedance function Z_0 of the parallel combination is wanted instead of the admittance function Y_0, it can be calculated by Eqs. 100a, 101, and 102.

If three or more circuit branches, instead of two, are in parallel, the same method applies. The only difference is that there are more G's and B's to add together to get the combined admittance Y_0.

The methods of combining admittances and impedances and of changing from one form to the other are applicable regardless of how the particular admittance or impedance functions are originally obtained. Thus for some circuits, such as a resistance and a capacitance in parallel, the parallel formulation leading to an admittance function may be the simpler, whereas for a series circuit, the formulation leading to the impedance function may be simpler. But once the admittance or impedance function is obtained, it can be inverted or combined in the ways shown in this article.

18. Illustrative example of series-parallel circuit

In Fig. 17 is shown a circuit typical of the form encountered frequently in both the power and communications fields. In the power field it might be the equivalent circuit of an induction motor with a phase advancer;

in communications it might represent any of numerous transformer circuits or other forms of coupling circuits. The numerical values are quite arbitrary and vary with any particular application. The questions are:

(a) What is the effective value of the current delivered by the source?
(b) What is the effective value of the voltage across *cd*, and what is its phase relative to that of the source?

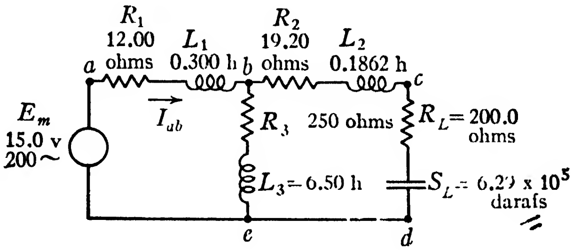


FIG. 17. Series parallel circuit for example of Art. 18.

Solution: The general plan of attack is to find the resultant impedance of the entire circuit as viewed from the source, from which the current in *ab* is readily found, as the source voltage is known. This current then divides between paths *be* and *bcde* in the ratio of their admittances. The voltage *cd* is calculated from the current in *cd* times the impedance between *c* and *d*.

Calculation of the equivalent impedance of the entire circuit as viewed from the source begins at the portion of the circuit most distant from the source, in this case at the portion *bcd*.

$$Z_{cd} = R_L - j\frac{S_L}{\omega} = 200.0 - j\frac{6.29 \times 10^5}{2\pi 200} = 200.0 - j500.0 \text{ vector ohms,} \qquad [35c]$$

$$Z_{bc} = R_2 + j\omega L_2 = 19.2 + j2\pi 200 \times 0.1862 = 19.2 + j234.0 \text{ vector ohms,} \qquad [33d]$$

$$Z_{bd} = Z_{cd} + Z_{bc} = 219.2 - j266.0 \text{ vector ohms,} \qquad [88a]$$

$$\left.\begin{aligned} \mathbf{Y}_{bd} &= \frac{1}{Z_{bd}} = \frac{219.2 + j266.0}{(219.2 - j266.0)(219.2 + j266.0)} = \frac{219.2 + j266.0}{219.2^2 + 266.0^2} \\ &= \frac{219.2 + j266.0}{48{,}000 + 70{,}800} = \frac{219.2 + j266.0}{118{,}800} \\ &= (1.840 + j2.24)10^{-3} \text{ vector mho} \\ &= 2.90 \times 10^{-3}\underline{/50.6^\circ} \text{ vector mho.} \end{aligned}\right\} \qquad [92c]$$

The calculation of the reciprocal of a complex number is likely to be one of the tedious bits of arithmetic in alternating-current calculations. There are several methods of making the calculation, two of which are illustrated here. The calculation of $\mathbf{Y}_{be}$ uses the polar form, which can be stated as follows:

$$\mathbf{Y}_{be} = \frac{1}{Z_{be}} = \frac{1}{Z_{be}\epsilon^{j\theta_{be}}} = \frac{1}{Z_{be}}\epsilon^{-j\theta_{be}}. \qquad [105]$$

This form is particularly convenient for use with a vector slide rule, which is a useful device for dealing with the arithmetic of complex numbers.

$$\left.\begin{aligned} Z_{be} &= R_3 + j\omega L_3 = 250 + j2\pi 200 \times 6.50 = 250 + j8170 \\ &= 8170\underline{/88.3^\circ} \text{ vector ohms;} \end{aligned}\right\} \quad [33e]$$

$$\left.\begin{aligned} \mathbf{Y}_{be} &= \frac{1}{8170}\underline{/-88.3^\circ} = 1.224 \times 10^{-4}\underline{/-88.3^\circ} \\ &= 1.224 \times 10^{-4}[\cos(-88.3^\circ) + j\sin(-88.3^\circ)] \\ &= 1.224 \times 10^{-4}[\cos 88.3^\circ - j\sin 88.3^\circ] \\ &= (0.036 - j1.224)10^{-4} \text{ vector mho.} \end{aligned}\right\} \quad [92d]$$

If the total Y between b and e is $\mathbf{Y}_{23}$,

$$\left.\begin{aligned} \mathbf{Y}_{23} &= \mathbf{Y}_{be} + \mathbf{Y}_{bd} = [1.840 + 0.004 + j(2.240 - 0.122)]10^{-3} \\ &= (1.844 + j2.118)10^{-3} \text{ vector mho} \\ &= 2.81 \times 10^{-3}\underline{/48.9^\circ} \text{ vector mho;} \end{aligned}\right\} \quad [97a]$$

$$\left.\begin{aligned} \mathbf{Z}_{23} &= \frac{1}{\mathbf{Y}_{23}} = 234 - j269 \text{ vector ohms} \\ &= 356\underline{/-48.9^\circ} \text{ vector ohms;} \end{aligned}\right\} \quad [100b]$$

$$Z_{23} = \frac{1}{Y_{23}} = 356 \text{ ohms.} \quad [100c]$$

Now $\mathbf{Z}_{ab}$ may be added to $\mathbf{Z}_{23}$ to obtain the equivalent impedance $\mathbf{Z}_0$ of the entire circuit viewed from the terminals:

$$\mathbf{Z}_{ab} = 12.00 + j2\pi 200 \times 0.300 = 12.00 + j376 \text{ vector ohms;} \quad [33f]$$

$$\mathbf{Z}_0 = \mathbf{Z}_{ab} + \mathbf{Z}_{23} = 246 + j107 \text{ vector ohms} = 268\underline{/23.5^\circ} \text{ vector ohms.} \quad [88b]$$

The effective current in ab measured with respect to the voltage phase angle as zero is

$$\mathbf{I}_{ab} = \frac{15.0\underline{/0^\circ}}{1.41 \times 268\underline{/23.5^\circ}} = 0.0397\underline{/-23.5^\circ} \text{ vector amp.} \quad [27p]$$

The effective current in cd is given by

$$\left.\begin{aligned} \mathbf{I}_{cd} &= \mathbf{I}_{ab}\frac{\mathbf{Y}_{bd}}{\mathbf{Y}_{23}} = 0.0397\underline{/-23.5^\circ}\,\frac{2.90 \times 10^{-3}\underline{/50.6^\circ}}{2.81 \times 10^{-3}\underline{/48.9^\circ}} \\ &= 0.0410\underline{/-23.5^\circ + 50.6^\circ - 48.9^\circ} \\ &= 0.0410\underline{/-21.8^\circ} \text{ vector amp.} \end{aligned}\right\} \quad [106]$$

The effective voltage $\mathbf{V}_{cd}$ is given by $\mathbf{Z}_{cd}\mathbf{I}_{cd}$:

$$\mathbf{Z}_{cd} = 200 - j500 = 538\underline{/-68.2^\circ} \text{ vector ohms,} \quad [35d]$$

$$\left.\begin{aligned} \mathbf{V}_{cd} &= \mathbf{Z}_{cd}\mathbf{I}_{cd} = 538\underline{/-68.2^\circ} \times 0.0410\underline{/-21.8^\circ} \\ &= 22.1\underline{/-90.0^\circ} \text{ vector v.} \end{aligned}\right\} \quad [27q]$$

The word " vector " has been prefixed to the units of the results for emphasis. Hereafter this is not done. The form of the result indicates whether it is a complex number or merely a magnitude.

In this solution two interesting effects peculiar to alternating-current circuits are observed. The first is that the current I_{cd} is greater than the current I_{ab} from the source. This is possible because the two components I_{be} and I_{cd} into which I_{ab} divides are out of phase by the angle between Y_{be} and Y_{bd}, that is, by

$$50.6° - (-88.3°) = 138.9°, \qquad [107]$$

so that the sum of these two components can reasonably be less than either. The second peculiarity is that the voltage V_{cd} across a portion of the circuit is greater than that impressed on the circuit by the source. This is also a peculiarity of alternating-current circuits in which the impedance of a part of a simple circuit may be greater than the impedance of the entire circuit because of the fact that the imaginary components may be either positive or negative

In many problems, particularly in communications, where the behavior of a circuit over a band of frequencies must be determined, a calculation like the foregoing must be made for each of a number of frequencies. This is admittedly somewhat laborious but may frequently be justified economically by the importance of the results. Again, an analysis of an arithmetical problem from the point of view of reducing the work to a minimum frequently enables one to make an apparently very laborious calculation relatively simple and short. Actual numerical results are very important to the engineer, for until the results of any formal analysis are reduced to actual numerical values, he cannot apply them to the design or predetermination of performance of a given piece of equipment.

19. Power and power factor in terms of admittances

Additional forms now can be written for expressions of Arts. 10 and 14:

$$\cos\theta_z = \frac{R}{Z} = \frac{G}{Y} = \cos\theta_y. \qquad [28c]$$

$$\left.\begin{aligned} P_{av} &= VI\cos\theta_z = VI\cos\theta_y \\ &= I^2R = V^2G \\ &= \frac{V^2}{Z}\cos\theta_z = V^2Y\cos\theta_y = I^2Z\cos\theta_z = \frac{I^2}{Y}\cos\theta_y. \end{aligned}\right| \qquad \blacktriangleright[56j]$$

When merely the power factor is desired, either θ_z or θ_y may be used.

20. Active and reactive power

At the terminals of a circuit whose admittance is $Y\epsilon^{j\theta_y}$, the vector current I is displaced in phase from the vector-voltage drop V by the angle θ_y. By the laws of combination and resolution of sine waves, the

current wave can be resolved into two component waves, one in phase with the voltage and one in quadrature with it. Thus, if the voltage is taken as having zero phase angle,

$$v = V_m \cos \omega t, \tag{5f}$$

the current

$$i = I_m \cos (\omega t + \theta_y) \tag{108}$$

can be resolved into the *inphase* component

$$i_i = I_m \cos \theta_y \cos \omega t \tag{109}$$

and the *quadrature* or *reactive* component

$$i_q = I_m \sin \theta_y \cos \left(\omega t + \frac{\pi}{2}\right), \tag{110}$$

which leads the voltage by $\pi/2$ radians. This is shown as follows:

$$\left.\begin{aligned} i &= I_m \cos (\omega t + \theta_y) \\ &= I_m \cos \omega t \cos \theta_y - I_m \sin \omega t \sin \theta_y \\ &= I_m \cos \theta_y \cos \omega t + I_m \sin \theta_y \cos \left(\omega t + \frac{\pi}{2}\right) \\ &= i_i + i_q. \end{aligned}\right\} \tag{108a}$$

Vectorially, this resolution becomes

$$\mathbf{I}_m = I_m \cos \theta_y + jI_m \sin \theta_y \tag{108b}$$

as shown in Fig. 18.

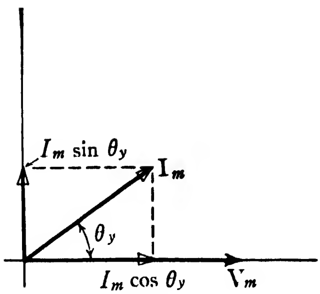


FIG. 18. Vector diagram of in-phase and quadrature currents.

Half the product of V_m and the in-phase component $I_m \cos \theta_y$ of the vector current is the average power. Half the product of V_m and the quadrature component $I_m \sin \theta_y$ is the *reactive power*, usually denoted by Q and measured in units of volt-amperes-reactive, abbreviated to *var.**

Thus

$$Q \equiv \frac{V_m I_m}{2} \sin \theta_y = VI \sin \theta_y. \tag{111}$$

The factor $\sin \theta_y$ is the *reactive factor*. Though the reactive power has a physical significance, given below, its chief usefulness arises from its relation to the quadrature or reactive current. In the transfer of power,

* Kilovolt-amperes-reactive, abbreviated kvar, are used for large quantities.

only the inphase component of current is of direct usefulness, the reactive component contributing only to the double-frequency term of the instantaneous power in Eq. 56b. The heating loss in conductors, however, depends on the square of the entire effective current, to which the reactive component contributes. Thus the heating loss is

$$RI^2 = RI^2 \cos^2 \theta_y + RI^2 \sin^2 \theta_y, \qquad [112]$$

and is therefore increased by the presence of reactive current. Some reactive current is usually present in alternating-current circuits because of its usefulness as a voltage regulator, or because of the inherent nature of many electrical devices such as induction machinery

When considering power in a circuit, one usually is interested also in the amount of reactive current (which is contributing nothing to the average power) relative to the amount of inphase current. Since Q is the reactive current times the voltage, and P_{av} is the inphase current times the voltage (all currents and voltages effective values), Q is a measure of the reactive current just as P_{av} is a measure of the inphase current. Thus

$$\frac{Q}{P_{av}} = \frac{I \sin \theta_y}{I \cos \theta_y} = \tan \theta_y \qquad [113]$$

shows how Q and P_{av} are related to the phase angle θ_y of the current with respect to the voltage. This use of Q to show the amount of reactive current present in a circuit contributes principally to the practical utility of Q.

The physical significance of Q is interesting, however, from a theoretical and sometimes a practical point of view, as appears from manipulation of the last term of Eq. 56b which contains $\cos (2\omega t + \psi + \phi)$. Since

$$2\omega t + \psi + \phi = 2(\omega t + \psi) + (\phi - \psi) = 2(\omega t + \psi) + \theta_y, \qquad [114]$$

the cosine can be expanded into

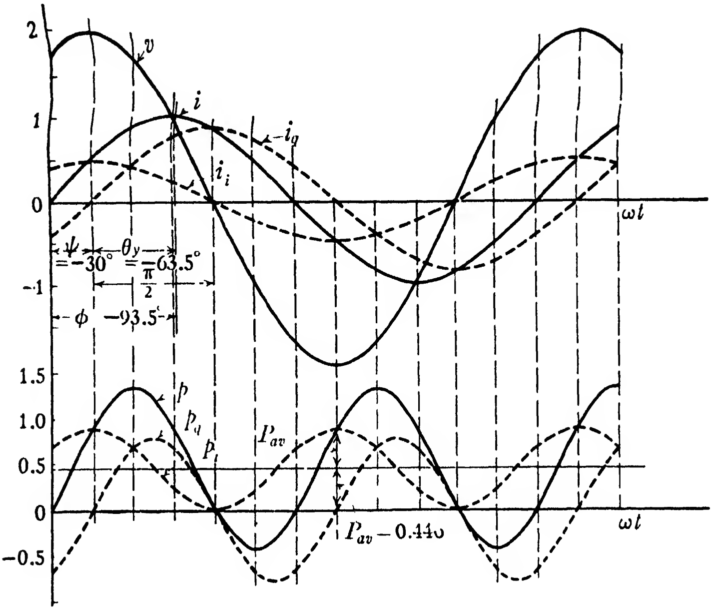
$$\left.\begin{aligned} \cos (2\omega t + \psi + \phi) &= \cos [2(\omega t + \psi) + \theta_y] \\ &= \cos 2(\omega t + \psi) \cos \theta_y - \sin 2(\omega t + \psi) \sin \theta_y, \end{aligned}\right\} \qquad [115]$$

by means of which Eq. 56b can be rewritten

$$\left.\begin{aligned} p(t) = V\{&I \cos \theta_y[1 + \cos 2(\omega t + \psi)] \\ &- I \sin \theta_y \sin 2(\omega t + \psi)\}. \end{aligned}\right\} \qquad [56k]$$

In this equation the term containing $\cos \theta_y$ is the power resulting from the component of the current that is in phase with the voltage V, and the term containing $\sin \theta_y$ is the power resulting from the quadrature or reactive component of current. The power resulting from the inphase

current oscillates about the average power $VI \cos \theta_y$, varying betwee $2P_{av}$ and zero at double the frequency of the source. This power never becomes negative, however. The power resulting from the quadrature component, on the other hand, varies at double frequency about zero and represents a surging of energy from source to circuit and vice versa, with no net transfer of energy over any integral number of cycles in



$\psi = -30°,\ \phi = -93.5°,\ \theta_y = -63.5°$

Plots on v, i axes $\quad v = 2 \cos (\omega t - 30°) \qquad i_i = 0.446 \cos (\omega t - 30°)$

$\qquad i = \cos (\omega t - 93.5°) \qquad i_q = 0.895 \cos (\omega t - 120°)$

Plots on power axes $\quad P_{av} = 0.446 \qquad p_i = 0.446 [1 + \cos 2(\omega t - 30°)]$

$\qquad p = 0.446 + \cos (2\omega t - 123.5°)$

$\qquad p_q = 0.895 \sin 2 (\omega t - 30°)$

FIG. 19. Time plot of inphase and quadrature currents and power.

either direction. The amplitude of this power oscillation resulting from the quadrature component of current is $VI \sin \theta_y$, which is seen to be Q. Hence the reactive power Q can be interpreted physically as the amplitude of the power oscillation resulting from the component of current in quadrature with the voltage. In Fig. 19 are shown the voltage, the current and its two components, and the various components of the power as given by Eq. 56k for a numerical example.

Reactive power can be measured by an instrument similar in con-

struction to the wattmeter, but with an inductor instead of a resistor in series with the potential coil, and with a condenser shunting the coil, in order to make the current in the potential coil exactly 90 degrees out of phase with the voltage across the potential circuit.

A word regarding the sign convention relating to Q is appropriate here. As defined by Eq. 111, Q is considered positive when θ_y is positive, that is, when the current leads the voltage drop or when the reactive component is 90 degrees ahead of the voltage. An equally plausible convention is to define Q in terms of θ_z, in which case a leading current gives rise to a *negative* Q. Both conventions are in use, but current practice tends toward the convention of Eq. 111; in fact, this is the tentative international standard.[3] Accordingly this convention is followed here, with the warning to the student that in the literature the alternative convention is frequently used and that consequently he should be cognizant of it.*

21. Products of V and I; vector power

Before attempting to attach significance to a product of V and I, it is well to recall how these complex quantities are introduced into alternating-current-circuit analysis and to what limitations they are subject. The student should therefore review here Arts. 2 and 3 of this chapter. In those articles the complex quantities E, V, I, Z, and so on, are shown to come from the introduction of the exponential functions. These are substituted for the original trigonometric expressions in terms of which the instantaneous currents and voltage are expressed, in order that the ωt's can be eliminated from the solutions.

When the subject of power is introduced, however, one is dealing with products of instantaneous currents and voltages, and it is evidently necessary to return to complete expressions for them. Thus, while the expression $VI \cos \theta_z$ has a definite, useful meaning, as has been shown, the meaning and use of the complex product VI, if any, can be determined only by investigating its relation to vi. It turns out, in fact, from what follows that $\overline{\mathrm{V}}\mathrm{I}$ ($\overline{\mathrm{V}}$ being the conjugate of V) and not VI is the significant and useful complex product. Thus the fact is emphasized that in the complex number method, or *symbolic* method as it is often called, of handling alternating-current calculations, the complex numbers V and I are not the actual voltage and current functions but are merely derived or related conceptions that are very useful for certain purposes. They have significance only for those purposes for which they can be shown to

[3] A. E. Kennelly, "Actions on Electric and Magnetic Units," *A.I.E.E. Trans.*, LIII (1934), 402–405.

* The convention of positive Q for lagging current is used in publications by engineers of the Westinghouse Electric and Manufacturing Company.

have significance, and any new use must be justified by proof in terms of the actual voltage and current functions v and i. Expression of the *product vi in terms of complex functions is therefore investigated.* The instantaneous power absorbed by a circuit branch, whose voltage drop is v and whose current i, is

$$p = vi, \tag{116}$$

in which

$$v = V_m \frac{\epsilon^{j(\omega t+\psi)} + \epsilon^{-j(\omega t+\psi)}}{2} = \frac{\mathrm{V}_m\epsilon^{j\omega t} + \overline{\mathrm{V}}_m\epsilon^{-j\omega t}}{2}, \tag{7d}$$

$$i = I_m \frac{\epsilon^{j(\omega t+\phi)} + \epsilon^{-j(\omega t+\phi)}}{2} = \frac{\mathrm{I}_m\epsilon^{j\omega t} + \overline{\mathrm{I}}_m\epsilon^{-j\omega t}}{2}. \tag{21d}$$

Putting Eqs. 7d and 21d in Eq. 116 gives

$$\left.\begin{aligned} p &= \tfrac{1}{2}[\mathrm{V}\overline{\mathrm{I}} + \overline{\mathrm{V}}\mathrm{I} + \mathrm{VI}\epsilon^{j2\omega t} + \overline{\mathrm{V}}\overline{\mathrm{I}}\epsilon^{-j2\omega t}] \\ &= \frac{VI}{2}\,[\epsilon^{j\psi}\epsilon^{-j\phi} + \epsilon^{-j\psi}\epsilon^{j\phi} + \epsilon^{j(\psi+\phi)}\epsilon^{j2\omega t} + \epsilon^{-j(\psi+\phi)}\epsilon^{-j2\omega t}] \\ &= VI\,[\cos(\psi-\phi) + \cos(2\omega t + \psi + \phi)], \end{aligned}\right\} \tag{117}$$

the same result as Eq. 56b obtained directly from the trigonometric expressions. Not the final result, however, but certain of the intermediate steps are to be used. The terms $\frac{1}{2}(\mathrm{V}\overline{\mathrm{I}} + \overline{\mathrm{V}}\mathrm{I})$ of Eq. 117 are the average power since they yield the term $VI\cos(\psi-\phi)$ or $VI\cos\theta_z$. Also it can be seen that $\mathrm{V}\overline{\mathrm{I}}$ is the conjugate of $\overline{\mathrm{V}}\mathrm{I}$. Hence,

$$P_{av} = \tfrac{1}{2}(\mathrm{V}\overline{\mathrm{I}} + \overline{\mathrm{V}}\mathrm{I}) = \mathcal{R}_e[\mathrm{V}\overline{\mathrm{I}}] = \mathcal{R}_e[\overline{\mathrm{V}}\mathrm{I}]. \tag{118}$$

Equation 118 is frequently of practical use. The last forms are also useful if the components of V and I are written out thus:

$$\mathrm{V} = V_1 + jV_2 \tag{119}$$

and

$$\mathrm{I} = I_1 + jI_2 \tag{120}$$

as shown in Fig. 20.

$$\left.\begin{aligned} P_{av} &= \mathcal{R}_e[(V_1 + jV_2)(I_1 - jI_2)] \\ &= \mathcal{R}_e[(V_1 - jV_2)(I_1 + jI_2)] \\ &= V_1I_1 + V_2I_2. \end{aligned}\right\} \tag{▶121}$$

If V and I are in rectangular form, Eq. 121 is convenient for computation since it gives the average power as the sum of the products of the real portions and the imaginary portions of V and I.

An even more useful result follows from the term $\bar{V}I$, which expanded becomes

$$\left.\begin{aligned}\bar{V}I &= VIe^{j(\phi-\psi)} = VIe^{j\theta_y}\\ &= VI\,(\cos\theta_y + j\sin\theta_y)\\ &= P_{av} + jQ.\end{aligned}\right\} \qquad [122]$$

Equation 122 is very useful when the power reactive Q as well as the average or active power P_{av} is wanted. The quantity $P_{av} + jQ$, usually written $P + jQ$, is called the *vector power*, for which Eq. 122 may be taken as the definition. Vector power is used extensively in power work and is one of the principal quantities in terms of which the power transmission engineer thinks.

The product

$$VI = \sqrt{P^2 + Q^2} \qquad ▶[123]$$

is also often used, particularly in expressing the ratings for loads on alternating-current machines. In such machines, the limitations on output are such that the capacity is limited by the volt-ampere* product rather than the power.

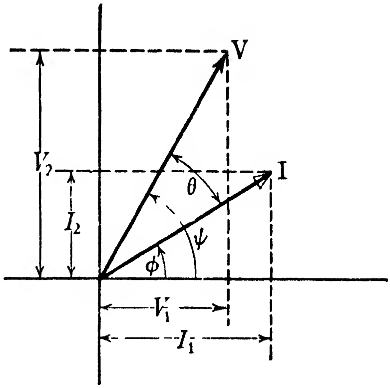


FIG. 20. Real and imaginary components of V and I.

The concept of vector power is useful in circuit calculations because of its relation to the currents, voltages, and circuit parameters. Two of these relations are developed below. Their application is illustrated in Art. 22 by a numerical example.

The first relation is that for any circuit whose terminal impedance is Z or terminal admittance is Y:

$$P + jQ = \bar{\mathrm{V}}\mathrm{I} = I^2R - jI^2X = V^2G + jV^2B. \qquad ▶[124]$$

This is readily proved as follows:

$$\bar{\mathrm{V}}\mathrm{I} = \overline{\mathrm{ZI}}\mathrm{I} = (\overline{R + jX})\bar{\mathrm{I}}\mathrm{I} = (R - jX)I^2 = I^2R - jI^2X = \bar{\mathrm{Z}}I^2, \quad [124a]$$

$$\bar{\mathrm{V}}\mathrm{I} = \bar{\mathrm{V}}\mathrm{VY} = \bar{\mathrm{V}}\mathrm{V}(G + jB) = V^2(G + jB) = V^2G + jV^2B = V^2\mathrm{Y}. \quad [124b]$$

The second relation is that in any circuit consisting of series and parallel combinations of n single branches, the vector power for the entire

* The kilovolt-ampere (abbreviated kva) is used for large quantities.

circuit is equal to the sums of the vector powers for the individual branches. Stated mathematically,

$$\overline{V}_0 I_0 = \sum_{k=1}^{n} \overline{V}_k I_k. \qquad \blacktriangleright[125]$$

Equation 125 is proved by showing that it holds for parallel combinations and for series combinations of branches. Thus for two branches having admittances Y_1 and Y_2, and carrying currents I_1 and I_2, respectively,

$$\overline{V}_0 I_0 = \overline{V}_0(I_1 + I_2) = \overline{V}_0 I_1 + \overline{V}_0 I_2. \qquad [125a]$$

For two branches in series having impedances Z_1 and Z_2 across which voltage drops V_1 and V_2 occur respectively,

$$\overline{V}_0 I_0 = (\overline{V}_1 + \overline{V}_2) I_0 = \overline{V}_1 I_0 + \overline{V}_2 I_0. \qquad [125b]$$

In Eq. 125, the vector power $\overline{V}_k I_k$ for the kth branch can be computed by any of the three forms given in Eq. 124.

Equation 125 holds even for circuits containing star and mesh combinations of branches.[4] This equation is often useful for checking the arithmetical solutions of circuits made in terms of currents and voltages.

Vector power can readily be obtained from the admittance locus of a circuit. For, as shown by Eq. 124b,

$$P + jQ = \overline{V} I = \overline{V} V Y = V^2 Y, \qquad [124c]$$

whence it is seen that Y as read off its locus has merely to be multiplied by the square of the magnitude of the effective applied voltage to give the vector power. As is shown in Ch. IX, the Y locus is usually a circle. There are many adaptations of circle diagrams for power charts and associated graphical methods of analysis.[5]

22. Illustrative example of the computation of vector power

As an illustration of the various methods of calculating vector power and also of the use of vector power in verifying arithmetical work, Eq. 124 is applied to the numerical example of Art. 18.

At the terminals

$$\left.\begin{aligned} \overline{V}_0 I_0 &= (10.6 \angle 0)(0.0397 \angle -23.5^\circ) \\ &= 0.420 \angle -23.5^\circ = 0.385 \text{ w} - j0.168 \text{ var.} \end{aligned}\right| \qquad [122a]$$

[4] W. V. Lyon, "Reactive Power and Power Factor," *A.I.E.E. Trans.*, LII (1933), 763-770.

[5] O. G. C. Dahl, *Electric Circuits*, Vol. I: *Theory and Application* (New York: McGraw-Hill Book Company, Inc., 1928), Chs. x, xi, and xii, L. F. Woodruff, *Principles of Electric Power Transmission* (2d ed.; New York: John Wiley & Sons, 1938), Ch. vi.

The individual branches are now considered, and bc and cd are taken separately. For branch cd:

$$\left.\begin{aligned}(\overline{VI})_{cd} &= (22.1\underline{/90.0^\circ})0.0410\underline{/-21.8^\circ}\\ &= 0.903\underline{/68.2^\circ}\\ &= 0.334\text{ w} + j0.836\text{ var.}\end{aligned}\right\} \qquad [122b]$$

For branch bc:

$$\left.\begin{aligned}(\overline{VI})_{bc} &= I^2(R - jX)\\ &= (0.0410)^2(19.2 - j234.0) = 1.68 \times 10^{-3}(19.2 - j234.0)\\ &= 0.0322\text{ w} - j0.391\text{ var.}\end{aligned}\right\} \qquad [124d]$$

For branch be:

$$V_{be} = I_{cd} Z_{bcd} = 0.0410 \times 345 = 14.1 \qquad [27r]$$

$$\left.\begin{aligned}(\overline{VI})_{be} &= V_{be}^2 Y_{be} = (14.1)^2(0.036 - j1.224)10^{-4}\\ &= (7.2 - j244)10^{-4} = 0.0007\text{ w} - j0.0244\text{ var.}\end{aligned}\right\} \qquad [124e]$$

For branch ab:

$$(\overline{VI})_{ab} = I^2 \overline{Z} = 0.0397^2(12 - j376) = 0.0188\text{ w} - j0.590\text{ var.} \qquad [124f]$$

Adding the separate $\overline{VI}$'s:

Branch	*Vector Power* w var	
cd	$0.334 + j0.836$	
bc	$0.032 - j0.391$	
be	$0.001 - j0.024$	[125c]
ab	$0.019 - j0.590$	
$\Sigma\overline{VI}$	$0.386 - j0.169$	
$\overline{V}_0 I_0$	$0.385 - j0.168$	[122a]

This checks reasonably well for slide-rule calculations of this complexity.

23. Active loads

In the treatment thus far, the load, that is, the part of the circuit receiving power, is assumed to be composed entirely of passive elements: resistance, inductance, and capacitance. Quite as commonly the load is composed of active apparatus — motors of various kinds. The distinction of definition between active and passive apparatus is that active apparatus can accomplish sustained conversion from electrical energy to some other form of energy, or vice versa, whereas passive apparatus can accomplish only irreversible conversion of electrical energy into heat, or no sustained conversion at all. For example, a *dynamo* (a generic term for both motor and generator) is an active element. It can convert electrical energy into mechanical energy, or vice versa. There are various types of

such apparatus for alternating currents and various other types for direct currents. A resistor is a passive element. It can produce no energy transformation except when electrically connected to an active element (a generator) to receive electrical energy for conversion into heat. Capacitors and inductors also are passive elements. They can produce no energy transformation unless connected electrically to a generator. If the supply is from an alternating source, merely a temporary transfer of energy into the electric or magnetic field is accomplished. Energy is stored during one half of the cycle and returned to the source during the other half. No sustained transfer of energy into a capacitor can be accomplished unless the voltage of the capacitor can be increased indefinitely. No sustained transfer of energy into an inductor can be accomplished unless the current of the inductor can be increased indefinitely. Neither of these conditions can be attained physically.*

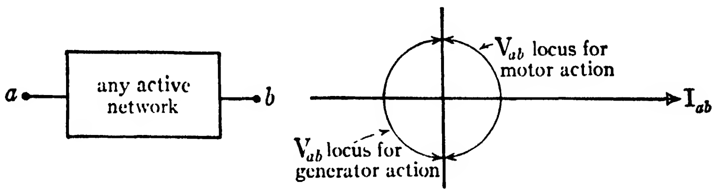


FIG. 21. Relation between voltage and current for an active network.

The characteristics of motors† are such that they cannot be represented in terms of a fictitious fixed static impedance or admittance. Representation by a static element is possible for a fixed operating condition, but since a new parameter must be determined for every new load condition, the method is not ordinarily very useful. Hence engineers usually treat active loads in terms of current and power factor, which in turn are known from test data (perhaps assembled in the form of graphs) as functions of operating conditions.

If a generator and a motor are electrically connected, and if the latter is connected to a load which can supply energy as well as receive it (for example, a motor-generator set), it is possible for the generator and the motor to exchange functions. The diagram for an active load, supplementing those of Fig. 7, is therefore as shown in Fig. 21.

If $\mathbf{V}_{ab}$ lies in the second or third quadrant, $V_{ab}I_{ab}\cos\theta$ is negative. The load therefore is absorbing negative power or actually is delivering power.

* The discussion of active and passive elements in Art. 3, Ch. I, should be reviewed.

† Motors and generators are studied in this series in the volume on rotating electric machinery.

24. Illustrative example of an active load

A single-phase induction motor which has an effective voltage of 230 volts at 60 cycles per second maintained across its terminals takes 5.0 kilowatts of power at 0.80 power factor. A condenser of adjustable capacitance is connected in parallel with the motor and is set to make the resultant power factor of motor and condenser unity. The power factor of the condenser is 0.02, independent of its capacitance. The questions are:

(a) If it is assumed that the condenser is equivalent to an ideal capacitance in parallel with a resistance, for what capacitance is it set? How much power does it take?

(b) What is the total vector power taken by the motor and condenser in parallel?

(c) What is the amplitude of the oscillation in instantaneous power taken by the motor? What is the frequency of this oscillation?

(d) What is the expression as a function of time for the instantaneous power taken by the motor and condenser in parallel? It may be assumed that time, t, is zero when the impressed voltage drop across the motor and condenser is a positive maximum.

The conditions are shown in Fig. 22.

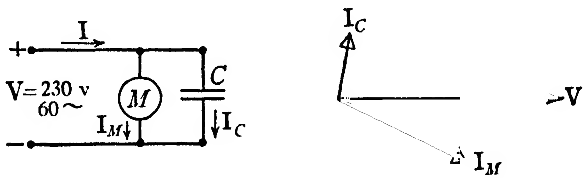


FIG. 22. Condenser in parallel with induction motor for power-factor correction.

Solution: For part (a), the reactive power taken by the motor is

$$\frac{0.6}{0.8} \times 5{,}000 = 3{,}750 \text{ var.} \quad [126]$$

The condenser current and the condenser capacitance are related by

$$3{,}750 = 230 I_C \quad [127]$$

or

$$I_C = \frac{230}{X_C} = 230 \times 377C. \quad [128]$$

Hence,

$$3{,}750 = (230)^2 \times 377C, \quad [129]$$

from which the capacitance of the condenser is

$$C = \frac{3{,}750}{(230)^2 \times 377} = 189 \times 10^{-6} = 189\ \mu f, \quad [129a]$$

and the power which it takes is

$$P_C = 0.02 \times 3{,}750 = 75\text{ w (approx.)}. \qquad [130]$$

For part (b), the total vector power is

$$P + jQ = 5{,}000 + 75 = 5{,}075\text{ w} + j0\text{ var}. \qquad [131]$$

For part (c), the power taken by the motor is

$$p_M = VI_M \cos\theta_z + VI_M \cos(2\omega t + \psi + \phi_M). \qquad [56m]$$

The amplitude of the power oscillation is

$$VI_M = \frac{5{,}000}{0.8} = 6{,}250\text{ va}, \qquad [123a]$$

and the frequency of the oscillation is

$$2 \times 60 = 120 \sim. \qquad [132]$$

For part (d), the instantaneous voltage and total current are

$$v = 230\sqrt{2}\cos 377t \qquad [5g]$$

and

$$i = \frac{5{,}075}{230}\sqrt{2}\cos 377t. \qquad [21e]$$

The total instantaneous power is

$$\left.\begin{aligned} p = vi &= 5{,}075 \times 2\cos^2 377t \\ &= 5{,}075 \times 2(\tfrac{1}{2} + \tfrac{1}{2}\cos 754t) \\ &= 5{,}075 + 5{,}075 \cos 754t. \end{aligned}\right\} \qquad [56n]$$

25. Analysis of Circuits in Which Only Magnitudes Are Known

In practice, problems often occur in which the known data are not in form suitable for direct substitution in vector equations of the type presented in the preceding articles. Thus in power-system studies only the *magnitudes* of voltages and currents at different points on a network may be known, but the circuit impedance and the phase angles between voltages at remote points must be calculated from the values that can be measured. Certain analytical methods that are useful in such problems are presented in this article by means of a numerical example rather than by developing literal expressions which become rather cumbersome. For the circuit of Fig. 23 the phase angle between the source and load voltages, and the vector value of the impedance Z are desired. The load is known to be inductive.

Solution: The first step is to determine the resistance R. All the energy loss in the line takes place in R and is determined by the difference between source and load power.

$$R = \frac{(558 - 443)10^3}{(43.5)^2} = \frac{115{,}000}{1{,}892} = 60.8\text{ ohms}. \qquad [133]$$

The load power factor can also be determined from the data:

$$\text{Load power factor} = \frac{443}{11 \times 43.5} = 0.926. \qquad [134]$$

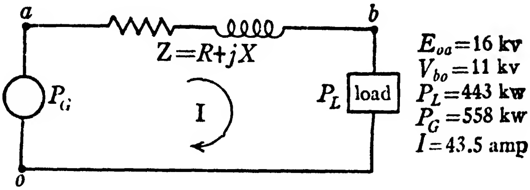


FIG. 23. Single-loop circuit in which E_{oa}, V_{bo}, I, P_G, and P_L are known from test data.

Of course, in any method of solution used, the load voltage V_{bo} can be taken as the reference axis, and the current I then lags the load voltage by $\cos^{-1} 0.926$, an angle of 22.2°.

25a. SOLUTION BY VECTOR LOOP-VOLTAGE EQUATION

The vector equation for loop voltages is

$$16{,}000\underline{/\alpha} = (43.5\underline{/-22.2^\circ})(60.8 + jX) + 11{,}000\underline{/0^\circ}, \qquad [135]$$

in which the phase angle α of the source electromotive force and the magnitude X of the circuit reactance are unknown. In any equation involving complex quantities, the real components on both sides must be equal and the imaginary components on both sides must be equal. Hence Eq. 135 may be separated into two equations, as follows:

$$16{,}000 \cos \alpha = 60.8(43.5) \cos (-22.2^\circ) + 43.5X \cos 67.8^\circ + 11{,}000, \qquad [135a]$$

$$16{,}000 \sin \alpha = 60.8(43.5) \sin (-22.2^\circ) + 43.5X \sin 67.8^\circ + 0. \qquad [135b]$$

These equations can be solved simultaneously for α and X, but it is simpler to determine α from

$$\cos (\alpha + 22.2^\circ) = \frac{P_G}{E_{oa}I} = \frac{558}{16 \times 43.5} = 0.802 \qquad [136]$$

$$22.2^\circ + \alpha = \pm 36.6^\circ, \qquad [137]$$

$$\alpha_1 = 14.4^\circ, \qquad [137a]$$

$$\alpha_2 = -58.8^\circ. \qquad [137b]$$

The desired reactance X may now be found by substituting the two values of α from Eqs. 137a and 137b in Eqs. 135a or 135b. Equation 135b gives

$$X_1 = \frac{42{,}300 \cos 14.4^\circ - 35{,}600}{43.5} = \frac{5{,}400}{43.5} = 124 \text{ ohms} \qquad [138]$$

and the theoretically correct but actually improbable value

$$X_2 = \frac{42{,}300 \cos (-58.8^\circ) - 35{,}600}{43.5} = \frac{-13{,}700}{43.5} = -315 \text{ ohms}. \qquad [139]$$

This procedure is in effect the same as computing X from the reactive power of the line. From Eq. 124,

$$Q = -I^2X. \qquad [140]$$

The reactive power of the load is

$$Q = 443 \tan(-22.2°) = -181 \text{ kvar}. \qquad [141]$$

The reactive power of the generator is

$$Q_{G1} = 558 \tan(-36.6°) = -414 \text{ kvar}, \qquad [142a]$$

or

$$Q_{G2} = 558 \tan 36.6° = 414 \text{ kvar}. \qquad [142b]$$

The reactive power of the line is therefore

$$Q_1 = -414 + 181 = -233 \text{ kvar}, \qquad [143]$$

or

$$Q_2 = 414 + 181 = 595 \text{ kvar}. \qquad [144]$$

The reactance of the line is therefore

$$X_1 = \frac{233{,}000}{(43.5)^2} = 123 \text{ ohms}, \qquad [145]$$

or

$$X_2 = -\frac{595{,}000}{(43.5)^2} = -315 \text{ ohms}. \qquad [146]$$

Equations 145 and 146 show that the reactance of the circuit may be 124 ohms inductive, or 315 ohms capacitive, either of which satisfies the test data as given. The selection between these can be determined from a power-factor test at the source, a lagging source power factor corresponding to the inductive reactance.

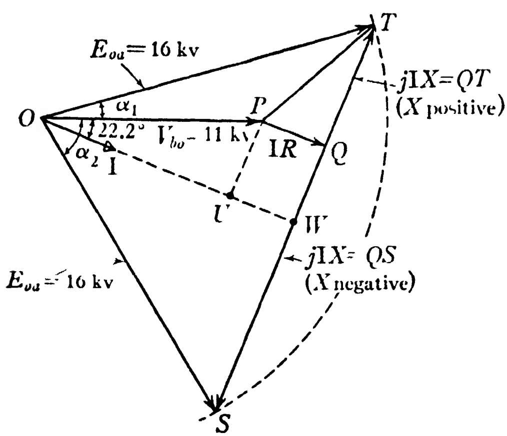


FIG. 24. Vector diagram of relations in circuit of Fig. 23.

25b. GRAPHIC SOLUTION BY VECTOR DIAGRAM

Two alternative methods of solution can be achieved by means of a vector diagram of the circuit. The load voltage, used as reference vector, is first drawn. From its tip, at point P, Fig. 24, a vector PQ is drawn parallel to the current vector I, and having a length equal to IR or 2.65 kv. This vector represents the IR drop in the circuit. As the phase angle of the IX drop differs from that of the current by 90°, its direction is represented on the vector diagram by drawing the straight line ST perpendicular to

PQ through Q. The magnitude of the source voltage is known to be 16 kv; an arc ST of a circle of radius 16 kv with center at O is therefore drawn next. The intersections of this arc with the line ST fix the two locations of the tip of the source-voltage vector. If the diagram is laid out carefully to scale, all desired results can be determined by measuring the vector and angles on the diagram.

25c. Analytic solution from vector diagram

If graph paper and drawing instruments are not at hand, or if it is desired to make an analytic calculation as a more precise method of determining α and X than the graphic method, a rough vector diagram may be constructed, following the method outlined in Art. 25b. The purpose of this diagram is to provide a means of visualizing geometric relations which may be used to determine the desired results. In the illustrative problem here, the current vector **I** (Fig. 24) is extended, as shown by the dotted line, to intersect the perpendicular ST at W', and UP is drawn through P parallel to TS, intersecting OW' at U. From inspection of the diagram,

$$OU' = (OP)\cos(22.2°), \tag{147}$$

$$OU' = 11(0.926) = 10.2 \text{ kv}, \tag{148}$$

$$UW' = PQ = 2.65 \text{ kv}, \tag{149}$$

$$OW' = OU' + UW' = 12.85 \text{ kv}, \tag{150}$$

$$W'T = \sqrt{OT^2 - OW'^2} = \sqrt{16^2 - 12.85^2} = 9.53 \text{ kv}, \tag{151}$$

$$QW' = PU' = OP\sin(22.2°) = 11(0.378) = 4.16 \text{ kv}, \tag{152}$$

$$\therefore \quad QT = W'T - W'Q = 9.53 - 4.16 = 5.37 \text{ kv}, \tag{153}$$

$$X_1 = \frac{5{,}370}{43.5} = 123 \text{ ohms, inductive.} \tag{154}$$

Likewise,

$$W'S = W'T = 9.53 \text{ kv}, \tag{155}$$

$$QS = QW' + W'S = 4.16 + 9.53 = 13.69 \text{ kv}, \tag{156}$$

$$X_2 = \frac{13{,}690}{43.5} = 315 \text{ ohms, capacitive} \tag{157}$$

The values of X agree with those derived in Arts. 25a and 25b. The source-voltage phase angle α is readily determined if desired.

25d. General discussion of these methods

The methods of analysis illustrated by the specific examples of Arts. 25a, b, and c are very important and are widely used by practicing engineers, especially the vector-diagram methods of Arts. 25b and 25c. A suggested exercise for the student is to change the problem by considering that R and X are known, and that in addition only E_{oa}, I, and the load power factor are known. Then, in Eq. 135, X is known but V_{bo} appears as an unknown in place of 11,000. However, since P_G is unknown, α cannot be determined by means of Eq. 136.

It is now in order to present rules for the application of these methods to network problems in general.

▶ 1. *In any equation involving complex quantities, the real parts of the individual terms can be equated, and the imaginary parts of the individual terms can be equated. This rule, which is illustrated by Eqs. 135, 135a, and 135b, gives the necessary information for determining the values of two unknowns in a single equation involving complex quantities.* ◀

▶ 2. *Since the voltage drop across a resistance is in phase with the current through the resistance, the vector representing the* $\mathbf{I}R$ *drop in a vector diagram is parallel to the vector representing the current* $\mathbf{I}$. ◀

▶ 3. *Since the voltage drop across an inductive reactance leads the current through the reactance by a phase angle of 90 degrees, the vector representing an inductive* $j\mathbf{I}X$ *drop is perpendicular to the vector representing the current* $\mathbf{I}$*, and is in a counterclockwise direction from it.* ◀

▶ 4. *Since the voltage drop across a capacitive reactance lags the current through the reactance by a phase angle of 90 degrees, the vector representing a capacitive* $j\mathbf{I}X$ *drop is perpendicular to the vector representing the current* $\mathbf{I}$*, and is in a clockwise direction from it.* ◀

▶ 5. *Since the vector sum of the voltage rises around any loop of a network equals the vector sum of the voltage drops around the loop, the voltage-rise vectors placed end to end from the origin must terminate at the same point in which the voltage-drop vectors placed end to end from the origin terminate.* ◀

▶ 6. *A vector diagram constructed in accordance with rules 2 to 5 may be considered as a plane geometric figure whose elements obey all the laws of plane geometry, trigonometry, and analytic geometry.* ◀

26. Steady-state response as a function of frequency

For the series RL circuit, the magnitude Z and angle θ_z of the steady-state impedance $\mathbf{Z}$ at various values of the source angular velocity ω are now examined. By Eqs. 33a and 34,

$$Z = \sqrt{R^2 + \omega^2 L^2}, \tag{33a}$$

$$\theta_z = \tan^{-1}\frac{\omega L}{R}. \tag{34}$$

The magnitude Z and the angle θ_z can be plotted as functions of ω. If instead of Z, however, the ratio Z/R is plotted, the generality of the result is greatly increased. If this ratio is expressed in terms of R, L, and ω,

$$\frac{Z}{R} = \frac{\sqrt{R^2 + \omega^2 L^2}}{R} = \sqrt{1 + \left(\frac{\omega L}{R}\right)^2}, \tag{158}$$

it is seen that Z/R, as well as θ_z, is a function of $\omega L/R$ and that one plot can be made to cover all possible series RL circuits at all angular frequencies. Figure 25 is such a plot of Z/R and θ_z.

Both the time constant L/R and the source angular frequency ω enter as factors of equal importance in determining Z/R and θ_z. When L/R or ω is very small, Z is nearly equal to R, and θ_z is nearly equal to zero. In the limiting case of direct voltage, ω equals zero, Z equals R, and θ_z equals zero. When the angular frequency is very large, Z is nearly equal to ωL and θ_z is nearly equal to $\pi/2$.

Next is considered the steady-state response of the series RS circuit as the frequency of the applied voltage is varied. As in the series RL

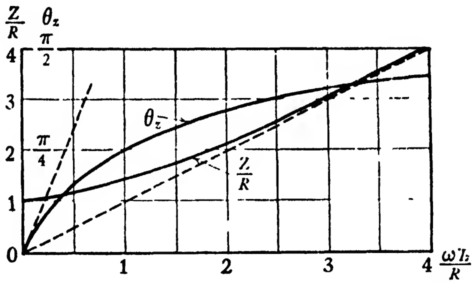


FIG. 25. Impedance-frequency relations for series RL circuit.

circuit, the magnitude and angle of the impedance function as given by Eqs. 35a and 36 are sought. Again Z is divided by R to obtain a plot which applies to all series RS circuits. Dividing Z from Eq. 35a by R gives

$$\frac{Z}{R} = \frac{\sqrt{R^2 + \left(\frac{S}{\omega}\right)^2}}{R} = \sqrt{1 + \left(\frac{S}{\omega R}\right)^2}. \qquad [159]$$

Figure 26 shows Z/R plotted as a function of $\omega R/S$. The curve for θ_z is plotted from

$$\theta_z = \tan^{-1}\left(\frac{-S}{\omega R}\right) = -\tan^{-1}\left(\frac{S}{\omega R}\right) = -\text{ctn}^{-1}\left(\frac{\omega R}{S}\right). \qquad [36b]$$

The angle θ_z is negative, and the reactance is also negative at all frequencies; so the current always leads the voltage. This phase lead of the current with respect to the voltage is characteristic of the capacitive circuit, just as the phase lag of the current is characteristic of the inductive circuit. Both the time constant R/S and the source angular frequency ω enter as factors of equal importance in determining Z/R and θ_z. When R/S or ω is very small, Z/R approaches infinity, and θ_z is nearly equal to

$-\pi/2$. In the limiting case of direct voltage, ω equals zero and Z equals infinity. When R/S or ω is very large, Z is very nearly equal to R and θ_z is very nearly equal to zero.

For the series RL and RS circuits, a study of the impedance characteristic suffices to give a visualization of the behavior of the circuit, since the admittance and hence the current are reciprocally related to the impedance. For the series RLS circuit, a more complete analysis is desirable. *The complete study of the steady-state response reduces to a study of I as a function of the applied voltage E, the angular frequency ω, and the circuit parameters R, L, and S.*

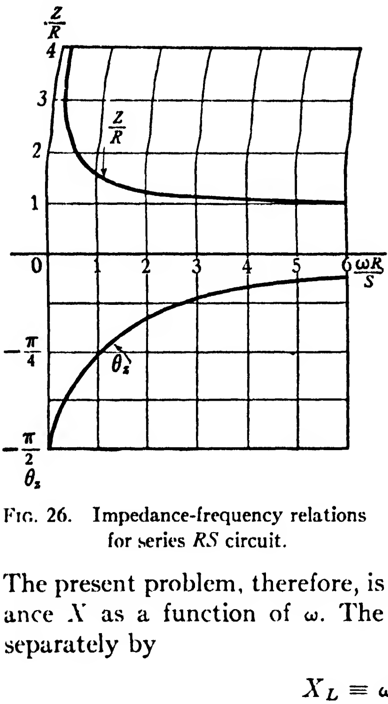


FIG. 26. Impedance-frequency relations for series RS circuit.

First the way in which the function I depends upon the angular frequency ω is considered. In the expression for the impedance of this circuit,

$$Z = R + j\left(\omega L - \frac{S}{\omega}\right)$$

$$= R + j(X_L + X_C) = R + jX, \quad [29]$$

the part

$$\omega L - \frac{S}{\omega} \equiv X \equiv X_L + X_C, \quad [30]$$

designated as the reactance, alone contains the angular frequency ω. The present problem, therefore, is approached first by study of the reactance X as a function of ω. The two terms in this reactance denoted separately by

$$X_L \equiv \omega L, \quad [31]$$

$$X_C \equiv -\frac{S}{\omega} = -\frac{1}{\omega C}, \quad [32]$$

are called, respectively, the inductive and capacitive (or elastive) reactances. The capacitive reactance is negative, whereas the inductive reactance is positive, at all frequencies. Accordingly, in the frequency range in which the resultant reactance of a combination of elements is positive, it is called *inductive* reactance; in the range in which it is negative, it is called *capacitive* reactance. In Fig. 27 the reactances X_L and X_C as well as the resultant reactance X are plotted versus angular frequency in order to make their individual and combined behaviors apparent. The

curve of X_L is given by a straight line with constant slope L, and X_C is represented by one branch of a rectangular hyperbola lying below the ω axis. Thus X_L is zero when ω is zero and increases linearly toward infinity as ω approaches infinity; that is, X_L varies directly as ω, while X_C is negative and infinite when ω is equal to zero and decreases in magnitude toward zero as ω approaches infinity. In other words, X_C varies inversely as ω. The net or resultant reactance X is given by the sum of the separate curves for X_L and X_C. It is clear that for small values of ω, the X curve almost coincides with the X_C curve, while for large values of ω the X curve approaches the X_L curve asymptotically; that is, for small ω, the reactance X is predominantly capacitive; for large ω, it is predominantly inductive. At some intermediate frequency the inductive and capacitive reactances must evidently be equal and opposite, and the net reactance zero. This situation occurs at an angular frequency corresponding to the condition

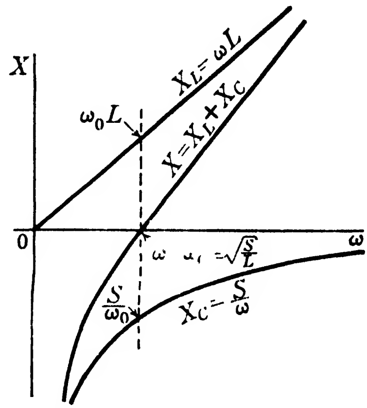


FIG. 27. Reactance-frequency relations for series RLS circuit.

$$\omega L - \frac{S}{\omega} = 0 \qquad [160]$$

or

$$\omega L = \frac{S}{\omega}. \qquad [160a]$$

The value of ω which satisfies this equation is

$$\omega_0 = \sqrt{\frac{S}{L}}, \qquad [161]$$

which is recognized as the angular frequency of free oscillation of this circuit for the nondissipative case, that is, for the case in which R equals zero. Thus, if the series RLS circuit is excited by a voltage having an angular frequency equal to the nondissipative angular frequency of free oscillation ω_0, the circuit as a whole acts as though it consisted of a resistance R alone. Of course this statement is true only so far as the steady-state behavior is concerned. It is easy to see from Eqs. 29 and 27a that for this condition the impedance Z has a minimum magnitude and, hence, if E is constant, the current I has a maximum magnitude. This circuit

condition is designated by the term *series resonance,** and is the condition for which the effective current per unit of applied voltage is a maximum. It appears later that there are circuits in which a complementary condition of a current minimum per unit of applied voltage or a voltage maximum per unit of applied current can exist; descriptive terminology must therefore be so formed that these complementary conditions may be clearly distinguished.

Whether or not the current response at the resonance angular frequency is exceptional depends evidently upon a comparison of this response with that which obtains at angular frequencies that are different from ω_0 but that nevertheless are adjacent to this value, that is, angular frequencies which differ from ω_0 by relatively small values.

For the purpose of studying the phenomenon of resonance, Eq. 27a may be written in the form

$$\mathbf{I} = \frac{\mathbf{E}}{R}\,\frac{1}{1 + j\dfrac{\omega_0 L}{R}\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)} \qquad [162]$$

by using Eq. 161. Since the study is principally to determine how the current response for an off-resonance frequency compares with the resonance value $\mathbf{E}/R$, the foregoing form of expression for $\mathbf{I}$ is convenient. Furthermore, using the resonant angular frequency ω_0 gives a symmetrical expression for the ratio of reactance to resistance, which is

$$\frac{X}{R} = \frac{\omega_0 L}{R}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{\omega_0 L}{R}\left[\frac{\omega}{\omega_0} - \frac{1}{\dfrac{\omega}{\omega_0}}\right]. \qquad [163]$$

This form further suggests that the ratio ω/ω_0 instead of ω alone be considered as the independent variable. In fact, Eq. 163 shows that the dependence of the current response upon frequency is wholly relative to the resonant frequency; that is, the response at a given angular frequency *depends only upon the value of this frequency relative to* ω_0. The percentage departure from resonance is also easily expressed. Thus, for example, ω/ω_0 equal to 1.1 or ω/ω_0 equal to 0.9 refers to frequencies ten per cent above or below resonance, while ω/ω_0 equal to unity refers to the resonance frequency.

The magnitude of the current response is considered first:

$$I = \frac{E}{R}\,\frac{1}{\sqrt{1 + \left(\dfrac{\omega_0 L}{R}\right)^2\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)^2}}. \qquad ▶[164]$$

* This term is often abbreviated by omitting the word "series."

Plots of this expression versus ω/ω_0 are shown in Figs. 28 and 29. The first illustrates the general character of the circuit behavior for the condition when $R/(\omega_0 L)$ is greater than $\sqrt{2}$; Fig. 29 shows how the response versus frequency looks when $R/(\omega_0 L)$ is less than $\sqrt{2}$. These conditions express

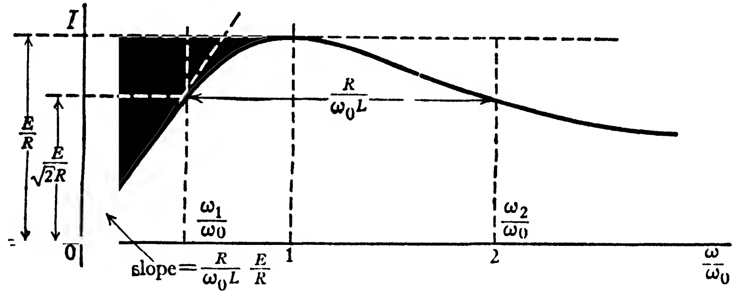


FIG. 28. Current-frequency relations for series RLS circuit, $\frac{R}{\omega_0 L} = \frac{3}{2}$.

a relation among the resistance, inductance, and angular frequency of resonance. The condition

$$\frac{R}{\omega_0 L} = \sqrt{2}, \qquad [165]$$

which separates the cases for which the curve of I versus ω/ω_0 has a point of inflection between the origin and the point with abscissa 1 (as in

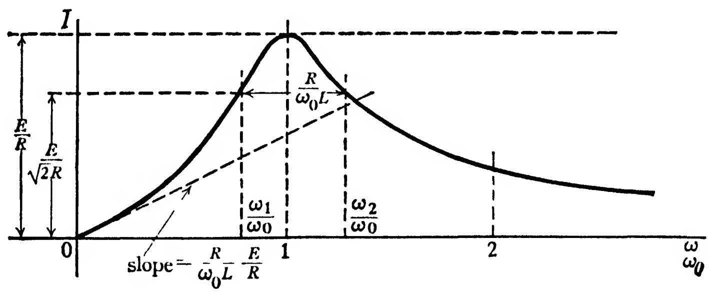


FIG. 29. Current-frequency relations for series RLS circuit, $\frac{R}{\omega_0 L} = \frac{1}{2}$

Fig. 29) from those in which there is no point of inflection in this range (as in Fig. 28) can be derived by finding the ω/ω_0 co-ordinate of the intersection of the tangent to the curve at the origin and the curve, and setting this equal to zero. Thus, letting ω/ω_0 equal x, the slope* of the

* In seeking maxima and minima of complex functions, it is sometimes advantageous to carry out the work in complex notation instead of in terms of absolute values. The method of doing so is explained in Art. 9, Ch IX.

tangent at the origin is

$$\left.\frac{dI}{dx}\right|_{x=0} = \frac{R}{\omega_0 L}\frac{E}{R}. \tag{166}$$

Using Eq. 164 gives

$$\frac{xR}{\omega_0 L} = \frac{1}{\sqrt{1 + \left(\frac{\omega_0 L}{R}\right)^2 \left(x - \frac{1}{x}\right)^2}}. \tag{167}$$

Solving this for x gives x^2 equal to zero, which is the intersection at the origin, and

$$x^2 = \frac{2\left(\frac{\omega_0 L}{R}\right)^2 - 1}{\left(\frac{\omega_0 L}{R}\right)^2}, \tag{168}$$

which if set equal to zero to make the point of inflection disappear gives the result

$$\frac{R}{\omega_0 L} = \sqrt{2}. \tag{165}$$

Angular frequencies ω_1 and ω_2 give equal values of I if

$$\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1} = -\frac{\omega_2}{\omega_0} + \frac{\omega_0}{\omega_2}, \tag{169}$$

or

$$(\omega_1 + \omega_2)\left(\frac{1}{\omega_0} - \frac{\omega_0}{\omega_1\omega_2}\right) = 0. \tag{170}$$

So

$$\omega_1 = -\omega_2 \tag{171}$$

or

$$\frac{1}{\omega_0} = \frac{\omega_0}{\omega_1\omega_2}. \tag{172}$$

Since negative angular velocities are not considered, the desired solution is

$$\omega_0 = \sqrt{\omega_1\omega_2}. \tag{172a}$$

Equal response occurs for two frequencies whose geometric mean is equal to the resonance frequency. The response curve is said to be *geometrically*

even, about ω_0. The interval between the two ω/ω_0 values, at which the response is $1/\sqrt{2}$ of its maximum value, is $R/(\omega_0 L)$. On an ω scale this frequency interval $R/(\omega_0 L)$ becomes an interval of R/L, which is equal to 2α. The smaller this interval, the more sharply does the circuit respond; that is, the more pronounced does its resonant character become. This result, therefore, shows that the sharpness of resonance phenomenon depends only upon twice the constant α, that is, only upon the resistance-to-inductance *ratio*, not upon the individual values of R and L, and not at all upon the elastance parameter S.

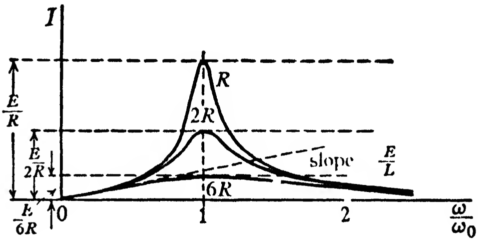


FIG. 30. Current-frequency relations for series RLS circuit, L constant, $\frac{R}{\omega_0 L} \approx 0.185$.

The response at resonance depends only on R, and varies inversely with R, while far from resonance it depends almost wholly upon (and varies inversely as) $\omega_0 L$, S/ω_0, or $\sqrt{LS}$. Thus, although the general character of the frequency discrimination depends only upon the ratio R/L as is shown by the curves in Figs. 28 and 29, the value of maximum response depends on R alone.

The effect of varying the parameters R and L separately is interesting to observe. This is illustrated by Figs. 30 and 31, the first of which shows a family of resonance curves for various values of R, with L constant, while the second shows a family of curves for various values of L, with R constant. Thus for a given circuit, if the resistance is made smaller, the circuit favors the resonance frequency more strongly; that is, remote from resonance frequency the response curve is essentially unchanged, whereas in the vicinity of this frequency the curve is pushed up to a peak. On the other hand, if R is held constant and the inductance is made larger, the circuit discriminates against the nonresonant frequencies more strongly; that is, at the resonance frequency the curves are unchanged with increase in L, but at all other frequencies they are pushed down. In each case the circuit becomes more selective with regard to frequency as R is decreased, or L is increased, or both.

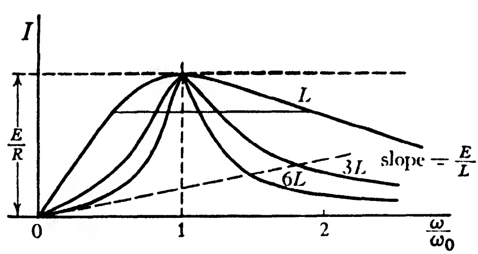


FIG. 31. Current-frequency relations for series RLS circuit, R constant, $\frac{R}{\omega_0 L} \approx 1.46$

27. Inductive and elastive voltages as functions of L, S, and ω

When resonance is obtained by variation of inductance, the maximum voltage across the inductance is obtained by maximizing the expression

$$V_L = IX_L = \frac{EX_L}{[R^2 + (X_L + X_C)^2]^{1/2}} \tag{173}$$

with respect to inductive reactance X_L:

$$\left.\begin{aligned}\frac{dV_L}{dX_L} &= E[R^2 + (X_L + X_C)^2]^{-1/2} \\ &\quad - EX_L[R^2 + (X_L + X_C)^2]^{-3/2}(X_L + X_C) = 0;\end{aligned}\right\} \tag{174}$$

whence,

$$X_L = -\frac{R^2 + X_C^2}{X_C} \tag{175}$$

for maximum V_L. If resonance is approached by increasing the inductance, the maximum V_L is reached after resonance is passed, that is, with a larger value of L than that required for resonance.

A similar derivation gives

$$X_C = -\frac{R^2 + X_L^2}{X_L} \tag{176}$$

for the value of elastive reactance X_C which corresponds to maximum voltage V_C across the elastance when resonance is obtained by variation of elastance. If resonance is approached by increasing the elastance (decreasing the capacitance), the maximum V_C is reached after resonance is passed, that is, with a larger value of S than is required for resonance. If resonance is approached by increasing the capacitance, the maximum V_C is reached before resonance is passed, that is, with a smaller value of C than is required for resonance.

Obviously the maximum V_L accompanying a variation of S, and the maximum V_C accompanying a variation of L, correspond with the current maximum and hence occur at resonance.

When resonance is obtained by variation of frequency, the maximum V_L occurs when

$$\omega = \sqrt{\frac{2S^2}{2LS - R^2}}, \tag{177}$$

and the maximum V_C occurs when

$$\omega = \sqrt{\frac{2LS - R^2}{2L^2}}. \tag{178}$$

The former occurs for ω when greater than ω_0 and the latter for ω when less than ω_0.

The foregoing results can be obtained also by differentiation and maximizing in complex notation.* For example,

$$\mathrm{V}_L = \frac{jX_L(E + j0)}{R + j(X_L + X_C)}. \tag{179}$$

In this and in similar cases when the variable occurs in the denominator, the differentiation is simpler if a minimum is sought for the reciprocal:

$$\left.\begin{aligned} \frac{1}{\mathrm{V}_L} &= \frac{1}{E}\frac{[R + j(X_L + X_C)]}{jX_L} = \frac{1}{E}\left[\frac{X_L + X_C}{X_L} - j\frac{R}{X_L}\right] \\ &= \frac{1}{E}\Big[U_r + jU_j\Big], \end{aligned}\right\} \tag{179a}$$

$$\frac{dU_r}{dX_L} = -\frac{X_L + X_C}{X_L^2} + \frac{1}{X_L} = -\frac{X_C}{X_L^2}, \tag{180}$$

$$\frac{dU_j}{dX_L} = \frac{R}{X_L^2}; \tag{181}$$

whence

$$\frac{R}{-X_C} = -\frac{X_L + X_C}{-R}, \tag{182}$$

or

$$X_L = -\frac{R^2 + X_C^2}{X_C} \tag{183}$$

as before. In this illustration there is no great advantage in obtaining the conditions for the maximum by this method. In other more complicated cases, however, the advantage sometimes may be very decided.

28. Steady-state response of the series *RLS* circuit in the vicinity of resonance

In the preceding articles of this chapter the steady-state responses to alternating impressed forces have been developed for the simple series and parallel combinations of circuit elements. Next, the steady-state behavior of the simple circuits in which resonance effects occur is further analyzed, because of the great practical importance of such circuits. Whereas in the preceding discussions of the steady-state behavior as a function of the impressed frequency, attention is given to the entire

* Article 9, Ch. IX.

frequency spectrum, in the following discussion attention is turned more particularly to the steady-state behavior of simple resonant circuits in the vicinity of the resonant frequency. This subject is of fundamental importance in communications work, particularly in radio, and finds numerous applications in power work.

It may be emphasized again that the alternating-current steady-state response relates only to the conditions in a circuit when the driving force has constant angular frequency and amplitude and has been applied for a time so long that the transient resulting from its application or a change in its magnitude or frequency has subsided to a negligible size. Thus when in this discussion the angular frequency or a parameter is spoken of as varying, not a variation with time but merely a change in its numerical value is meant. In other words, if the circuit response were being determined experimentally, the instruments would be read only after they had taken a steady deflection following any change in the source frequency or amplitude or in the circuit parameters. It may seem that undue emphasis is being placed on what may appear to be quite obvious, but the emphasis is justified because not infrequently steady-state analysis is applied to conditions that are not actually steady, with misleading or sometimes entirely erroneous results.

Further discussion of the series RLS circuit is first presented. In Art. 26, Eq. 162 was used as a basis for calculating the current I as a function of the angular frequency ω. While correct, this equation contains the expression $(\omega/\omega_0 - \omega_0/\omega)$ which near resonance becomes very small compared to either of its terms. In fact, four or five significant figures may be required to obtain a result within one per cent. To avoid this difficulty and also to make clearer some of the aspects of resonance phenomena, Eq. 162 can be rewritten using two new quantities δ and Q_0, which are defined and described as follows:

$$\delta \equiv \frac{\omega}{\omega_0} - 1 = \frac{\omega - \omega_0}{\omega_0}; \qquad [184]$$

$$Q_0 \equiv \frac{\omega_0 L}{R} = \frac{\text{reactance of coil at resonance}}{\text{resistance of coil}}. \qquad [185]^*$$

The quantity δ is seen to be the fraction of the resonance angular frequency ω_0, by which any given angular frequency ω differs from ω_0. Thus if ω is $1.05\omega_0$, δ is 0.05, that is, ω is five per cent greater than ω_0. The practical usefulness of δ is evident subsequently.

The symbol Q without the subscript zero is extensively used to denote the ratio of the reactance of a coil to its resistance at any given angular

* The same symbol is used for reactive power, but is not the same quantity.

frequency ω. Thus

$$Q \equiv \frac{\omega L}{R}, \qquad [186]$$

in which Q is obviously a function of ω; Q_0 is thus the value of Q at the resonant frequency ω_0. It is evident in what follows that the value of Q_0 has great influence on the magnitude of the resonance effect, a large Q_0 resulting in a marked resonance phenomenon. The Q of a coil may be termed its *quality* or *merit*, or ability to produce resonance effects.*

In terms of δ and Q_0, Eq. 162 becomes

$$\mathbf{I} = \frac{\mathbf{E}}{R} \frac{1}{1 + jQ_0\delta\left(\frac{2+\delta}{1+\delta}\right)}, \qquad [162a]$$

which for near-resonance frequencies, that is, when ω is nearly equal to ω_0 or δ is much less than unity, can be approximated by

$$\mathbf{I} \approx \frac{\mathbf{E}}{R} \frac{1}{1 + j2Q_0\delta}. \qquad ▶[162b]$$

The magnitude I is expressed exactly in terms of δ and Q_0 by

$$I = \frac{E}{R} \frac{1}{\sqrt{1 + Q_0^2\delta^2\left(\frac{2+\delta}{1+\delta}\right)^2}} \qquad [164a]$$

or approximately for near-resonance frequencies by

$$I \approx \frac{E}{R} \frac{1}{\sqrt{1 + (2Q_0\delta)^2}} \qquad ▶[164b]$$

* For frequencies extending well up into the audio range, the Q of a coil can be made almost arbitrarily high by using sufficient material. At higher frequencies the effect of distributed capacitance, skin effect, and so on, limit the value of Q which can be attained even with unlimited material. At frequencies roughly under a few thousand cycles per second, skin effect and distributed capacitance can often be neglected and for these conditions

$$\begin{aligned} Q &\propto \text{frequency}, \\ &\propto (\text{linear dimensions})^2 \quad (\text{for a given shape of coil}), \\ &\propto (\text{volume or weight})^{2/3} \quad (\text{for a given shape of coil}). \end{aligned}$$

The air-core coil having the best possible Q for a specified length and weight of wire is a circular air-core copper-wire coil having a square winding section b centimeters on a side and a mean diameter equal to $3.02b$ (*Circ. Nat. Bur. Stand.*, No. 74, p. 294), for which

$$\frac{Q}{\omega} = \frac{L}{R} = \frac{b^2}{635} \times (\text{space factor of winding}). \qquad [187]$$

Such a coil, with b equal to 2 centimeters, space factor of 0.5, weight 400 grams or 0.9 pound, has a Q at 1,000 cycles per second of about 20.

For the usual coils used at frequencies of 10^4 to 10^7 cycles per second, the value of Q is of the order of 100 to 300.

When E is taken as a real quantity, the real and imaginary parts, $\mathcal{R}_e[\mathrm{I}]$ and $\mathcal{I}[\mathrm{I}]$, respectively, of I are also of interest and are found to be

$$\left.\begin{aligned} \mathcal{R}_e[\mathrm{I}] &= \frac{E}{R}\,\frac{1}{1 + Q_0^2\delta^2\left(\dfrac{2+\delta}{1+\delta}\right)^2} && [188] \\ \mathcal{I}[\mathrm{I}] &= -\frac{E}{R}\,\frac{Q_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)}{1 + Q_0^2\delta^2\left(\dfrac{2+\delta}{1+\delta}\right)^2} && [189] \end{aligned}\right\}\ \text{exact expressions}$$

or approximately

$$\left.\begin{aligned} \mathcal{R}_e[\mathrm{I}] &\approx \frac{E}{R}\,\frac{1}{1 + (2Q_0\delta)^2} && [188a] \\ \mathcal{I}[\mathrm{I}] &\approx -\frac{E}{R}\,\frac{2Q_0\delta}{1 + (2Q_0\delta)^2}. && [189a] \end{aligned}\right\}\ \text{approximations for } \delta \ll 1$$

These equations are plotted in Figs. 32a and 32b for various values of Q_0. The solid curves are exact; the dash curves are from the approximate equations. The ordinates are expressed in terms of the magnitude I_0 (equal to E/R) of the current at resonance. A curve of

$$Z = R\sqrt{1 + \left(Q_0\delta\,\frac{2+\delta}{1+\delta}\right)^2} \quad \text{(solid curve)} \qquad [190]$$

and

$$Z \approx R\sqrt{1 + (2Q_0\delta)^2} \quad \text{(dash curve)} \qquad [190a]$$

is shown in Fig. 32c. This curve is merely the inverse of the I curve for E equal to unity.

For the approximate equations plotted with dash curves on Fig. 32, $\mathcal{R}_e[\mathrm{I}]$ and I are even functions and $\mathcal{I}[\mathrm{I}]$ is an odd function of δ. The quantities $\mathcal{R}_e[\mathrm{I}]$ and I have a maximum value E/R at δ equal to zero while $\mathcal{I}[\mathrm{I}]$ is zero there. On the other hand, $\mathcal{I}[\mathrm{I}]$ has a maximum and a minimum of magnitude $E/(2R)$ at approximately δ equal to $1/(2Q_0)$ and δ equal to $-1/(2Q_0)$, respectively, for δ much greater than 1, or exactly for $Q_0\delta[(2+\delta)/(1+\delta)]$ equal to ± 1. The value of $\mathcal{R}_e[\mathrm{I}]$ has the same general appearance as I but is a more sharply peaked curve. This fact can be seen readily from the approximate equations in which for large values of $Q_0\delta$, $\mathcal{R}_e[\mathrm{I}]$ drops off as $1/\delta^2$ whereas I drops off only as $1/\delta$.

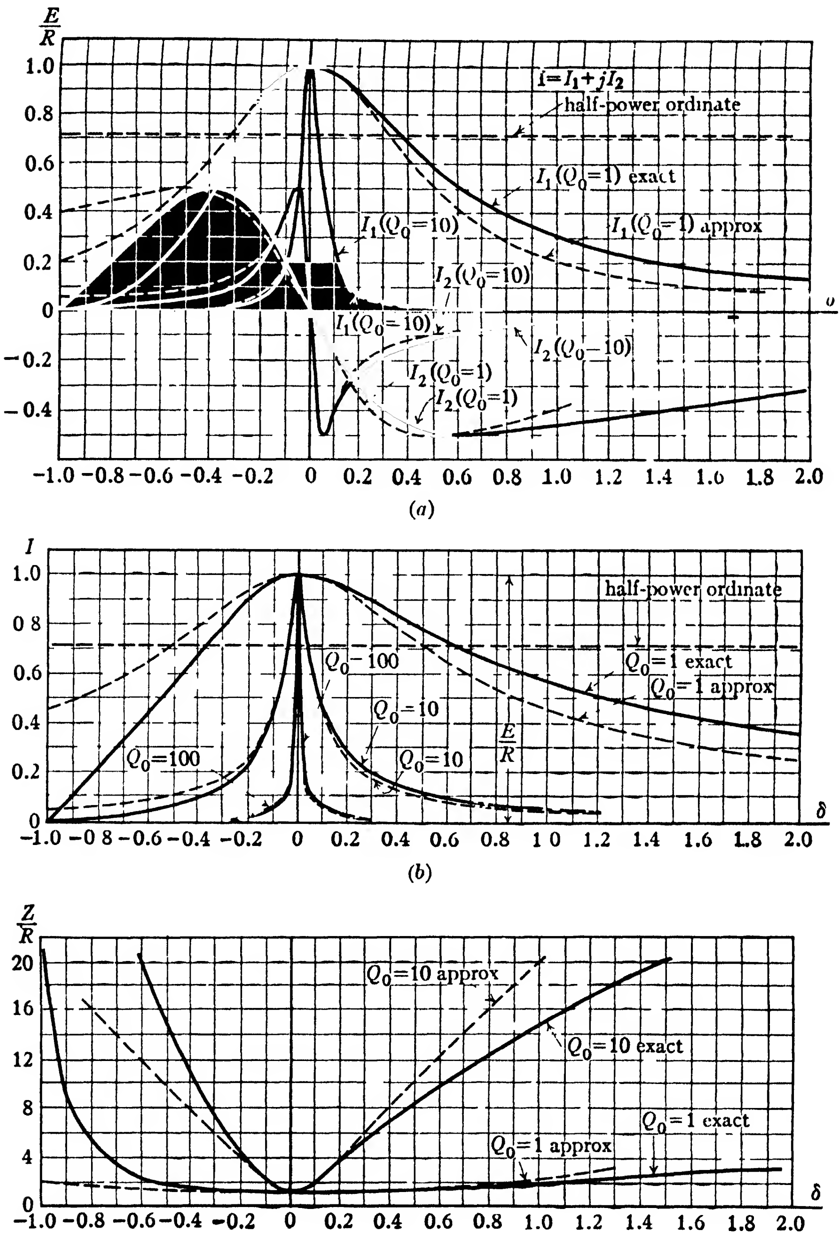


FIG. 32 Exact and approximate current, impedance, and frequency relations, series RLS circuit

29. Sharpness of resonance; half-power points

In most applications of resonant circuits it is important to have a measure of the sharpness of resonance or the width of the current peak. The most commonly used measure is the width of the peak at the *half-power points*, at which the power is one half and the current is $1/\sqrt{2}$ or 0.707 of its value at the resonance peak. In Art. 26 this width is shown to be R/L. This width is also easily expressed in terms of δ and Q_0. Thus from an inspection of the approximate expression of Eq. 164b for the current magnitude I, the current is seen to drop to $1/\sqrt{2}$ of its maximum when $2Q_0\delta$ is ± 1 or δ is $\pm 1/(2Q_0)$. Thus the increment of δ between these two half-power points is $1/Q_0$, or the increment $\Delta\omega$ in ω is approximately

$$\Delta\omega \approx \frac{\omega_0}{Q_0} = \frac{R}{L}. \qquad [191]$$

The exact result is obtained from Eq. 164a as follows: The current I equals $1/\sqrt{2}$ of its maximum value when

$$Q_0\delta\left(\frac{2+\delta}{1+\delta}\right) = \pm 1, \qquad [192]$$

from which

$$\delta_+ = -1 + \frac{1}{2Q_0} \pm \sqrt{1 + \frac{1}{4Q_0^2}} \quad \text{for the + sign} \qquad [193]$$

and

$$\delta_- = -1 - \frac{1}{2Q_0} \pm \sqrt{1 + \frac{1}{4Q_0^2}} \quad \text{for the − sign.} \qquad [194]$$

A consideration of the + and − signs shows that + signs are associated with the positive values for ω and are therefore the values of significance in this problem. The half-power points therefore occur at

$$\delta_+ = -1 + \frac{1}{2Q_0} + \sqrt{1 + \frac{1}{4Q_0^2}} \qquad [195]$$

for the ω above resonance, and at

$$\delta_- = -1 - \frac{1}{2Q_0} + \sqrt{1 + \frac{1}{4Q_0^2}} \qquad [196]$$

for the ω below resonance. The half-power points are separated in terms of δ by

$$\delta_+ - \delta_- = \frac{1}{Q_0}, \qquad [197]$$

which was the result obtained from the approximate analysis. The exact half-power δ's are greater than the approximate values by the difference between $\sqrt{1 + 1/(4Q_0^2)}$ and unity, but their difference $(\delta_+ - \delta_-)$ is the same by either the exact or the approximate analysis.

The substitution of 100 for Q_0, which is a typical value in radio-frequency work, shows that the exact and approximate values of the half-power δ's differ by about 1 part in 80,000 or about 12 cycles per second for a resonant frequency of 10^6 cycles per second. At lower frequencies the Q's of coils are likely to be smaller, so if the error may be significant the exact expressions should be used.

A word concerning the accuracy of measurements made at radio frequencies (the order of 10^5 cycles per second and upwards) may be helpful here. In general, and subject to exceptions, it may be said that errors of one per cent are permissible and difficult to avoid, and five per cent may be quite allowable. At these frequencies the values of the parameters often change rather rapidly with frequency. Consequently, for most purposes in such work one is seldom justified in using the exact relations, the approximate ones often being more nearly accurate than experimental data.

30. Inductive and elastive voltages in the vicinity of resonance

In the foregoing paragraphs the behavior of the steady-state current in the vicinity of resonance in a series *RLS* circuit is considered. The voltages appearing across the inductive and elastive elements of the series *RLS* circuit are also of particular interest because in the vicinity of resonance they may become many times the magnitude of the applied voltage. To the power engineer these large voltages are important, for one reason, because of the stress they may impose on insulation. To the communication engineer they are frequently of great importance because they occur only over a limited frequency range near resonance and can therefore be used to obtain sharp discrimination between frequencies which are desired and others which are to be suppressed. The tuning of radio circuits is an excellent illustration of this use.

In the series *RLS* circuit the inductive steady-state voltage V_L and elastive voltage V_C are given respectively by the terms $j\omega L\mathrm{I}$ and $-j(S/\omega)\mathrm{I}$ of Eq. 27a. In terms of the applied electromotive force E, the angular frequency ω, and the circuit parameters, the vector current is given by Eq. 162. Putting Eq. 162 in the foregoing expressions for V_L and V_C gives

$$\mathrm{V}_L = \frac{\mathrm{E}}{R} \frac{j\omega L}{1 + j\dfrac{\omega_0 L}{R}\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)} \tag{198}$$

and

$$\mathbf{V}_C = \frac{\mathbf{E}}{R} \frac{-j\dfrac{S}{\omega}}{1 + j\dfrac{\omega_0 L}{R}\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)}. \qquad [199]$$

As in the case of the current and for the same reason, these expressions can be put in a form more useful for examining their behavior in the vicinity of resonance by being rewritten in terms of the quantities δ and Q_0. Thus by using Eqs 184 and 185

$$\mathbf{V}_L = \mathbf{E}\frac{jQ_0(1+\delta)}{1 + jQ_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)} \qquad [198a]$$

and

$$\mathbf{V}_C = \mathbf{E}\frac{-j\dfrac{Q_0}{1+\delta}}{1 + jQ_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)}. \qquad [199a]$$

For small values of δ, that is, when δ is much less than unity (near-resonance conditions) the values of $\mathbf{V}_L$ and $\mathbf{V}_C$ are well approximated by

$$\mathbf{V}_L \approx -\mathbf{V}_C \approx \mathbf{E}\frac{jQ_0}{1 + j2Q_0\delta}. \qquad ▶[200]$$

Inspection of Eq. 200 indicates that V_L and V_C have maxima when δ equals zero, or at resonance. Actually this indication is only approximately correct, as is shown by an evaluation of δ from Eqs. 198a and 199a for the exact maxima. The magnitudes V_L and V_C are

$$V_L = \frac{EQ_0(1+\delta)}{\sqrt{1 + Q_0^2\delta^2\left(\dfrac{2+\delta}{1+\delta}\right)^2}} = \frac{EQ_0(1+\delta)^2}{\sqrt{(1+\delta)^2 + Q_0^2\delta^2(2+\delta)^2}} \qquad [198b]$$

and

$$V_C = \frac{E\dfrac{Q_0}{1+\delta}}{\sqrt{1 + Q_0^2\delta^2\left(\dfrac{2+\delta}{1+\delta}\right)^2}} = \frac{EQ_0}{\sqrt{(1+\delta)^2 + Q_0^2\delta^2(2+\delta)^2}}, \qquad [199b]$$

from which their maxima can be obtained by the usual process of equating the derivative to zero. This process, after some algebraic simplifications, yields the following:

For maximum V_L

$$\delta = \pm \frac{1}{\sqrt{1 - \frac{1}{2Q_0^2}}} - 1, \qquad [201]$$

for maximum V_C

$$\delta = \pm \sqrt{1 - \frac{1}{2Q_0^2}} - 1; \qquad [202]$$

and for minimum V_C

$$\delta = -1. \qquad [203]$$

Only the + sign need be used, since frequency is always considered as positive. The − sign, however, gives an identical numerical value for ω with a negative sign.

The maximum values of V_L and V_C are the same:

$$\text{Maximum } V_L \text{ or } V_C = E\frac{Q_0}{\sqrt{1 - \frac{1}{4Q_0^2}}} = E\frac{2Q_0^2}{\sqrt{4Q_0^2 - 1}}. \qquad [204]$$

Because Q_0 is necessarily positive, for maximum V_L, δ is always positive, that is, ω is greater than ω_0, or the maximum occurs above the resonant frequency. This fact appears anomalous until one recalls that V_L changes because of change in the inductive reactance as well as because of change in current. Thus as ω increases above ω_0, V_L is increased by the factor ωL, whereas, when ω is nearly equal to ω_0, I changes very little, and the maximum V_L, therefore, occurs when ω is greater than ω_0. A parallel line of reasoning shows why the maximum V_C occurs when ω is less than ω_0. The value δ equal to -1 shows that the curve for V_C has zero slope at the origin.

Figures 33a and 33b show curves of V_L and V_C as functions of both δ and ω/ω_0.

By an examination of Eqs. 201 and 202 the maxima of V_L and V_C are seen to occur very near to ω_0 when Q_0 is large, that is, when the circuit is highly oscillatory, and shift away from ω_0 as Q_0 decreases. When Q_0 is $1/\sqrt{2}$ the maxima have decreased to E, and the ω for maximum V_L has become infinite, while that for maximum V_C has become zero. Thus the voltage maxima of the type shown in Figs. 33a and 33b occur only when Q_0 is greater than $1/\sqrt{2}$, and are pronounced only if Q_0 is much greater than unity.

In practical circuits the resistance R is usually that of the coil having inductance L. In that case the magnitude of the measurable voltage at the coil terminals is not V_L, but is the magnitude of the sum of two vectors

whose lengths are V_L and IR, which are in quadrature. In resonance circuits, however, since IR is usually small compared with V_L, the magnitude of the voltage measured across the coil terminals ordinarily differs very little from V_L except at relatively low frequencies.

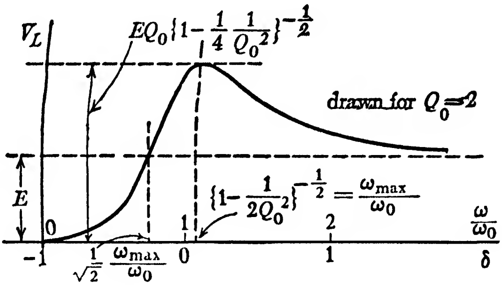


FIG. 33a Inductance voltage — frequency relation, series *RLS* circuit.

In the foregoing discussion of the series *RLS* circuit, the source is one that impresses a constant-amplitude voltage on the circuit. At the resonant frequency the general characteristics of the response are low impedance, large current in phase with the impressed voltage, and large voltages across both the inductance and the elastance; at frequencies differing appreciably from the resonant frequency, the general characteristics

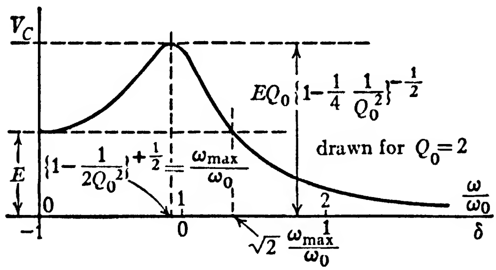


FIG. 33b. Capacitance voltage — frequency relation, series *RLS* circuit.

of the response are high impedance, small current largely in quadrature with the applied voltage, and component voltages not materially exceeding the source voltage. If, on the other hand, a current source of constant amplitude is applied to this circuit, the resonant-frequency response is characterized by a small minimum over-all voltage drop and relatively large inductive and elastive voltage drops, whereas at the off-resonant frequencies the total voltage drop becomes relatively large. The phenomenon of minimum response is usually described as *antiresonance*, and the series *RLS* circuit is thus said to exhibit *current resonance* and *voltage antiresonance*.

31. The parallel $GC\Gamma$ circuit in the vicinity of resonance

The parallel $GC\Gamma$ circuit is the dual of the series RLS circuit. If Eq. 27a for the series circuit and Eq. 82 for the dual parallel circuit are compared, the vector-current response of the series circuit to a unit applied vector voltage and the vector-voltage response of the parallel circuit to a unit applied vector current are seen to be mathematically identical. Thus the entire analysis of the series circuit is applicable to the parallel case, and the result can be used by merely interchanging current and voltage and by interchanging the parameters. Written in terms of δ and D_0, the voltage across the dual circuit when a current $\mathbf{I}$ is impressed is

$$\mathbf{V} = \frac{\mathbf{I}}{G} \frac{1}{1 + jD_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)}, \qquad [205]$$

in which

$$D_0 \equiv \frac{\omega_0 C}{G} \qquad [206]$$

corresponds to Q_0 in the series circuit and may be called a quality factor for a leaky condenser represented as C and G in parallel. The student may find it instructive to evaluate the dual circuit equations and sketch the characteristic curves that correspond to Figs. 32a to 32c, and Figs. 33a and 33b.

It can be seen both by the duality relations and by following through the analysis in detail that the dual circuit exhibits the phenomena of *voltage resonance* and *current antiresonance* in response to a *constant-amplitude applied current*. If, on the other hand, a *constant-amplitude voltage* is applied, this parallel circuit shows a current minimum at the resonant frequency.

32. The parallel coil-condenser circuit in the vicinity of resonance

Next is discussed another parallel circuit, shown in Fig. 34, of more practical interest than the $GC\Gamma$ circuit. This parallel circuit consists of a coil having a resistance R and an inductance L, in parallel with a condenser of elastance S. Practically, the loss in the condenser is usually negligible compared to the loss in the coil, so that the circuit elements shown in Fig. 34 approximate rather well an actual parallel coil-condenser combination.

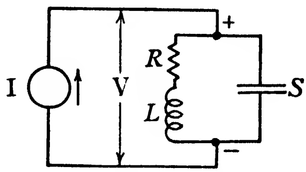


Fig. 34. Parallel coil-condenser circuit

The response of this circuit to a constant-amplitude current is most easily obtained by writing the impedance of the circuit and taking the reciprocal to get the admittance. Since the combined impedance of two impedances in parallel is given by their product divided by their sum, the impedance Z through which the source forces current is found to be

$$\mathrm{Z} = \frac{(R + j\omega L)\left(-j\dfrac{S}{\omega}\right)}{R + j\left(\omega L - \dfrac{S}{\omega}\right)}. \qquad [207]$$

Since the frequencies of principal interest are those in the vicinity of resonance, Eq. 207 is most conveniently rewritten in terms of Q_0 and δ:

$$\mathrm{Z} = \frac{SL\left[1 - j\dfrac{1}{Q_0(1+\delta)}\right]}{R\left[1 + jQ_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)\right]}. \qquad [207a]$$

The vector-voltage drop V across the circuit is

$$\mathrm{V} = \mathrm{IZ} = \mathrm{I}\frac{SL}{R}\,\frac{1 - j\dfrac{1}{Q_0(1+\delta)}}{1 + jQ_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)}. \qquad [208]$$

The current I_L in the inductive branch is

$$\left.\begin{aligned}\mathrm{I}_L &= \frac{\mathrm{V}}{R + j\omega L} = \frac{\mathrm{IZ}}{R + j\omega L} = \mathrm{I}\frac{-j\dfrac{S}{\omega}}{R + j\left(\omega L - \dfrac{S}{\omega}\right)}\\ &= \mathrm{I}\frac{-j\dfrac{\omega_0 L}{1+\delta}}{R\left[1 + jQ_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)\right]} = \mathrm{I}\frac{-j\dfrac{Q_0}{1+\delta}}{1 + jQ_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)}.\end{aligned}\right\} \qquad [209]$$

The current in the capacitive branch is

$$\left.\begin{aligned}\mathrm{I}_C &= \frac{\mathrm{V}}{-j\dfrac{S}{\omega}} = j\frac{\omega}{S}\mathrm{IZ} = \mathrm{I}\frac{R + j\omega L}{R + j\left(\omega L - \dfrac{S}{\omega}\right)}\\ &= \mathrm{I}\frac{1 + jQ_0(1+\delta)}{1 + jQ_0\delta\left(\dfrac{2+\delta}{1+\delta}\right)}.\end{aligned}\right\} \qquad [210]$$

In the vicinity of resonance where δ is much less than unity, Eqs. 208 to 210 can be simplified to

$$\mathbf{V} \approx \mathbf{I}\frac{SL}{R}\frac{1 - j\dfrac{1}{Q_0}}{1 + j2Q_0\delta}, \qquad \blacktriangleright[208a]$$

$$\mathbf{I}_L \approx \mathbf{I}\frac{-jQ_0}{1 + j2Q_0\delta}, \qquad \blacktriangleright[209a]$$

$$\mathbf{I}_C \approx \mathbf{I}\frac{1 + jQ_0}{1 + j2Q_0\delta}. \qquad \blacktriangleright[210a]$$

While Eqs. 209a and 210a are good approximations for $\mathbf{I}_L$ and $\mathbf{I}_C$, their sum, which is small compared to either $\mathbf{I}_L$ or $\mathbf{I}_C$ with the Q_0's encountered in usual practice, is incorrect.

Equations 208a and 210a can be further simplified when Q_0 is much greater than unity.

$$\mathbf{V} \approx \mathbf{I}\frac{SL}{R}\frac{1}{1 + j2Q_0\delta}, \qquad \blacktriangleright[208b]$$

$$\mathbf{I}_C \approx \mathbf{I}\frac{jQ_0}{1 + j2Q_0\delta}. \qquad \blacktriangleright[210b]$$

When δ is much less than unity, the condition when Q_0 is equal to infinity is very nearly equivalent to neglecting R in the numerator of Z.

To indicate the order of the approximation involved in Eqs. 208b and 210b at the resonant frequency because of the assumption that Q_0 is infinity, a circuit is considered in which the coil has a Q_0 of 100, a typical value for a radio circuit. For this case $\mathbf{V}$ is in error in magnitude by about one part in 10^4 and in angle by about 0.01 radian or 0.57 degree. If Q_0 is 10, as might, for example, be the case at low audio frequencies, $\mathbf{V}$ is in error by about one part in 200 in magnitude and by 0.1 radian or 5.7 degrees in angle. The current $\mathbf{I}_C$ is in error by the same amounts. The errors introduced by the assumption that δ is negligible compared to unity are of the same order as those occurring in the series *RLS* circuit and are readily determined by inspection for any given case.

The close correspondence between the approximate equations for the series and parallel physical circuits is of interest. For example, Eqs. 208b and 162b, Eqs. 209a and 200, and Eqs. 210b and 200 may be compared. With current and voltage interchanged, and inductive and capacitive subscripts interchanged, the equations are identical except for the constant factor *SL* in Eq. 208b. Consequently within the accuracy of the approximate equations, the same curves relabeled, and in one case rescaled, apply to both the series and parallel coil-condenser circuits. A summary of the resonant properties of the two circuits is given here for comparison.

SUMMARY OF CHARACTERISTICS OF COIL-CONDENSER SERIES AND PARALLEL CIRCUITS

	Series Coil-Condenser Circuit	Parallel Coil-Condenser Circuit
Source →	Constant amplitude E	Constant amplitude I
Near resonant frequency $\omega \approx \omega_0$ $\delta \approx 0$	Current resonance Low net impedance Large current Large voltages across inductive and capacitive elements	Voltage resonance High net impedance Large voltage Large currents in branches
Away from resonant frequency ω differs considerably from ω_0	High net impedance Small current Voltages across elements of the order of E or smaller	Low net impedance Small voltage Currents in elements of the order of I or smaller
Source →	Constant amplitude I	Constant amplitude E
Near resonant frequency $\omega \approx \omega_0$ $\delta \sim 0$	Voltage antiresonance, that is, voltage minimum	Current antiresonance, that is, current minimum
Away from resonant frequency ω differs considerably from ω_0	Large voltage across circuit	Large current in line

33. Illustrative Example of Conditions near Resonance

A resonant circuit for a radio receiver is to be designed. The variable tuning condenser to be used has a capacitance range of 11.0 to 150 micromicrofarads. The frequency range for which the circuit is to be resonant is to be from 0.500×10^6 to 1.600×10^6 cycles per second. The questions are:

(a) For what inductance should the coil be designed, bearing in mind that in mass production of the coils the inductance may vary ± 5 per cent?

(b) What is the allowable effective resistance of the coil if the half-power points for a single tuned circuit are to be separated by not more than 5,000 cycles per second?

(c) What is the response of the circuit to disturbing frequencies differing from the resonant frequency by (1) 10,000 cycles per second, (2) 50,000 cycles per second?

Solution: The inductance of the coil required for resonance at the lower frequency limit and the upper capacitance limit is

$$L = \frac{1}{\omega^2 C} = \frac{1}{(2 \times 3.14 \times 0.500 \times 10^6)^2 \times 150 \times 10^{-12}} = 0.000675 \text{ h.} \qquad [160b]$$

If the nominal value of L is increased by 5 per cent to allow for manufacturing variations,

$$L = 0.000675 + 0.000034 = 0.000709 \text{ h,} \qquad [211]$$

or about 0.71 mh for design purposes. Tuning at all other frequencies up to a frequency of $1.600 \times 10^6 \sim$ evidently is possible with this coil because

$$C = \frac{1}{(2 \times 3.14 \times 1.600 \times 10^6)^2 \times 0.000709} = 14.1 \times 10^{-12} \text{ farad,} \qquad [160c]$$

which is within the range of the condenser by more than 5 per cent.

The allowable effective resistance of the coil is

$$R = 2 \times 3.14 \times 5{,}000 \times 0.000709 = 22.2 \text{ ohms} \qquad [191a]$$

For determining the response for the off-resonance conditions, the resonant frequency of $0.500 \times 10^6 \sim$ gives the worst deviation. For this frequency

$$Q_0 = \frac{2 \times 3.14 \times 500{,}000 \times 0.000709}{22.2} = 100, \qquad [185a]$$

$$\delta_1 = \frac{10{,}000}{500{,}000} = 0.02, \qquad [184a]$$

$$\delta_2 = \frac{50{,}000}{500{,}000} = 0.10, \qquad [184b]$$

$$(2Q_0\delta_1)^2 = (200 \times 0.02)^2 = 16, \qquad [212]$$

$$(2Q_0\delta_2)^2 = (200 \times 0.1)^2 = 400. \qquad [212a]$$

From Eq. 164b

$$\frac{I_1}{I_0} \approx \frac{1}{\sqrt{1 + 16}} \approx 0.24, \qquad [213]$$

$$\frac{I_2}{I_0} \approx \frac{1}{\sqrt{1 + 400}} \approx 0.050. \qquad [214]$$

34. Instantaneous current-voltage loci

In preceding articles, graphical methods are introduced for showing the relations among the steady-state complex quantities. The instantaneous quantities are plotted also as functions of time.

In many cases the instantaneous conditions at the terminals of a network are more conveniently studied in terms of an explicit relation between the voltage and current. Thus, since $e(t)$ and $i(t)$ are given by Eqs. 5 and 21 as explicit functions of time, a definite relationship between $e(t)$ and $i(t)$ is implied. This may be put into the form of an explicit

relation by eliminating the time variable between the two equations Plots of the resulting expression are useful in experimental studies by means of the cathode-ray oscillograph. If the deflection of the cathode-ray spot in one direction is made proportional to $e(t)$ and the deflection in the direction normal to the first direction is made proportional to $i(t)$, the resulting curve is exactly that obtained by eliminating (t) between the two expressions. Such curves are referred to in what follows as *instantaneous current-voltage loci* or merely *current-voltage loci*.

For simplicity a zero instant may be selected such that ψ in Eq. 5 is zero. This is merely a convenience and in no way detracts from the generality of the results. Then

$$e(t) = E_m \cos \omega t, \tag{5g}$$
$$i(t) = I_m \cos \theta \cos \omega t + I_m \sin \theta \sin \omega t. \tag{21f}$$

The current may be expressed as

$$i(t) = i_a(t) + i_b(t), \tag{215}$$

with

$$i_a(t) \equiv I_m \cos \theta \cos \omega t, \tag{216}$$

$$i_b(t) \equiv I_m \sin \theta \sin \omega t. \tag{217}$$

Here $i_a(t)$ is the component of $i(t)$, which is in phase with the voltage, and $i_b(t)$ is the quadrature component or the component whose phase is 90 degrees behind the voltage. The inphase component of the current is expressible linearly in terms of the voltage; namely, Eqs. 5g and 216 give

$$i_a(t) = \frac{I_m \cos \theta}{E_m} e(t), \tag{216a}$$

or

$$i_a(t) = \frac{R}{R^2 + X^2} e(t). \tag{216b}$$

The following relations assist in eliminating time between $e(t)$ and $i_b(t)$:

$$\frac{e(t)}{E_m} = \cos \omega t, \tag{218}$$

$$\frac{i_b(t)}{I_m \sin \theta} = \sin \omega t. \tag{219}$$

Squaring and adding Eqs. 218 and 219 give

$$\frac{e^2(t)}{E_m^2} + \frac{i_b^2(t)}{I_m^2 \sin^2 \theta} = 1, \tag{220}$$

which is the equation of an ellipse in normal form with semiaxes equal to E_m and $I_m \sin \theta$, respectively.

Thus the plot of $e(t)$ versus $i_a(t)$ is the straight line of Fig. 35, while the plot of $e(t)$ versus $i_b(t)$ is the ellipse, Eq. 220. Since the net current i is the sum of i_a and i_b, the plot of $e(t)$ versus $i(t)$ is the graphical sum of the straight line and the ellipse, which is an ellipse sheared in the straight line. The sheared ellipse still has its center at the origin but its principal axes no longer coincide with the co-ordinate axes. The resulting ellipse is inscribed in the rectangle $2E_m$ by $2I_m$ and is tangent to this rectangle at four points, which correspond to those instants at which either the voltage or current passes through its maximum values. At these points the corresponding values of current or voltage respectively are simply expressible in terms of the impedance angle, as indicated in the figure. The points

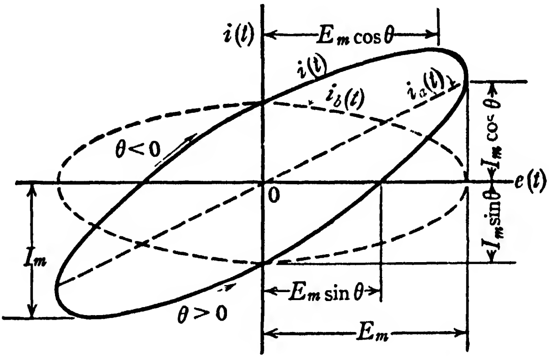


FIG 35. Instantaneous current-voltage locus.

corresponding to instants at which either voltage or current passes through zero are similarly easily expressed. Thus eight points through which the resulting ellipse must pass are readily set down. For an approximate sketch this suffices, so that the desired locus is easily obtained.

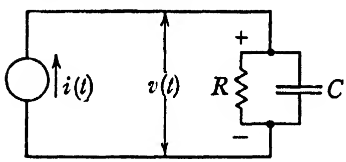
Several additional facts are worth noting: If the impedance angle θ is positive, the current lags the voltage, and the ellipse is traced in the counterclockwise direction. When θ is negative, the direction is reversed. The major axis of the ellipse does *not* coincide with the line representing $i_a(t)$. Neither does it coincide with the diagonal of the rectangle, as one might suppose from an offhand inspection. The orientation of the axes of the sheared ellipse bears a rather complicated relation to the co-ordinate axes of the figure and need not be investigated here. In the special case of zero power factor (when θ is $\pm\pi/2$), the inphase current is zero and the resulting ellipse is in the normal form. On the other hand, when the power factor is unity (θ equal to zero), the quadrature component of current becomes zero and the ellipse degenerates into the straight

line given by $i_a(t)$ alone. The elliptic locus may alternatively be derived by taking the current as a reference and splitting the voltage into in-phase and quadrature components. The result is, of course, the same, and there is no need to present a detailed discussion of this process here.

The elliptic form of the current-voltage characteristic results from the choice of a sinusoidal source, combined with linear circuit elements, and, in general, the type of characteristic depends upon the type of source as well as the type of circuit. *In experimental work such an instantaneous locus may occasionally depart widely from an ellipse because of the circuit* exaggerating the effects of harmonic frequencies in the source voltage. This is particularly true of circuits in which resonance effects occur.

PROBLEMS

1. A noninductive resistance of 15 ohms, an impedance coil which has a resistance of 2 ohms and an inductance of 0.1 h, and a condenser of 120 μf capacitance are connected in series across 310-v (amplitude) mains having a frequency of 60 ~. What are the current amplitude and its phase relation with respect to the impressed voltage? What is the instantaneous value of the impressed voltage when the energy of the electrostatic field is 0? when the energy of the electromagnetic field is zero?



$i(t) = I_m \cos(\omega t + \phi)$ $\omega = 2\pi f$
$R = 50{,}000$ ohms $C = 0.004\ \mu f$
f = frequency

FIG. 36. Elements of amplifier circuit, Prob. 2.

2. The plate circuit of a five-element vacuum tube or pentode can frequently be considered as a current generator since its internal resistance is so high that a load impedance connected in the plate circuit has a negligible effect on the plate current. In using such a tube as an amplifier, it is useful to have as functions of frequency

(a) the amplitude of $v(t)$ in terms of I_m,
(b) the phase angle of $v(t)$ with respect to that of $i(t)$, for the circuit of Fig. 36. Both functions are to be plotted on one set of axes, up to 5,000 ~.

3. A coil having a resistance of 100 ohms and an inductance of 1.0 h, and a condenser having a capacitance of 9.6 μf, are in series with a sinusoidal electromotive force of amplitude 141 v and frequency 60 ~.

(a) What are the expressions for the total complex impedance of the circuit in rectangular, polar, and exponential form?
(b) What are the complex expressions for current and for voltage drops (amplitudes) across the coil and condenser, referred to the impressed voltage as an axis of reference?
(c) The results of (b) are to be illustrated on a vector diagram and each vector is to be identified with reference to a circuit diagram which shows the directions of the current and of the voltage drops.

4. A source of 110 v and 60 ~ supplies power to a 1,000-ohm resistor and a 1.0-μf capacitor in series.

(a) What is the maximum power supplied to the capacitor?
(b) At what instant is this reached?

(c) What is the maximum power supplied to the resistor?
(d) At what instant is this reached?
(e) What is the average power supplied to the capacitor?
(f) What is the average power supplied to the resistor?

5. What is the effective value of a triangular-shaped alternating-current wave whose maximum value is 5 amp?

6. Two coils are in series across 220-v mains having a frequency of 60 ~. There is the same voltage (effective magnitude) across each coil, but one coil has a resistance of 15.0 ohms and the other coil has a resistance of 7.00 ohms. The total power taken by the two coils is 550 w. What is the inductive reactance of each coil?

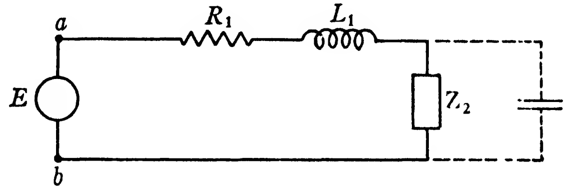


FIG. 37. Transmission line and load, Prob. 7.

7. A sinusoidal source of 60 ~ frequency is connected at a and b (Fig. 37) to a circuit whose resistance R_1 is 0.80 ohm and inductance L_1 is 3.72 mh. A test at a–b shows that with 2,300 v applied, the power input is 190 kw at a lagging power factor of 78%.
(a) What is the vector impedance Z_2 which corresponds to the given conditions?
(b) If a lossless condenser having a capacitance of 57.2 μf is connected as shown by the dotted lines in parallel with Z_2, the rest of the circuit remaining unchanged, what is the resulting kilowatt input and power factor at a b?

8. A single-phase distribution line 1 mile long operates at 60 ~, delivering 2,300 v at the load. What is the magnitude of the voltage drop in the line when the load is 75 kw at a power factor of 70% lagging? The conductors are 4 AWG hard-drawn copper, spaced 6 ft apart on centers, operating at a temperature of 20 C. Leakage, capacitance, and the effect of flux linkages within the conductors may be neglected.

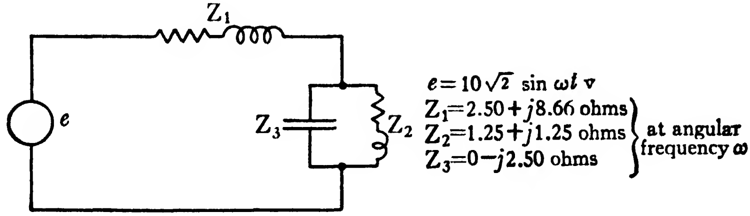


FIG. 38. Series-parallel circuit for Prob. 9.

9. The circuit of Fig. 38 is in the steady state.
(a) What is the maximum instantaneous power supplied to Z_1, Z_2, and Z_3, respectively?
(b) What is the maximum instantaneous total power supplied by the source?

10. In the circuit of Fig. 39 the total vector power from the source is

$$\bar{E}I = 5{,}200 \text{ w} - j2{,}500 \text{ var.} \quad [221]$$

The circuit operates in the steady state at a frequency of 60 ~.

(a) *What is the value of the inductance L_1 in henrys?*
(b) *What is the effective value of the electromotive force E?*
(c) What is the power factor of the circuit at *ab*? Does the current I lead or lag the electromotive force E?

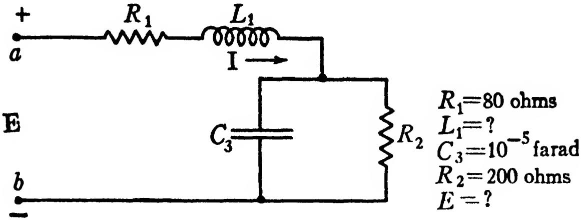


FIG. 39. Series-parallel circuit for Prob. 10.

11. An impedance takes 250 kva at a lagging power factor of 0.5 from a constant-voltage source. A second impedance is placed in parallel with the first impedance and takes 100 kva at an adjustable power factor.

(a) What is the highest over-all power factor that can be obtained?
(b) What is the active power supplied to the second impedance in case (a)?
(c) What is the reactive power supplied to the second impedance in case (a)?
(d) What is the vector power of the combination in case (a)?

12. A coil connected across 230-v mains having a frequency of 25 ~ draws 5.0 amp at 0.15 power factor. If this same coil is connected in series with a condenser across 230-v mains having a frequency of 60 ~,

(a) What capacitance must the condenser have in order to make the current 5 amp lagging?
(b) What is the average power supplied to the circuit?
(c) What is the power factor of the circuit?
(d) What is the power factor of the coil?
(e) What is the maximum instantaneous power supplied to the circuit?
(f) What is the vector power supplied to the circuit? to the coil? to the condenser?

13. An inductive load of 2,000 kw at 0.8 power factor is connected to a 2,300-v power system of frequency 60 ~. A condenser is placed in parallel with this load to make the resultant power factor of the inductive load and the condenser in parallel equal to unity. The power factor of the condenser with the necessary auxiliary apparatus which goes with it is 5%.

(a) If the impressed voltage of the circuit is 2,300 v, what are the currents taken by the load and by the condenser?
(b) A vector diagram is to be sketched showing the effective values of the vectors which represent (1) the voltage impressed on the circuit, (2) the current taken by the load, and (3) the current taken by the condenser
(c) What is the complex expression for the resultant current taken by the load and the condenser in parallel?
(d) What are the active power and the reactive power (1) of the load, (2) of the condenser, and (3) of the condenser and the load in parallel?

14. A coil has a resistance of 2 ohms and an inductance of 0.1 h. It is desired to use this coil in series with a condenser to show the phenomena of resonance. If the only source available is 115 v, 60 ~, what is the necessary capacitance of the condenser? If the condenser is designed for a maximum rms voltage of 700 v, what resistance must be inserted in the circuit in order that at resonance the voltage across the condenser shall not exceed this amount?

15. A wattmeter and an ammeter are connected to measure respectively the average power and the effective current taken by a series circuit which contains constant resistance, constant inductance, and constant capacitance. This circuit is connected to a source of sinusoidal voltage which has a constant amplitude. The frequency of this voltage is slowly increased from 20 ~ to above 60 ~, at such a rate that the transient current produced by the change is negligible.

When the frequency reaches 25 ~, the wattmeter indicates 314.4 w. As the frequency is increased, the readings of the ammeter and the wattmeter increase. At 60 ~ they are, respectively, 35.0 amp and 3,851 w. Any further increase in the frequency causes the wattmeter reading to decrease.

(a) What is the resistance of the circuit?
(b) What is the amplitude of the applied voltage?
(c) What are the capacitance and inductance of the circuit?
(d) What is the resonant angular frequency of the circuit?
(e) What is the characteristic damped angular frequency of the circuit?

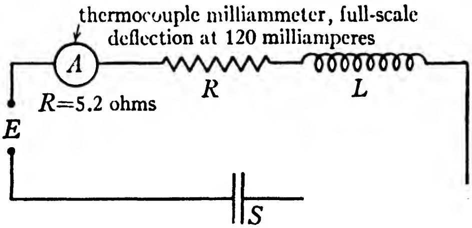


FIG. 40. Circuit of frequency meter, Prob. 16.

16. The circuit shown in Fig. 40 is to be used for indicating frequency fluctuations in a 5,000-~ voltage supply. This circuit is to be adjusted for resonance at a frequency slightly above 5,000 ~. It is to be so designed that 5,000 ~ is the below-resonance half-power point.

It may be assumed that a coil having Q_0 of 100 at 5,000 ~ can be built at a reasonable cost. The circuit is to be designed using such a coil so that frequency deviations of ± 50 ~ from 5,000 cause the milliammeter deflection to vary roughly from one-fourth to three-fourths full-scale deflection.

The design should give the values of the resistance R and inductance L of the coil, the value of the elastance S, and the value of the voltage to be used. Any approximations may be used which give results within 5 or 10%, as this is merely a preliminary design.

Does the current at resonance exceed the current required for full-scale deflection of the milliammeter?

The deflection of the thermocouple milliammeter is proportional to the square of the current.

17. When two voltages

$$v_1 = A \sin \omega t \qquad [2b]$$

and

$$v_2 = B \sin(\omega t + \psi') \tag{2c}$$

are impressed upon the control plates of a cathode-ray oscillograph tube, the image appearing upon the screen is a plot of v_2 versus v_1 in rectangular co-ordinates. For an arbitrary phase angle ψ', the plot is an ellipse. As ψ' is made to approach 0, the ellipse becomes slimmer and finally degenerates into a straight line when ψ' is 0. The cathode-ray tube may therefore be used to detect an inphase condition between two sine voltages.

(a) Does the sensitivity of this method of detecting phase coincidence depend upon the ratio of the amplitudes A/B?
(b) What is the optimum value of this ratio?
(c) When the optimum ratio is used, is the phase discrepancy given by the approximate expression

$$\psi' \approx \frac{2b}{a} \text{ radians} \begin{cases} \text{applicable for small } \psi' \\ \text{exact in the limit } \psi' \to 0 \end{cases} \tag{222}$$

in which a and b are the semimajor and semiminor axes of the ellipse?

18. The following voltages are impressed across the respective pairs of plates of a cathode-ray oscillograph:

$$v_1 = V_{m1} \cos(\omega t - \psi), \tag{5h}$$

$$v_2 = V_{m2} \cos 2\omega t. \tag{5i}$$

(a) What is the general expression for the form of figure which should appear on the fluorescent window?

(b) The forms of the figures are to be sketched for values of ψ of 0, $\pi/4$, and $\pi/2$ radians, taking V_{m1} and V_{m2} equal and v_1 measured horizontally

CHAPTER V

Transient Analysis of Simple Alternating-Current Circuits

1. Transient response of the series RL circuit with alternating-voltage source

As was stated in Art. 3, Ch. IV, the procedure for determining the transient portion of the solutions for circuits having alternating voltages or currents impressed is essentially no different from that for direct voltages or currents. For the series RL circuit, as shown in Arts. 4 and 5, Ch. III, the procedure is to solve the force-free equilibrium equation

$$L\frac{di_t}{dt} + Ri_t = 0, \qquad [1]$$

which is found on p. 172 to be

$$i_t = A\epsilon^{-(R/L)t}. \qquad [2]$$

Exactly the same reasoning is used to evaluate A in the alternating-current case as in the direct-current case. Thus at any given instant the steady-state solution gives a definite current, determined only by the source voltage and the circuit parameters. When some change is made in the circuit parameters or in the source voltage, the actual current in general is not that given by the steady-state solution corresponding to the changed conditions. Because of the inductance this actual current cannot change instantaneously; hence it cannot immediately have the new steady-state value. By means of the transient component of current, the actual current function gradually approaches its steady-state value.

These general principles are now applied to the series RL case under consideration. From the original statement of the problem, the value of the current immediately prior to switching is $i(0-)$. This value changes only infinitesimally during the switching period, so that

$$i(0-) = i(0) = i(0+). \qquad [3]$$

The steady-state component $i_s(t)$ of the current at time t is

$$i_s = \mathcal{R}_e[\mathrm{I}_m\epsilon^{j\omega t}] = I_m \cos(\omega t + \phi). \qquad [4]$$

When t is zero,

$$i_s(0) = I_m \cos\phi. \qquad [5]$$

But at all times

$$i = i_s + i_t = I_m \cos(\omega t + \phi) + A\epsilon^{-(R/L)t}. \qquad [6]$$

Hence for the particular instant when t is equal to zero,

$$i(0) = i(0-) = i_s(0) + i_t(0) = I_m \cos\phi + A, \qquad [6a]$$

and

$$A = i(0-) - I_m \cos\phi. \qquad [7]$$

The actual current at any time t is therefore

$$i = I_m \cos(\omega t + \phi) + [i(0-) - I_m \cos\phi]\epsilon^{-(R/L)t}. \quad [6b]$$

Thus the transient component which is retired gradually by the factor $\epsilon^{-(R/L)t}$ enables the actual current to meet the two following requirements: (1) that its initial value be $i(0-)$, and (2) that the steady-state component of current $i_s(0)$ at the initial instant be $I_m \cos\phi$. If it happens that

$$i(0-) = I_m \cos\phi, \qquad [8]$$

no transient component is required. Furthermore, if $i(0-)$ is greater than I_m, the switching cannot be accomplished without a transient.

2. Illustrative example of series RL circuit

The foregoing analysis may be illustrated by computing the current in the circuit used for the steady-state illustration of Art. 7, Ch. IV. In the circuit of Fig. 4, p. 262 (omitting the capacitance), the switch is thrown to position 1 when the voltage is 45 degrees beyond its positive maximum, and the current in the circuit is -0.200 ampere at that instant.

Solution: If time is measured from the instant of switching, the expression for the applied voltage is

$$e = 155.2 \cos(\omega t + 45°), \qquad [9]$$

and the steady-state component of current is

$$i_s = 0.376 \cos(\omega t + 45° - 76.6°) = 0.376 \cos(\omega t - 31.6°). \qquad [4a]$$

When t is zero,

$$i_s(0) = 0.376 \cos(-31.6°) = 0.321 \text{ amp}. \qquad [5a]$$

Using Eq. 6a gives

$$-0.200 = 0.321 + i_t(0), \qquad [6c]$$

from which

$$A = i_t(0) = -0.521 \text{ amp}. \qquad [7a]$$

The coefficient R/L in the exponent of the exponential is

$$\frac{R}{L} = \frac{95.6}{0.128} = 747 \text{ sec}^{-1}. \qquad [10]$$

The current in the circuit after the switch operation is, therefore,

$$i = 0.376 \cos (\omega t - 31.6°) - 0.521\epsilon^{-717t} \text{ amp}. \quad [6d]$$

This calculation is readily shown graphically as in Fig. 1. The vector $\mathbf{E}_m\epsilon^{j\omega t}$ is plotted for the first instant as 155.2 v at an angle of 45°. The current vector $\mathbf{I}_m\epsilon^{j\omega t}$ is

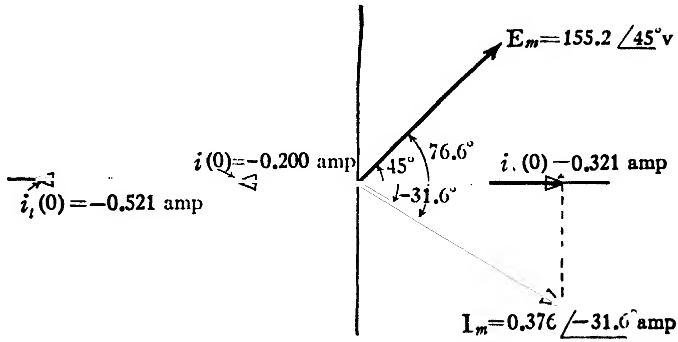


FIG. 1. Vector diagram of transient and steady state components at initial instant, for example of Art. 2.

76.6° behind the voltage vector, so that at the first instant it is 0.376 amp at an angle of −31.6°.

At a subsequent instant, such, for example, as one half a cycle later, the diagram of Fig. 1 will have changed to that shown in Fig. 2. The voltage and current vectors

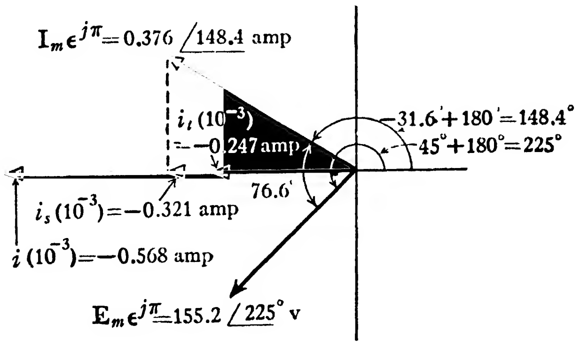


FIG. 2. Diagram of Fig 1 advanced ½ cycle.

have rotated an angle of π radians or 180°. A time of $\pi/(2\pi 500)$ or 0.001 sec has elapsed, and the transient component of current has decreased to

$$i_t(10^{-3}) = -0.521\epsilon^{-0.747} = -0.521 \times 0.474 = -0.247 \text{ amp}. \quad [2a]$$

The current in the circuit is

$$i(10^{-3}) = -0.321 - 0.247 = -0.568 \text{ amp}. \quad [6e]$$

Figure 3 shows a plot of Eq. 6d as a function of time.

In this example the transient component of current at the end of one cycle or 0.002 sec has decreased to $\epsilon^{-747 \times 0.002}$ or 0.224 of its initial value. At the end of two cycles it has decreased to 0.224 of this or 0.0500 of its initial value. Thus the transient has largely (95 per cent) disappeared by the end of two cycles.

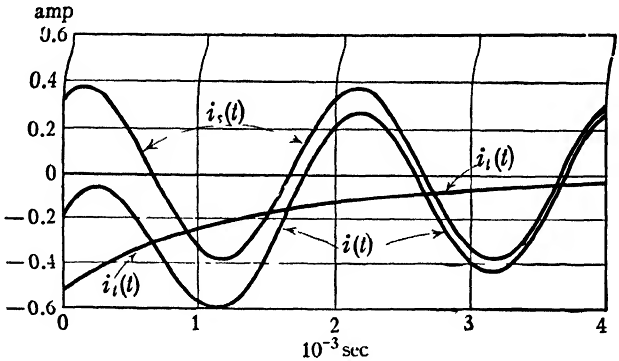


FIG. 3 Time plot of instantaneous components of current, for example of Art 2

3. FURTHER CONSIDERATION OF TRANSIENTS IN SERIES RL CIRCUITS

It is interesting to express the decay of the transient component per cycle in terms of the angle θ_z by which the steady-state current lags the source voltage. From Eq. 34, p. 264,

$$\frac{\omega L}{R} = \tan \theta_z \qquad [11]$$

or

$$\frac{R}{L}t = \frac{\omega t}{\tan \theta_z}. \qquad [11a]$$

At the end of one period or cycle ωt is 2π; hence

$$\epsilon^{-(R/L)t} = \epsilon^{-2\pi/(\tan \theta_z)} \qquad [12]$$

at the end of one period. From Eq. 12 it can be seen that if θ_z is small, $\tan \theta_z$ is small, and $\epsilon^{-2\pi/(\tan \theta_z)}$ is small.

In general, if the transient time constant L/R is large compared to one period of the source voltage, the transient persists over a number of cycles; whereas if L/R is small compared to one period, the transient disappears in a fraction of a cycle. This idea is frequently important practically. In communication systems, intelligence is often transmitted by varying the amplitude of an alternating voltage. Evidently, for faithful reproduction of the intelligence, the circuit should respond rapidly to changes in this voltage, that is, the transient accompanying the change

should disappear quickly. In this way the amplitude of the current can be made to follow closely the changing amplitude of the applied voltage.

Another aspect of the transient performance of the series RL circuit is important. The size of the transient is determined by the difference between the actual current and the steady-state component at the initial instant. If these are equal, there is no transient. The maximum transient is produced when the discrepancy between them is a maximum. In the example of Art. 2 the transient would be zero if the switch were thrown when

$$i(0) = i_s(0) = -0.200 \text{ amp}. \qquad [6f]$$

This situation would occur if at this instant the steady-state current vector made an angle

$$\cos^{-1}\left(-\frac{0.200}{0.376}\right) = 180° \pm 57.8° = \pm 122.2° \qquad [13]$$

with the axis of reals. The corresponding voltage vector angles are

$$\pm 122.2° + 76.6° = 198.8° \text{ or } -45.6°. \qquad [14]$$

Under the special condition of zero initial current, the initial value of the transient component is evidently equal in magnitude and opposite in sign to the initial steady-state component.

These considerations are often of great practical importance. For example, when a short circuit occurs on a power system, the currents are essentially the same as those just treated. Circuit breakers to interrupt these currents are very large and costly, so that it is of considerable importance to know just how large are the currents they may be called upon to interrupt. Also the forces set up in machines may be severe under short-circuit conditions. Predetermination of the size of the transient and of its duration is therefore an important problem for the designer.

4. Transient response of the series RS circuit with alternating-voltage source

The transient part of the charge solution is obtained by integrating the equation for the force-free state,

$$R\frac{dq_t}{dt} + Sq_t = 0. \qquad [15]$$

The solution is

$$q_t = A\epsilon^{-(S/R)t}, \qquad [16]$$

as shown in Art. 8, Ch. III. The steady-state component q_s is

$$q_s = \mathcal{R}_e[Q_m\epsilon^{j\omega t}] = Q_m \cos(\omega t + \phi_q). \qquad [17]$$

The complete solution is, therefore, given by

$$q = Q_m \cos(\omega t + \phi_q) + A\epsilon^{-(S/R)t}. \qquad [18]$$

Paralleling the discussion of current in the series RL circuit, the initial value of the transient charge must be such that, when added to the initial value of the steady-state charge, the sum is the known value of the actual condenser charge $q(0)$ at that instant. Thus for switching when time t is zero:

$$q(0) = Q_m \cos\phi_q + A; \qquad [19]$$

so

$$A = q(0) - Q_m \cos\phi_q. \qquad [19a]$$

Substituting this value for A in Eq. 18 gives

$$q = Q_m \cos(\omega t + \phi_q) + [q(0) - Q_m \cos\phi_q]\epsilon^{-(S/R)t}. \qquad ▶[20]$$

The engineer usually finds it more convenient to deal with current rather than charge, however; so the result is now expressed in terms of current, which is merely the time derivative of the charge.

Differentiating Eq. 20 with respect to time produces for the complete current solution

$$i = -\omega Q_m \sin(\omega t + \phi_q) - \frac{S}{R}[q(0) - Q_m \cos\phi_q]\epsilon^{-(S/R)t}. \qquad [21]$$

It is more convenient, however, to express the result in terms of the steady-state current I_m and the angle ϕ rather than in terms of Q_m and ϕ_q.

If the relations

$$Q_m = \frac{I_m}{\omega} \qquad [22]$$

and

$$\phi_q = \phi - \frac{\pi}{2} \qquad [23]$$

are used, the term $Q_m \cos\phi_q$ in Eq. 21 can be written

$$Q_m \cos\phi_q = \frac{I_m}{\omega}\cos\left(\phi - \frac{\pi}{2}\right) = \frac{I_m}{\omega}\sin\phi. \qquad [24]$$

Equation 21 can now be written in terms of the quantities I_m and ϕ by using a cosine form for the steady-state term and Eq. 24 in the transient term. Thus

$$i = I_m \cos(\omega t + \phi) - \frac{S}{R}\left[q(0) - \frac{I_m}{\omega}\sin\phi\right]\epsilon^{-(S/R)t}. \qquad ▶[21a]$$

It is sometimes convenient to state the solution in terms of initial current. The reasoning in this case with regard to initial conditions is the same physically as in the corresponding direct-current case; hence Art. 8 of Ch. III should be reviewed at this point. The essential fact in the argument is that whereas the condenser charge is a continuous function so that $q(0+)$ and $q(0-)$ are equal, the current is in general not continuous, so that $i(0+)$ does not equal $i(0-)$. Hence $i(0+)$, which is the significant value in determining the initial value of the transient current, must be evaluated in terms of $q(0-)$ or $Sq(0-)$.

In this way it is found that

$$\left.\begin{aligned} i(0+) &= \frac{e(0+) - Sq(0-)}{R} = i_s(0+) + i_t(0+) \\ &= I_m \cos\phi + i_t(0+); \end{aligned}\right\} \qquad [25]$$

whence

$$\left.\begin{aligned} i_t(0+) &= i(0+) - I_m\cos\phi = \frac{e(0+) - Sq(0-)}{R} - I_m\cos\phi \\ &= \frac{E_m\cos\psi - Sq(0-)}{R} - I_m\cos\phi, \end{aligned}\right\} \qquad [26]$$

and therefore

$$\left.\begin{aligned} i &= I_m\cos(\omega t + \phi) + [i(0+) - I_m\cos\phi]\epsilon^{-(S/R)t} \\ &= I_m\cos(\omega t + \phi) + \left[\frac{E_m\cos\psi - Sq(0-)}{R} - I_m\cos\phi\right]\epsilon^{-(S/R)t}. \end{aligned}\right\} \blacktriangleright[21b]$$

In words, the initial value $i(0+)$ of the current is that forced through the resistance by the difference between the instantaneous value $e(0+)$ of the voltage source at the initial instant and the voltage $Sq(0-)$ on the condenser resulting from its initial charge $q(0-)$.

From Eq. 21b the conditions under which no transient component appears can be determined. Evidently this requires that the coefficient of ϵ vanish or that

$$\frac{E_m\cos\psi - Sq(0-)}{R} = I_m\cos\phi. \qquad [27]$$

A simple method of determining whether or not there is a value of ψ which fulfills this condition is to plot the curves for these expressions, as in Fig. 4a. If the curves cross for any values of ψ, the voltage can be impressed when it has this phase angle, and the value of $i(0)$ equals $I_m \cos\phi$. Under these conditions it is possible to impress the voltage on the circuit without producing a transient current.

The curves of Fig. 4b show that $Sq(0)$ may be sufficiently large in value to prevent the two curves from touching at any point. Under such conditions a transient current always is present just after the instant of impressing the voltage.

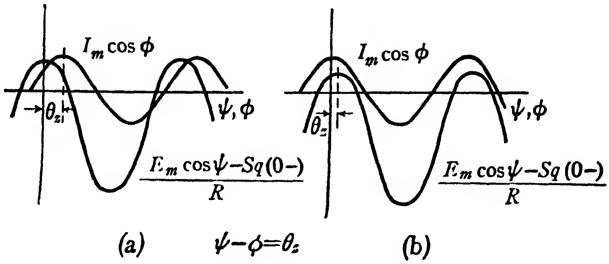


FIG. 4. Plots for determining conditions under which a transient can occur in a series RS circuit.

5. ILLUSTRATIVE EXAMPLE OF SERIES RS CIRCUIT

A capacitance of 9.00 microfarads in series with a resistance of 1,110 ohms is connected to a 60-cycles-per-second supply of 150 volts amplitude, at the instant when the supply voltage is +60 volts and decreasing. Previous to the switching, the condenser is charged to −200 volts. Figure 5 shows the circuit with the directions for which instantaneous quantities are considered positive.

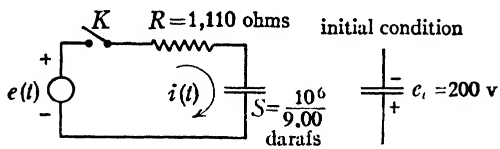


FIG. 5. Series RS circuit, for example of Art. 5.

Solution: For the steady state the vector current and voltage are calculated, based on t equal to zero for the switching instant. The voltage vector angle ψ is a positive angle (since e is decreasing with time at the instant of switching).

$$\psi = \cos^{-1}\frac{60}{150} = \cos^{-1} 0.400 = 66.4°; \tag{28}$$

whence

$$\mathbf{E}_m = 150\underline{\ 66.4°}. \tag{29}$$

The impedance is

$$\left.\begin{aligned} Z &= R - j\frac{S}{\omega} = R - j\frac{1}{\omega C} = 1{,}110 - j\frac{10^6}{9.00 \times 2\pi 60} \\ &= 1{,}110 - j295 \text{ ohms,} \end{aligned}\right\} \tag{30}$$

$$Z = \sqrt{1{,}110^2 + 295^2} = 1{,}148 \text{ ohms}, \qquad [30a]$$

$$\theta_z = \tan^{-1}\frac{-295}{1{,}110} = \tan^{-1}(-0.266) = -14.8°, \qquad [31]$$

$$I_m = \frac{150}{1{,}148} = 0.131 \text{ amp}, \qquad [32]$$

$$\phi = \psi - \theta_z = 66.4° - (-14.8°) = 81.2°. \qquad [33]$$

Calculating the charge Q_m gives

$$Q_m = \frac{I_m}{\omega} = \frac{0.131}{2\pi 60} = 3.46 \times 10^{-4} \text{ coulomb}, \qquad [22a]$$

$$\phi_q = \phi - \frac{\pi}{2} = 81.2° - 90.0° = -8.8°. \qquad [23a]$$

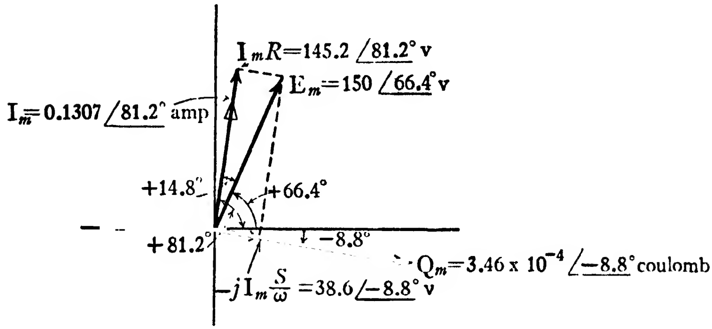


FIG. 6. Vector diagram of steady-state solution for example of Art. 5.

The component vector-voltage amplitudes across the resistance and capacitance are

$$\mathbf{I}_mR = 0.131\,\underline{/81.2°} \times 1{,}110 = 145\,\underline{/81.2°} \text{ v}, \qquad [34]$$

$$-j\mathbf{I}_m\frac{S}{\omega} = 0.131\,\underline{/81.2°} \times 295\,\underline{/-90°} = 38.6\,\underline{/-8.8°} \text{ v}. \qquad [35]$$

These quantities are shown in Fig. 6. This completes the steady-state solution.

The transient solution is obtained by means of physical reasoning which parallels the theory, rather than by mere substitution in the formulas.

The initial current $i(0+)$ for the instant immediately following the closure of the switch is obtained by calculating the net voltage applied to the resistance. From Fig. 5 it can be seen that the initial electromotive force and the capacitance voltage due to the initial charge add to give a voltage of 60 + 200, or 260 v, acting to produce an initial positive current $i(0+)$ of 260/1,110, or 0.234 amp. The initial steady-state component of current is obtained by noting that the vector $\mathbf{I}_m$ plotted in Fig. 6 is identical with the time vector $\mathbf{I}_m\epsilon^{j\omega t}$ at the initial instant; hence

$$i_s(0) = \mathcal{R}_e[\mathbf{I}_m] = 0.131 \cos 81.2° = 0.131 \times 0.153 = 0.0200 \text{ amp}. \qquad [36]$$

Hence

$$i_t(0+) = 0.234 - 0.020 = 0.214 \text{ amp}. \qquad [25a]$$

The coefficient of t in the exponent of the transient term is

$$-\frac{S}{R} = -\frac{1}{RC} = -\frac{10^6}{1{,}110 \times 9.00} = -100 \text{ sec}^{-1}. \qquad [37]$$

Therefore the actual current is

$$\left.\begin{aligned} i &= \mathcal{R}_e[0.131\underline{/81.2^\circ}\epsilon^{j\omega t}] + 0.214\epsilon^{-100t} \\ &= 0.131 \cos (377t + 81.2^\circ) + 0.214\epsilon^{-100t} \text{ amp.} \end{aligned}\right\} \qquad [21c]$$

Or, if desired, the solution for the charge can readily be obtained independently:

$$\left.\begin{aligned} q_s(0) &= \mathcal{R}_e[Q_m\epsilon^{j\omega 0}] = 3.46 \times 10^{-4} \cos(-8.8^\circ) \\ &= 3.46 \times 10^{-4} \times 0.988 = 3.42 \times 10^{-4} \text{ coulomb.} \end{aligned}\right\} \qquad [17a]$$

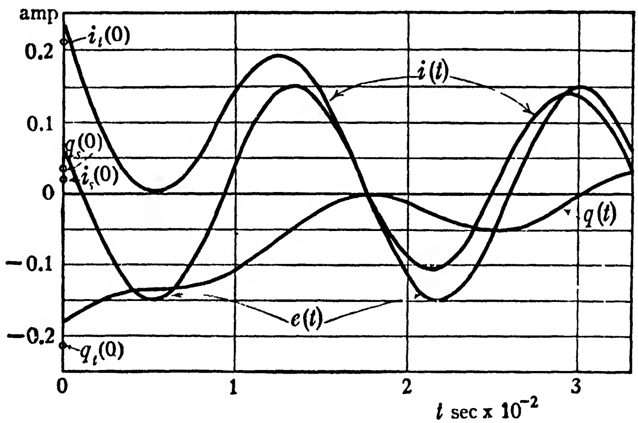


FIG. 7. Time plots of current and charge, for example of Art. 5.

The initial charge is

$$q(0) = -\frac{200}{S} = -200C = -\frac{200 \times 9.00}{10^6} = -18.0 \times 10^{-4} \text{ coulomb.} \qquad [38]$$

Since the charge cannot change instantaneously,

$$q(0) = q_s(0) + q_t(0) \qquad [39]$$

and

$$q_t(0) = -18.0 \times 10^{-4} - 3.4 \times 10^{-4} = -21.4 \times 10^{-4} \text{ coulomb.} \qquad [39a]$$

Therefore

$$\left.\begin{aligned} q &= \mathcal{R}_e[3.46 \times 10^{-4}\underline{/-8.8^\circ}\epsilon^{j\omega t}] - 21.4 \times 10^{-4}\epsilon^{-100t} \\ &= 3.46 \times 10^{-4} \cos (377t - 8.8^\circ) - 21.4 \times 10^{-4}\epsilon^{-100t} \text{ coulomb.} \end{aligned}\right\} \qquad [18a]$$

Equation 18a can be checked with Eq. 21c by differentiating q with respect to time.

Thus

$$
\begin{aligned}
i &= \frac{d}{dt}q = -\omega 3.46 \times 10^{-4} \sin(\omega t - 8.8^\circ) + 0.214\epsilon^{-100t} \\
&= -2\pi 60 \times 3.46 \times 10^{-4} \cos(\omega t - 8.8^\circ - 90^\circ) + 0.214\epsilon^{-100t} \\
&= 0.131 \cos(377t + 81.2^\circ) + 0.214\epsilon^{-100t} \text{ amp,}
\end{aligned}
\qquad [21d]
$$

which checks with Eq. 21c. Figure 7 shows a plot of e, q, and i from Eqs. 21c and 18a.

6. Relation of decay per cycle to phase angle, series RS circuit

It can be shown that a large value of $S/(\omega R)$ leads to a highly damped transient, because when $\omega R/S$ is small the time constant R/S is small compared to the period $2\pi/\omega$. Equation 36, p. 264, which gives

$$
\frac{-\dfrac{S}{R}}{\omega} = \tan\theta_z, \qquad [40]
$$

illustrates the situation more clearly. In this case

$$
\tan\theta_z \approx \tan\left(-\frac{\pi}{2}\right) = -\infty. \qquad [41]
$$

At the end of one period the ratio of the transient to its initial value is found by comparing

$$
i_t(0+) = \{[i(0+) - I_m \cos\phi]\epsilon^{-(S/R)t}\}_0 = i(0+) - I_m \cos\phi \qquad [26a]
$$

and

$$
\left.
\begin{aligned}
i_t\left(\frac{2\pi}{\omega}\right) &= [i(0+) - I_m \cos\phi]\epsilon^{-(2\pi)(S/\omega R)} \\
&= [i(0+) - I_m \cos\phi]\epsilon^{2\pi \tan\theta_z}.
\end{aligned}
\right\} \qquad [42]
$$

The ratio of Eq. 42 to Eq. 26a is $\epsilon^{2\pi \tan\theta_z}$. When θ_z is -60 degrees or $-\pi/3$ radians, for example, this is $\epsilon^{-2\pi\sqrt{3}}$ or 0.0000187; in other words, the transient current has practically disappeared at the end of one cycle. On the other hand, for a transient which is still appreciable at the end of the first period, that is, for one which has decreased by the factor ϵ^{-1}, or when the period $2\pi/\omega$ is equal to R/S, the time constant of transient, then

$$
\tan\theta_z = \frac{-\dfrac{S}{R}}{\omega} = \frac{-\dfrac{\omega}{2\pi}}{\omega} = -\frac{1}{2\pi} \qquad [40a]
$$

and θ_z is -9 degrees. Thus when the lead angle is not small, the transient duration is seen to be small.

7. Transient response of the series RLS circuit with alternating-voltage source

In the preceding cases emphasis is laid on the fact that the form of the transient or force-free component of the solution depends only upon the circuit parameters and that the amplitude of the transient is such that *the initial conditions are satisfied.* The entire discussion of the *RLS* transient given in Arts. 18 to 22, inclusive, of Ch. III is applicable without change to the present case if one merely replaces the steady-state charge and current by the alternating steady-state functions $q_s(t)$ and $i_s(t)$. These substitutions are indicated in more detail in the three cases which follow.

Case 1: *Overdamped circuit*

The transient components have the forms

$$q_t(t) = A_1\epsilon^{(-\alpha+\beta)t} + A_2\epsilon^{(-\alpha-\beta)t}, \tag{43}$$

$$i_t(t) = (-\alpha+\beta)A_1\epsilon^{(-\alpha+\beta)t} + (-\alpha-\beta)A_2\epsilon^{(-\alpha-\beta)t}, \tag{44}$$

in which A_1 and A_2 are determined by the fact that both charge and current remain unchanged during the switching operation and that therefore the sum of the steady-state and transient components of each immediately after switching must equal, respectively, the known values of each at the instant immediately prior to switching; that is,

$$q(0-) = q_s(0+) + q_t(0+), \tag{39b}$$

$$i(0-) = i_s(0+) + i_t(0+), \tag{6g}$$

or, using Eqs. 43 and 44,

$$q_t(0+) = A_1 + A_2 = q(0-) - q_s(0+), \tag{43a}$$

$$i_t(0+) = (-\alpha+\beta)A_1 - (\alpha+\beta)A_2 = i(0-) - i_s(0+), \tag{44a}$$

in which $q(0-)$ and $i(0-)$ are the actual charge and current immediately prior to switching.

In Eqs. 43a and 44a $q_s(0+)$ and $i_s(0+)$ are determined from any convenient form given in Art. 3, Ch. IV; $q(0-)$ and $i(0-)$ are known initial conditions, and α and β are fixed by the circuit parameters. The coefficients A_1 and A_2, the same as used in the corresponding solution, Art. 19, Ch. III, can therefore be determined by solving Eqs. 43a and 44a simultaneously.

The complete solutions are given by the sums of steady-state and transient components, or

$$q = q_s + A_1\epsilon^{(-\alpha+\beta)t} + A_2\epsilon^{(-\alpha-\beta)t}, \tag*{▶[45]}$$

$$i = i_s + (-\alpha+\beta)A_1\epsilon^{(-\alpha+\beta)t} + (-\alpha-\beta)A_2\epsilon^{(-\alpha-\beta)t}. \tag*{▶[46]}$$

Case 2: Underdamped, or oscillatory, circuit

The initial transient components are derived from Eqs. 39b and 6g, which, being general, apply to this case thus:

$$q_t(0+) = B_1 = q(0-) - q_s(0+), \tag{47}$$

$$i_t(0+) = -\alpha B_1 - \omega_d B_2 = i(0-) - i_s(0+). \tag{48}$$

Equations 47 and 48 are readily solved for B_1 and B_2. The complete solutions can be written in terms of known constants as

$$q = q_s + \epsilon^{-\alpha t}\mathcal{R}_e[\mathrm{B}\epsilon^{j\omega_d t}], \qquad \blacktriangleright[49]$$

$$i = i_s + \epsilon^{-\alpha t}\mathcal{R}_e[(-\alpha + j\omega_d)\mathrm{B}\epsilon^{j\omega_d t}]. \qquad \blacktriangleright[50]$$

The complex constant

$$\mathrm{B} = B_1 + jB_2 \tag{51}$$

is the same as used in the corresponding solution, Art. 20, Ch. III.

Case 3: Critically damped circuit

The initial transient components are obtained from Eqs. 39b and 6g thus:

$$q_t(0+) = a_1 = q(0-) - q_s(0+), \tag{52}$$

$$i_t(0+) = a_2 - \alpha a_1 = i(0-) - i_s(0+), \tag{53}$$

from which a_1 and a_2, the same coefficients as used in the corresponding solution of Art. 22, Ch. III, are easily obtained. The complete solutions for charge and current are

$$q = q_s + (a_1 + a_2 t)\epsilon^{-\alpha t}, \qquad \blacktriangleright[54]$$

$$i = i_s + (a_2 - \alpha a_1 - \alpha a_2 t)\epsilon^{-\alpha t}. \qquad \blacktriangleright[55]$$

It is emphasized again that the entire discussion regarding the transient components given in Arts. 18 to 22 of Ch. III is applicable to the transient components in this article. The difference in the steady-state components is the only dissimilarity between the direct- and alternating-current cases.

8. Illustrative example of series *RLS* circuit

A numerical example serves to illustrate the application of this analysis to a particular case. Figure 8 shows a circuit to which a 400-cycles-per-second source is connected at the instant when the electromotive force is zero and has a negative derivative. The condenser is initially charged to +400 volts, but the initial current is zero. The questions are:

(a) What are the steady-state charge and current?

(b) What are the transient components of charge and current?

Solution. Steady state.

$$\left.\begin{aligned} Z &= R + j\left(\omega L - \frac{S}{\omega}\right) = 15.6 + j\left(2\pi 400 \times 0.165 - \frac{6.85 \times 10^5}{2\pi 400}\right) \\ &= 15.6 + j(414 - 273) = 15.6 + j141 \text{ ohms.} \end{aligned}\right\} \qquad [56]$$

$$Z = 142 \text{ ohms,} \qquad [56a]$$

$$\theta_z = \tan^{-1}\frac{141}{15.6} = \tan^{-1} 9.04 = 83.7°, \qquad [57]$$

$$I_m = \frac{141}{142} = 0.993 \text{ amp,} \qquad [58]$$

$$Q_m = \frac{I_m}{\omega} = \frac{0.993}{2\pi 400} = 3.96 \times 10^{-4} \text{ coulomb.} \qquad [59]$$

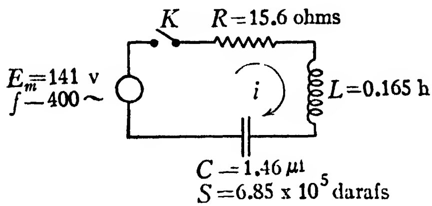


FIG. 8. Series RLS circuit, for example of Art. 8.

If the vectors are chosen to correspond to the switching instant, that is, the instant when the electromotive force is zero and going negative - a useful choice for the transient part of the problem then

$$\mathbf{E}_m = 141\underline{/90°} \text{ v,} \qquad [60]$$

$$\mathbf{I}_m = 0.993\underline{/90° - 83.7°} = 0.993\underline{/6.3°} \text{ amp,} \qquad [58a]$$

$$\mathbf{Q}_m = 3.96 \times 10^{-4}\underline{/6.3° - 90°} = 3.96 \times 10^{-4}\underline{/-83.7°} \text{ coulomb.} \qquad [59a]$$

The vector-voltage drops across the resistance, inductance, and capacitance are, respectively,

$$R\mathbf{I}_m = 15.6 \times 0.993\underline{/6.3°} = 15.5\underline{/6.3°} \text{ v,} \qquad [61]$$

$$j\omega L\mathbf{I}_m = j414 \times 0.993\underline{/6.3°} = 411\underline{/96.3°} \text{ v,} \qquad [62]$$

$$-j\frac{S}{\omega}\mathbf{I}_m = -j273 \times 0.993\underline{/6.3°} = 271\underline{/-83.7°} \text{ v.} \qquad [63]$$

All these vector quantities are plotted in Fig. 9.

Solution: Transient

The first calculation is a determination of the nature of the transient response, whether it is overdamped, critically damped, or oscillatory. For this determination ω_0^2

is compared with α^2:

$$\omega_0^2 = \frac{S}{L} = \frac{6.85 \times 10^5}{0.165} = 4.15 \times 10^6, \tag{64}$$

$$\alpha^2 = \left(\frac{R}{2L}\right)^2 = \left(\frac{15.6}{2 \times 0.165}\right)^2 = 47.2^2 = 2{,}230. \tag{65}$$

Evidently ω_0^2 is very much greater than α^2, and within one part in several thousand

$$\omega_d \approx \omega_0 = \sqrt{4.15 \times 10^6} = 2{,}040 \text{ radians per second.} \tag{66}$$

The transient charge and current are then

$$q_t = \mathcal{R}_e[\mathrm{B}\epsilon^{(-47.2+j2{,}040)t}] \tag{67}$$

and

$$i_t = \mathcal{R}_e[(-47.2 + j2{,}040)\mathrm{B}\epsilon^{(-47.2+j2{,}040)t}]. \tag{68}$$

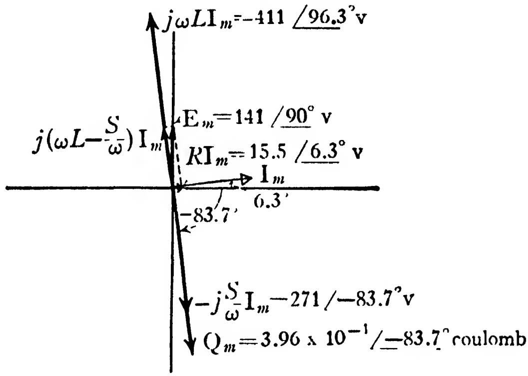


FIG. 9. Vector diagram of steady state solution, for example of Art. 8

To evaluate the complex constant B, one must know the initial values of the actual charge and current. From the statement of the problem,

$$i(0) = 0, \tag{69}$$

$$q(0) = \frac{+400}{S} = 400C = 400 \times 1.46 \times 10^{-6} = 5.84 \times 10^{-4} \text{ coulomb.} \tag{70}$$

The values of $q_s(0)$ and $i_s(0)$ are

$$\left.\begin{aligned} q_s(0) &= \mathcal{R}_e[\mathrm{Q}_m] = Q_m \cos\phi_q \\ &= 3.96 \times 10^{-4} \cos(-83.7^\circ) = 3.96 \times 10^{-4} \times 0.1097 \\ &= 4.34 \times 10^{-5} \text{ coulomb;} \end{aligned}\right\} \tag{17b}$$

$$\left.\begin{aligned} i_s(0) &= \mathcal{R}_e[\mathrm{I}_m] = I_m \cos\phi \\ &= 0.993 \cos 6.3^\circ = 0.993 \times 0.994 \\ &= 0.986 \text{ amp.} \end{aligned}\right\} \tag{5b}$$

$$\mathcal{R}_e[\mathrm{B}] = B_1 = q(0) - q_s(0) = (5.84 - 0.43)10^{-4} = 5.41 \times 10^{-4} \tag{71}$$

and

$$\left.\begin{aligned} \mathcal{I}[\mathbf{B}] = B_2 &= \frac{1}{\omega_d}[-i(0) + i_s(0) - \alpha B_1] \\ &= \frac{1}{2{,}040}[0 + 0.986 - 47.2 \times 5.41 \times 10^{-4}] \\ &= \frac{0.986 - 0.026}{2{,}040} = 4.70 \times 10^{-4}. \end{aligned}\right\} \qquad [72]$$

Therefore

$$\left.\begin{aligned} \mathbf{B} &= B_1 + jB_2 = (5.41 + j4.70)10^{-4} \\ &= \sqrt{5.41^2 + 4.70^2} \times 10^{-4} \,\Big/ \tan^{-1}\left(\frac{4.70}{5.41}\right) \\ &= 7.16 \times 10^{-4} \underline{/40.9^\circ}, \end{aligned}\right\} \qquad [73]$$

$$\left.\begin{aligned} (-\alpha + j\omega_d)\mathbf{B} &= (-47.2 + j2{,}040)7.16 \times 10^{-4} \underline{/40.9^\circ} \\ &= 2{,}040 \underline{/91.3^\circ} \times 7.16 \times 10^{-4} \underline{/40.9^\circ} \\ &= 1.460 \underline{/132.2^\circ}. \end{aligned}\right\} \qquad [74]$$

With the numerical values calculated, Eqs. 49 and 50 become

$$\left.\begin{aligned} q = \mathcal{R}_e[&3.96 \times 10^{-4} \underline{/-83.7^\circ} \epsilon^{j2\pi 400t} \\ &+ 7.16 \times 10^{-4} \underline{/40.9^\circ} \epsilon^{(-47.2 + j2{,}040)t}] \text{ coulomb}, \end{aligned}\right\} \qquad [49a]$$

$$i = \mathcal{R}_e[0.993 \underline{/6.3^\circ} \epsilon^{j2\pi 400t} + 1.460 \underline{/132.2^\circ} \epsilon^{(-47.2 + j2{,}040)t}] \text{ amp}. \qquad [50a]$$

As a check on the arithmetical work, these are evaluated for the situation when t is 0. Thus

$$\left.\begin{aligned} q(0) &= 3.96 \times 10^{-4} \cos(-83.7^\circ) + 7.16 \times 10^{-4} \cos(40.9^\circ) \\ &= (0.436 + 5.42)10^{-4} \\ &= 5.86 \times 10^{-4} \text{ coulomb}; \end{aligned}\right\} \qquad [49b]$$

$$\left.\begin{aligned} i(0) &= 0.993 \cos 6.3^\circ + 1.460 \cos 132.2^\circ \\ &= 0.987 - 0.983 \\ &= 0.004 \text{ amp}. \end{aligned}\right\} \qquad [50b]$$

These check within reasonable ten-inch slide-rule accuracy the actual initial values of 5.84×10^{-4} coulomb and 0.000 amp.

9. Special cases of general oscillatory solution

Because of the variety of interesting results that can appear in a highly oscillatory series *RLS* circuit when an alternating voltage is suddenly applied, the general solution is evaluated in more detail for some of these cases. The particular initial conditions and the relations between

circuit parameters considered are shown by the following summary:

(1) $\omega = \omega_0;\ \omega L = \dfrac{S}{\omega};\ Z = R;\ \theta_z = 0$		
(2) $\omega \gg \omega_0;\ \omega L \gg \dfrac{S}{\omega} \gg R;\ Z \approx j\omega L;\ \theta_z \approx \dfrac{\pi}{2}$	(a) $\psi = 0$	$\omega_d \approx \omega_0$
(3) $\omega \ll \omega_0;\ R \ll \omega L \ll \dfrac{S}{\omega}$;	(b) $\psi = -\dfrac{\pi}{2}$	$q(0) = 0$ $i(0) = 0$
$Z \approx -j\dfrac{S}{\omega};\ \theta_z \approx -\dfrac{\pi}{2}$		

Case (1*a*)

This is the series-resonance case the steady-state response of which is discussed in Art. 25, Ch. IV. The solutions for q and i by Eqs. 20 and 21, p. 261 and Eqs. 258 and 259, p. 228, are:

$$q \approx \mathcal{R}_e[\mathrm{Q}_m\epsilon^{j\omega t} - \mathrm{Q}_m\epsilon^{(-\alpha+j\omega)t}], \tag{75}$$

$$i \approx \mathcal{R}_e[\mathrm{I}_m\epsilon^{j\omega t} - \mathrm{I}_m\epsilon^{(-\alpha+j\omega)t}]. \tag{76}$$

By the conditions for this case, ϕ is zero, so

$$\mathrm{I}_m = \frac{E_m}{R}; \tag{77}$$

and, since

$$\mathrm{Q}_m = -j\frac{\mathrm{I}_m}{\omega}, \tag{78}$$

$$q \approx \frac{E_m}{\omega R}(1 - \epsilon^{-\alpha t})\sin\omega t, \tag{75a}$$

$$i \approx \frac{E_m}{R}(1 - \epsilon^{-\alpha t})\cos\omega t. \tag{76a}$$

The current, plotted in Fig. 10, consists of oscillations having the same frequency as the source voltages which are inclosed by an exponential envelope that gradually allows the amplitudes to grow from zero to the maximum value E_m/R. The initial rate of growth of the amplitude is given by the tangent to the envelope at the origin and is equal to $(di/dt)_0$ or $E_m/(2L)$. If the growth continued at this rate, the amplitude would be fully built up in time $1/\alpha$, the time constant of the circuit. Actually the oscillations have reached only 0.632 of their final value at this time.

Case (2*a*)

Here the impressed frequency is very large compared to the characteristic frequency. Consequently, the reactance is inductive and large com-

pared to the resistance R. Hence the steady-state current lags the voltage by nearly 90 degrees. The solutions are:

$$q \approx -\frac{E_m}{\omega^2 L}(\cos \omega t - \epsilon^{-\alpha t} \cos \omega_0 t), \tag{75b}$$

$$i \approx \frac{E_m}{\omega L}\left(\sin \omega t - \frac{\omega_0}{\omega}\epsilon^{-\alpha t} \sin \omega_0 t\right). \tag{76b}$$

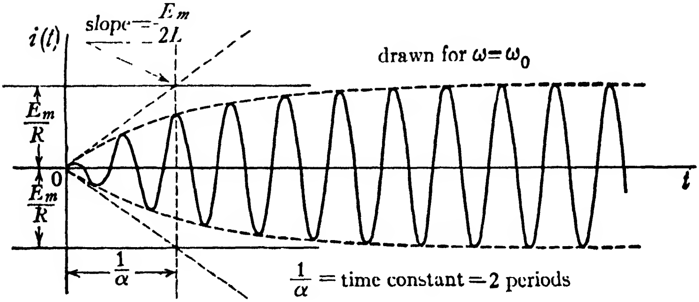


FIG. 10. Time plot of current for Case (1a) of Art. 9.

The transient component, since ω_0/ω is relatively small, is entirely negligible in the current but not negligible in the charge. The charge is plotted in Fig. 11.

The slowly varying transient component acts as an axis about which

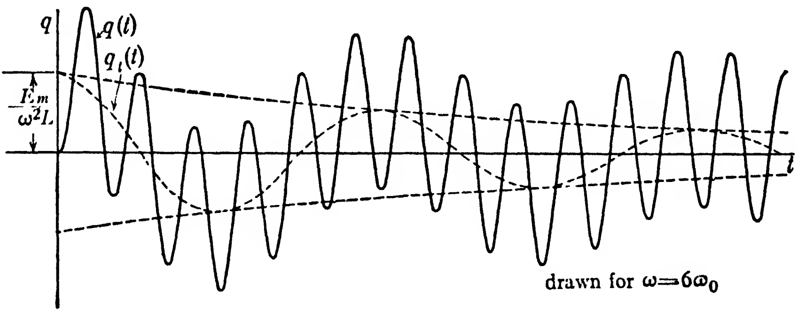


FIG. 11. Time plot of charge for Case (2a) of Art. 9.

the steady-state component charge oscillates. During the first few cycles the oscillations reach almost twice the r normal maximum value. For the current, the circuit acts substantially as though it consisted of inductance only.

Case (3a)

Here the resultant reactance is capacitive and large compared to the resistance. The steady-state current leads the voltage by approximately 90 degrees. Hence the solutions become:

$$q \approx \frac{E_m}{S}(\cos \omega t - \epsilon^{-\alpha t} \cos \omega_0 t), \quad [75c]$$

$$i \approx -\frac{E_m \omega}{S}\left(\sin \omega t - \frac{\omega_0}{\omega} \epsilon^{-\alpha t} \sin \omega_0 t\right). \quad [76c]$$

The charge solution has the same general character as for the preceding case. The current solution, however, is quite different. Since ω is very small compared to ω_0, the transient-component oscillations in the current are now large compared to those of the steady state. The general character of the resulting current is illustrated in Fig. 12. The decaying transient is of relatively high frequency and oscillates about the slowly varying steady-state current as an axis. The net result looks chaotic until the steady state gradually emerges and becomes recognizable. The ratio of the initial transient-component amplitude to that of the steady state is given by ω_0/ω.

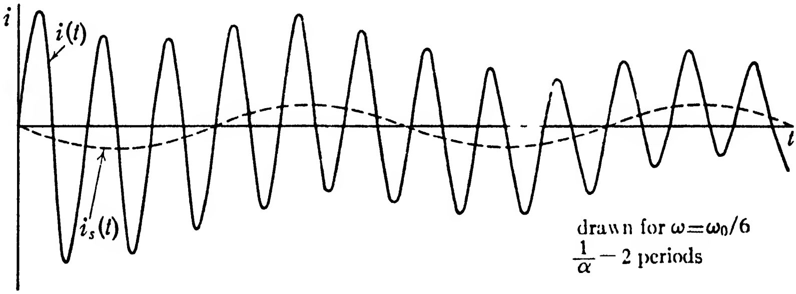


FIG. 12. Time plot of current for Case (3a) of Art. 9

Case (1b)

This is again the resonance case but for a switching instant chosen at the time the voltage passes through zero with positive slope. The results differ from that for Case 1a only by being 90 degrees behind them.

Case (2b)

The solutions are:

$$q \approx -\frac{E_m}{\omega^2 L}\left(\sin \omega t - \frac{\omega}{\omega_0} \epsilon^{-\alpha t} \sin \omega_0 t\right), \quad [75d]$$

$$i \approx -\frac{E_m}{\omega L}(\cos \omega t - \epsilon^{-\alpha t} \cos \omega_0 t). \quad [76d]$$

Here the transient component of charge is large compared to the steady-state charge whereas the transient component of current is not of exceptional magnitude. Figure 13 shows the resulting charge and current plots.

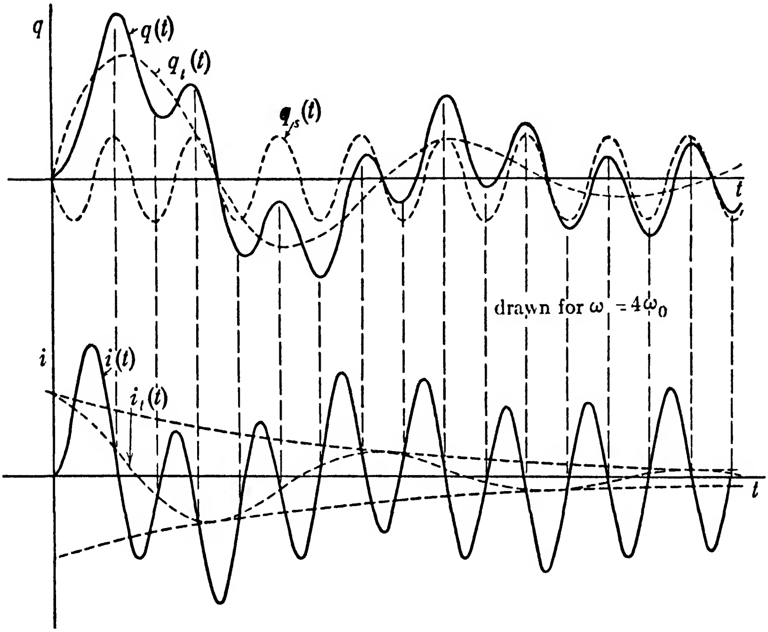


FIG. 13. Time plots of charge and current for Case (2b) of Art. 9.

Case (3b)

Here the solutions are:

$$q \approx \frac{E_m}{S}\left(\sin \omega t - \frac{\omega}{\omega_0}\,\epsilon^{-\alpha t}\sin \omega_0 t\right), \tag{75e}$$

$$i \approx \frac{E_m\omega}{S}\left(\cos \omega t - \epsilon^{-\alpha t}\cos \omega_0 t\right). \tag{76e}$$

The transient in the charge is here negligible. This case is similar to that for the series *RS* circuit for the same voltage phase at the switching instant. The transient component of current has the same initial amplitude as the steady-state current (but starts in opposite phase) and oscillates about the latter as an axis, as is shown in Fig. 14.

10. Transient response of the parallel $GC\Gamma$ circuit with alternating-current source

The transient response of this circuit and of the parallel GC or other duals of the series circuits whose transient response has been discussed in preceding articles can be obtained readily by analogy and so are omitted here in order to avoid repetition of details.

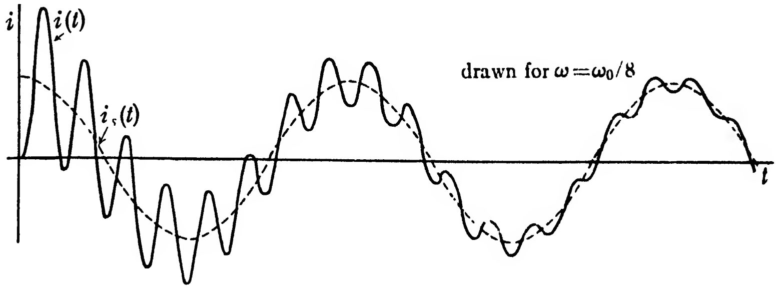


Fig. 14. Time plot of current for Case (3b) of Art. 9

11. The exponential function and its relation to linear physical systems

The response of simple linear circuits has by now been examined in sufficient detail to make worth while a few general observations by way of obtaining perspective. A survey of the solutions obtained thus far shows that the exponential function is adequate to describe all the time functions encountered, except for the constants expressing steady-state currents and charges arising from constant impressed forces. Some rather fundamental relation thus apparently exists between the exponential function and the behavior of linear circuits. Furthermore, this relation applies not only to linear electric circuits but to many and diverse other phenomena in nature as well.

The exponential function appears in two principal forms, the first with a real argument and the second with an imaginary argument. It also appears with the combination of the two, that is, with a complex argument. In each of these cases this function has been found to be the solution of a linear differential equation.

In the first form — with a real argument (and in the cases discussed, a negative argument) — the exponential function serves to describe a quantity that gradually decreases with time, and decreases at a rate which at every instant is proportional to the quantity. The series RL and RS transients are circuit examples. In other scientific fields the changes of concentrations, the rates of chemical reactions, the motions of inertia-

friction or elastance-friction systems, the variation of pressure or density with a distance, the rates of propagation of living organisms, and many other phenomena follow the same law or variation: the exponential law with a real exponent.

The second form — the exponential with an imaginary argument — is essentially periodic. Its close relation to sinusoidal or simple harmonic motion is evident. In fact, the cosine or sine function is merely the sum or difference of two conjugate exponentials with imaginary arguments. It is convenient mathematically to replace a sine or cosine by the real or imaginary part of a single exponential with an imaginary argument. In general, these two functions, the sinusoidal and the imaginary exponential, are interchangeable, the choice for a particular problem being one of mathematical convenience. This periodic function also describes many phenomena besides those occurring in electric circuits. Thus the motion of a well-pivoted pendulum executing small amplitude oscillations is practically sinusoidal. The vibrations of machines in all their great variety are resolvable into groups of sinusoids. Likewise, electromagnetic waves, the vibrations of strings, and the motions accompanying propagated waves in gases, in liquids, and in solids — each can be resolved into a series of sine functions of different frequencies, phases, and amplitudes. So naturally and efficiently do the usual physical systems respond to sinusoidal functions that great care is taken in the design of sources of electrical periodic forces, such as alternators and vacuum-tube oscillators, to make their output approximate very closely the sinusoidal or simple harmonic form.

The exponential function with a combined real and imaginary argument, or a complex argument, is characteristic of nearly all the physical systems mentioned in the preceding paragraph. In fact, this function describes in general the way in which such systems behave when disturbed and left to themselves. Nearly all, if not all, physical systems that do not contain sources of energy absorb and reconvert into other forms any energy of motion which they may possess. The periodic motions which they may have because of some momentary disturbing force therefore gradually die out as the energy of motion is gradually converted into another form, usually heat. This periodic motion of gradually decreasing magnitude is what the exponential of complex argument characterizes. Only when energy is supplied by some external source at a rate equal to that at which the energy is being dissipated by the vibrating system under consideration are the oscillations sustained or of constant amplitude.

From the foregoing very brief discussion the family of exponential functions and the linear differential equations from which they arise can be seen to have a far greater importance than merely that due to their

association with simple linear electrical circuits. For the electrical engineer the immediate and sufficient justification for a thorough understanding of the exponential functions is their importance in electric circuit analysis. He is very likely to find, however, that by merely redefining the symbols in terms of other physical quantities, he can apply this understanding to a wide variety of problems encountered in engineering and science.

PROBLEMS

1. A voltage $\mathcal{R}_e[100\epsilon^{j(377t+\pi/2)}]$ v is impressed on a series circuit which has an inductance of 0.0106 h and a resistance of 2.00 ohms. After steady conditions have been reached, the resistance of the circuit is doubled at an instant when the applied potential is passing through 0.

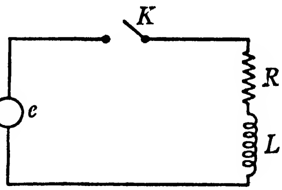
What is the magnitude of the current in the circuit

(a) at an infinitesimally short time before the resistance is increased?
(b) at an infinitesimally short time after the resistance is increased?
(c) one-half cycle after the resistance has been increased?

2. A sinusoidal voltage source having a frequency of 60 ~ is impressed on a coil that has a resistance of 10 ohms and an inductance of (a) 10, (b) 100, (c) 500 mh. What portion of the initial value of the transient component of current remains after one cycle?

3. The following questions refer to Fig. 15.

(a) What is the equation for the steady-state current at any time t after the voltage is applied?
(b) What is the initial value of the transient component of current if the voltage is 0 when the switch is closed? The transient and steady-state components of current for this condition are to be sketched.
(c) At what point on the voltage wave should the switch be closed to make the transient a maximum? What is the magnitude of the instantaneous voltage at this instant?



E_m = 9,000 v
R = 0.0030 ohm
L = 0.0025 h
f = 60 ~

FIG. 15 Series RL circuit for Prob. 3.

4. A sinusoidal voltage with a frequency of 60 ~ and an amplitude of 100 v is impressed on the series combination of a condenser with a capacitance of 1 μf and negligible resistance, and a noninductive resistor of resistance R The voltage is impressed at such a time that, when the switch is closed, the steady-state component of the current is going through its instantaneous positive maximum. The condenser is initially uncharged.

What are the transient and steady-state currents as functions of time (a) when R is 100 ohms? (b) when R is 10,000 ohms?

5. A condenser having a capacitance of 50 μf and a noninductive resistor having a resistance of 20 ohms are connected in series A sinusoidal voltage source having a frequency of 60 ~ and an amplitude of $230\sqrt{2}$ v is impressed across the combination at (a) the time the voltage is passing through 0 and increasing, (b) $\frac{1}{12}$, (c) $\frac{1}{6}$, (d) $\frac{1}{4}$, (e) $\frac{1}{3}$, (f) $\frac{5}{12}$ of a cycle after the voltage is passing through 0 and increasing. What are the initial currents and the initial voltages across each part of the circuit? What are the equations for charges and currents? Both the initial charges and currents are 0.

6. A sinusoidal voltage source having a frequency of 60 ~ is impressed on a 50-μf condenser in series with a noninductive resistor having a resistance of (a) 1.0, (b) 10, (c) 100, (d) 1,000 ohms. At what time in the cycle must the voltage be applied in order to have no transient?

7. A coil which has a resistance of 10 ohms and an inductance of 0.1 h is connected in parallel with a series combination of condenser and resistor. What should be the capacitance and resistance so that the total current supplied is *always* proportional to the instantaneous value of the applied voltage from the instant the voltage is impressed on the circuit?

8. A voltage $\mathcal{R}_e[100\epsilon^{+j(377t+\psi)}]$ v is applied to a coil and condenser in series. Owing to losses in the condenser, the condenser is equivalent to a capacitance of 50.0 μf in series with a resistance of 1.00 ohm. The coil is equivalent to a series circuit with an inductance of 0.0061 h and a resistance of 9.0 ohms.

After steady conditions have been established, the coil is short-circuited at an instant when the instantaneous applied potential has a maximum positive value.

(a) What is the amplitude of the steady-state current before the coil is short-circuited?
(b) What is the equation of the steady-state current in the condenser after the coil is short-circuited?
(c) What is the magnitude of the transient component of current an infinitesimally short time after the coil is short-circuited?

9. An electromotive force $230\sqrt{2}\sin(377t + \psi')$ v is impressed on a condenser in series with a noninductive resistor and a coil, when t is 0. At 60 ~, the reactance of the condenser is -100 ohms, the resistance of the resistor is 100 ohms, the resistance of the coil is 500 ohms, and the reactance of the coil is 500 ohms. Both the initial current and charge are 0. The phase angle ψ' is (a) 0°, (b) 30°, (c) 60°, (d) 90°, (e) 120°, (f) 150°. What are the equations of the current?

After the steady state is reached, the coil is short-circuited at an instant when the electromotive force is 0 and increasing. What are the equations of the currents in the coil and in the condenser from the moment of short circuit?

10. Is it possible to impress a sinusoidal voltage on a series RLS circuit at such a point in the cycle that no transient current results (a) for the condenser uncharged? (b) for the condenser precharged?

CHAPTER VI

Steady-State Analysis of Alternating-Current Circuits Involving Two Unknowns

1. Introduction

The principles developed in Ch. IV for application to circuits consisting of simple series and parallel combinations of single elements can be extended to the general case of far more complicated circuits. The multiplicity of terms and the notation thus involved, however, appear rather formidable unless one has learned to visualize their significance by practice with situations having an intermediate degree of complexity. It is appropriate, therefore, to develop first the theory for circuits which can be most readily analyzed by the simultaneous solution of two independent equations. Some of the circuits to be treated in this way can, of course, by the use of proper combinations of elements, be studied by the single-loop or single-node methods presented in Ch. IV. A brief experience with the procedure involved will, however, enable one to decide at the outset whether the solution of two simultaneous equations appears to be the most favorable method of attack. The circuits discussed in this chapter are of two general classes: two-loop circuits and two-node circuits. As with the relatively simple circuits discussed in Ch. IV, the two-loop and two-node circuits are composed of the three fundamental elements R, L, and S, but here any combination of the three may appear in series or in parallel, to form a composite branch; while many such branches may be connected in series, in parallel, or in other ways to give a network having two loops or two nodes. In this chapter, as in all preceding chapters, the discussion is limited to circuits containing only linear elements; so that the relations among currents, voltages, their derivatives, and their integrals are linear, the constants of proportionality being the constant parameters R, L, and S or their reciprocals.

The equations of equilibrium for an electrical network are best formulated through Kirchhoff's laws for, while these are not the only basis upon which the network equilibrium can be stated, the directness and simplicity of their form make them a very satisfactory one. The circuit laws of Kirchhoff are two: one expressing voltage conditions around a loop, the other expressing current conditions at a node. Similarly, there are two corresponding methods of writing the network equations: one in terms of branch or loop voltages, and the other in terms of currents approaching and leaving nodes in the network. Either method can be

applied to any network, but one usually results in simpler equations, solutions than the other, the choice in any particular case depending on the form of the network and the nature of the known data Both method are developed in detail in this chapter.*

2. Differential equations of the two-loop network

The circuit of Fig. 1 contains all the features encountered in the two-loop case. This circuit consists of two loops *abcd* and *abfg* with a source of voltage in each, and a common branch *ab*. There is magnetic coupling between the two loops by the mutual inductance M in addition to the coupling of the common branch *ab*. Each loop and the common branch

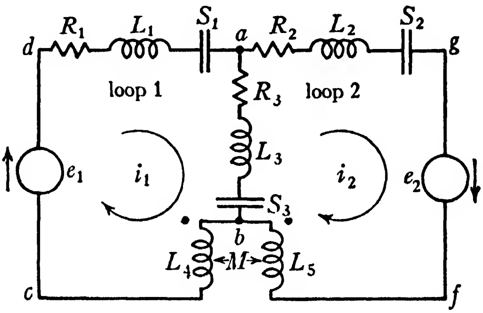


FIG. 1. Circuit diagram containing all features encountered in a two-loop network.

contain independently all three parameters R, L, and S; hence the network is general for the two-loop case.

The direction of voltage induced in one coil of a mutual inductor by a rate of change of current in the other coil must be designated, so that the proper signs may be given to such voltages in the loops of a network. One convenient system of designating the relative polarities of the coils — that used in practice for marking instrument transformers — places a dot adjacent to the terminals of corresponding polarity of the coils having mutual inductance. Current entering either coil through the dot-marked terminal produces magnetic flux in the same direction as that produced by current entering the other coil through its dot-marked terminal. In Fig. 2a, for example, the current I_A entering the dot-marked terminal of coil A produces an upward-directed flux in the core. The current I_B entering the dot-marked terminal of coil B produces flux in the same direction. If one of the coils were wound in the opposite direction, it would be necessary to place the marking dot at the terminal opposite

* The treatment for direct currents, Art. 3, Ch. II, should be reviewed.

the one shown. The relative polarity of the coils, as indicated by the dots, has great importance in the solution of network problems. A positive rate of change of the current I_A produces at the dot-marked terminal of coil B a potential higher than that of the unmarked terminal. This result is evident from the fundamental relation that coil B tries to establish a current which opposes the change of flux linking it. Such a current would be directed outward from the dot-marked terminal and could result only from a potential rise from the unmarked terminal to the marked terminal of coil B. The marks at the two coil terminals thus indicate that a positive rate of change of current entering either dot-marked terminal produces a potential rise from the other coil's unmarked terminal to its dot-marked terminal, or a potential drop from its marked terminal to its unmarked terminal.

In Fig. 2b is shown an experimental method for determining the relative polarities of two coils with mutual inductance and with windings that cannot be inspected A direct voltage source is connected so that it energizes the coil A of the mutual inductor w' e: switch K is closed. As

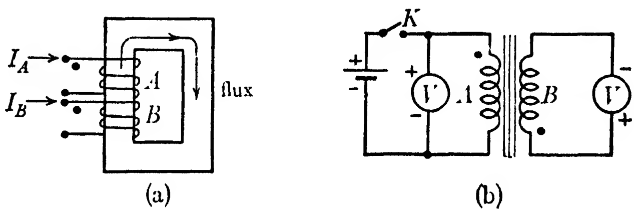


FIG. 2. Polarity designation of coils having mutual inductance

a basis for the marking of one coil, a direct-current voltmeter is connected across the terminals of coil A. The terminal which is connected to the positive voltage supply is determined and marked with a dot; then the voltmeter is disconnected from coil A, switch K is opened, and the voltmeter is connected across the terminals of coil B. With the circuit at rest, the switch K is again closed and the direction of swing of the voltmeter needle is noted. If the swing is positive, the terminal of coil B connected to the positive terminal of the voltmeter is marked with a dot; if the swing is negative, the terminal connected to the negative terminal of the voltmeter is so marked. For the conditions shown in Fig. 2b, both voltmeters would have a positive swing when switch K is closed.

In Fig. 1 the polarities as determined by any suitable method are indicated. The second step in determining the sign of the mutual-inductance voltage is described as the equations are written.

As in the simple resistance case, Kirchhoff's loop-voltage equations

can be written expressing the voltage drops in the circuit elements terms of the currents in the three branches *bcda, bfga,* and *ab,* or in term of the currents in loops *abcda* and *abfga.* Since Kirchhoff's loop-voltage law leads to two symmetrical equations — the simplest possible combination of expressions in this case — it is used for the analysis of the behavior of this circuit. The loop currents are taken as i_1 in loop *abcda* and i_2 in loop *agfba*, the arrow indicating in each the direction of a current which is described by a positive number. The branch current from a to b, if wanted, is easily obtained in terms of the loop currents as $i_1 - i_2$.

The voltage equations are written stating that the sum of the drops in potential around each loop in the arrow direction is zero. For loop **1**, which carries the current i_1,

$$\left.\begin{aligned} R_1 i_1 + L_1 \frac{di_1}{dt} + S_1 \int i_1 dt + R_3 i_1 + L_3 \frac{di_1}{dt} + S_3 \int i_1 dt + L_4 \frac{di_1}{dt} \\ - R_3 i_2 - L_3 \frac{di_2}{dt} - S_3 \int i_2 dt - M \frac{di_2}{dt} = e_1 \end{aligned}\right\} \quad [1]$$

and, for loop 2, which carries the current i_2,

$$\left.\begin{aligned} R_2 i_2 + L_2 \frac{di_2}{dt} + S_2 \int i_2 dt + L_5 \frac{di_2}{dt} + S_3 \int i_2 dt + L_3 \frac{di_2}{dt} \\ + R_3 i_2 - R_3 i_1 - L_3 \frac{di_1}{dt} - S_3 \int i_1 dt - M \frac{di_1}{dt} = e_2. \end{aligned}\right\} \quad [2]$$

It is most important that the signs in these two equations be clearly understood; if they are, the student is not likely to have the further difficulties so often encountered on this subject in network analysis. Since the drop in potential across a resistance occurs in the direction of the current, the term $R_1 i_1$ is positive by the direction conventions adopted. In Eq. 1, the term $R_3 i_2$ is negative since the drop in potential through R_3 due to a positive i_2 is opposite to the arrow direction in loop 1, which is the direction of numerically positive potential drops in this loop. The term $L_1[di_1/dt]$ is positive because a positive derivative of i_1 results in a higher potential at the left end of L_1 than at the right end, and thus in a potential drop in the arrow direction of loop 1. On the other hand, a positive di_2/dt in L_3 makes its upper end negative with respect to its lower end and therefore introduces a rise, or negative drop, in potential in the arrow direction in loop 1. Consideration of S_1 shows that a positive i_1 builds up positive charge on the left plate of S_1 and negative charge on its right plate, thereby producing a positive potential drop in loop 1. The e terms can be thought of as follows: If e_1 is a 10-volt battery having the

voltage rises indicated by the arrow,* then in going from c to d a rise in potential of 10 volts is encountered, or a potential drop of -10 volts. In this case e_1, which is numerically $+10$ volts, enters into the equation prefixed by a negative sign if on the same side of the equation as the drops, but with a positive sign if on the other side of the equality sign.

Perhaps the term that must be thought out most carefully is that involving M. As a standard convention in this text, M is taken as a positive number; hence the sign of the M term in the equation describes the direction of the effect of M in the circuit. It is desired to fix the sign of the $M[di_2/dt]$ term. A positive di_2/dt makes f positive with respect to h; whence, from the coil polarity marks, c is positive with respect to b. Therefore a positive di_2/dt produces a negative potential drop in the arrow direction in loop 1, so that the potential drop in loop 1 contributed by di_2/dt is $-M[di_2/dt]$. Other methods of fixing the signs and other conventions may be used. The important point is that the student shall have at least one unambiguous method by which he can fix signs correctly.

As with a resistance network having similar geometry, Eqs. 1 and 2 can be much simplified in appearance. First, like terms are collected and rewritten as

$$\left.\begin{aligned}(R_1 + R_3)i_1 + (L_1 + L_3 + L_4)\frac{di_1}{dt} + (S_1 + S_3)\int i_1 dt \\ - R_3 i_2 - (L_3 + M)\frac{di_2}{dt} - S_3\int i_2 dt = e_1,\end{aligned}\right\} \qquad [1a]$$

$$\left.\begin{aligned}(R_2 + R_3)i_2 + (L_2 + L_3 + L_5)\frac{di_2}{dt} + (S_2 + S_3)\int i_2 dt \\ - R_3 i_1 - (L_3 + M)\frac{di_1}{dt} - S_3\int i_1 dt = e_2.\end{aligned}\right\} \qquad [2a]$$

Next, all of the resistances of loop 1 are included in one term, all the self-inductances of loop 1 are included in another term, and so on, thus forming new parameters.

$$R_{11} \equiv R_1 + R_3 = \text{self- or total resistance in loop 1,} \qquad [3]$$

$$L_{11} \equiv L_1 + L_3 + L_4 = \text{self- or total inductance in loop 1,} \qquad [4]$$

$$S_{11} \equiv S_1 + S_3 = \text{self- or total elastance in loop 1,} \qquad [5]$$

$$R_{22} \equiv R_2 + R_3 = \text{self- or total resistance in loop 2,} \qquad [6]$$

$$L_{22} \equiv L_2 + L_3 + L_5 = \text{self- or total inductance in loop 2,} \qquad [7]$$

$$S_{22} \equiv S_2 + S_3 = \text{self- or total elastance in loop 2.} \qquad [8]$$

* As circuit diagrams become more complex, it is found that the use of + and − signs on the sources becomes awkward. The arrow beside the source indicates the direction for which the rise in potential of the electromotive force is described by a positive number

Similarly, for the parameters common to the two loops,

$$R_{12} \equiv -R_3 = \text{total resistance common to loops 1 and 2,} \tag{9}$$

$$L_{12} \equiv -(L_3 + M) = \text{total inductance common to loops 1 and 2,} \tag{10}$$

$$S_{12} \equiv -S_3 = \text{total elastance common to loops 1 and 2.} \tag{11}$$

The terms R_{12}, L_{12}, and S_{12} are often called *mutual* parameters. The minus signs associated with the individual circuit elements are included in the symbols for the mutual parameters. These minus signs result from the relative current directions considered as positive for reference in the two loops. For example, if i_2 were assumed positive in the opposite direction (counterclockwise), the signs prefixed to R_3, $(L_3 + M)$, and S_3 would all be positive in the mutual terms of Eqs. 1a and 2a, and the mutual parameters R_{12}, L_{12}, and S_{12} would correspondingly be positive. Hence if the loop-current directions are the same in a common branch, the mutual parameters are defined with the positive sign; if the loop-current directions are opposite, the mutual parameters are defined with the negative sign. This dependence of the sign of the elements comprising the mutual parameters on the assumed loop-current directions must be clearly understood. The proper signs always can be established by going back to the directions of the mutual voltages.

The double-subscript notation has a useful interpretation which is applied without change in Ch. VIII to the general ℓ-loop network. The first subscript designates the loop in which an effect is produced; the second designates the loop containing the cause. For example, because of the inductance L_{12} a voltage appears in loop 1 caused by a rate of change of current in loop 2. Because of the elastance S_{22} a voltage appears in loop 2 caused by the charge circulated in loop 2. It is seen by an examination of Eqs. 1a and 2a that

$$R_{12} = R_{21}, \tag{9a}$$

$$L_{12} = L_{21}, \tag{10a}$$

and

$$S_{12} = S_{21}, \tag{11a}$$

a relation which is true in general for all linear networks.

If Eqs. 3 to 11 are substituted in Eqs. 1a and 2a,

$$R_{11}i_1 + L_{11}\frac{di_1}{dt} + S_{11}\int i_1 dt + R_{12}i_2 + L_{12}\frac{di_2}{dt} + S_{12}\int i_2 dt = e_1, \quad \blacktriangleright[1b]$$

$$R_{21}i_1 + L_{21}\frac{di_1}{dt} + S_{21}\int i_1 dt + R_{22}i_2 + L_{22}\frac{di_2}{dt} + S_{22}\int i_2 dt = e_2. \quad \blacktriangleright[2b]$$

One further simplification can be made in writing if all the operations associated with one variable are collected into a single ***integrodifferential operator***. These operators,

$$\left.\begin{aligned} a_{11} &\equiv L_{11}\frac{d}{dt} + R_{11} + S_{11}\int dt \\ &\equiv \text{self-integrodifferential operator of loop 1,} \end{aligned}\right\} \quad [12]$$

$$\left.\begin{aligned} a_{12} = a_{21} &\equiv L_{12}\frac{d}{dt} + R_{12} + S_{12}\int dt \\ &\equiv \text{mutual integrodifferential operator,} \end{aligned}\right\} \quad [13]$$

$$\left.\begin{aligned} a_{22} &\equiv L_{22}\frac{d}{dt} + R_{22} + S_{22}\int dt \\ &\equiv \text{self-integrodifferential operator of loop 2,} \end{aligned}\right\} \quad [14]$$

are obtained by understanding, for example, that

$$a_{12}i_2 \equiv L_{12}\frac{di_2}{dt} + R_{12}i_2 + S_{12}\int i_2 dt. \quad [15]$$

By use of Eqs. 12 to 14 in Eqs. 1b and 2b, the following compact forms are obtained:

$$a_{11}i_1 + a_{12}i_2 = e_1, \quad ▶[1c]$$

$$a_{21}i_1 + a_{22}i_2 = e_2. \quad ▶[2c]$$

Equations 1c and 2c are of use primarily for expressing compactly the equilibrium equations of the network. In any actual solutions the a's must be expressed in terms of the network parameters and the differential and integral operators. The notation, however, is a great aid in the more advanced theory of complicated networks simply because it saves much writing, as can be seen by comparing Eqs. 1c and 2c with the fully expanded Eqs. 1 and 2.

3. Steady-state solution with two single-frequency impressed electromotive forces

In the solution of Eqs. 1c and 2c it is necessary that electromotive forces e_1 and e_2 be given as explicit functions of time. One of the simplest, and at the same time most widely useful, cases is the steady-state solution when e_1 and e_2 are sinusoidal functions of time with identical frequencies,

$$e_1 = E_{1m}\cos(\omega t + \psi_1), \quad [16]$$

$$e_2 = E_{2m}\cos(\omega t + \psi_2). \quad [17]$$

Because of the great importance of this case, a detailed development of the relations involved is presented here.

The discussion given in Art. 5, Ch. III, relating to the solution of a single linear differential equation with constant coefficients applies also to the solution of a simultaneous system of such equations. Thus the process of solution is not explicit in nature but is essentially a trial and verification method of finding a function that satisfies the system of simultaneous equations and the initial conditions. Fortunately, the experience of others permits the first trial solution to be the correct one. It is, of course, that the currents are sinusoidal functions of time having the same frequency as the sources. This similarity of form between applied force and response is characteristic not only of the one- and two-loop linear networks but of all linear networks and, in fact, of linear physical systems of all kinds. Thus it is assumed that

$$i_{1s}(t) = I_{1m} \cos(\omega t + \phi_1), \tag{18}$$

$$i_{2s}(t) = I_{2m} \cos(\omega t + \phi_2), \tag{19}$$

in which the amplitudes I_{1m} and I_{2m}, and the phase angles ϕ_1 and ϕ_2 are yet to be determined.

By using the reasoning developed in Art. 3 of Ch. IV for the single-loop case, Eqs. 16 to 19 may be written in the following simpler forms:

$$e_1 = \mathcal{R}_e[\mathbf{E}_{1m}\epsilon^{j\omega t}], \tag{16a}$$

$$e_2 = \mathcal{R}_e[\mathbf{E}_{2m}\epsilon^{j\omega t}], \tag{17a}$$

in which

$$\mathbf{E}_{1m} = E_{1m}\epsilon^{j\psi_1}, \tag{20}$$

and

$$\mathbf{E}_{2m} = E_{2m}\epsilon^{j\psi_2}, \tag{21}$$

$$i_{1s} = \mathcal{R}_e[\mathbf{I}_{1m}\epsilon^{j\omega t}], \tag{18a}$$

$$i_{2s} = \mathcal{R}_e[\mathbf{I}_{2m}\epsilon^{j\omega t}], \tag{19a}$$

in which

$$\mathbf{I}_{1m} = I_{1m}\epsilon^{j\phi_1}, \tag{22}$$

$$\mathbf{I}_{2m} = I_{2m}\epsilon^{j\phi_2}. \tag{23}$$

In this simplified form the values for voltage and steady-state current are substituted in Eqs. 1c and 2c, giving:

$$a_{11}\mathcal{R}_e[\mathbf{I}_{1m}\epsilon^{j\omega t}] + a_{12}\mathcal{R}_e[\mathbf{I}_{2m}\epsilon^{j\omega t}] = \mathcal{R}_e[\mathbf{E}_{1m}\epsilon^{j\omega t}], \tag{1d}$$

$$a_{21}\mathcal{R}_e[\mathbf{I}_{1m}\epsilon^{j\omega t}] + a_{22}\mathcal{R}_e[\mathbf{I}_{2m}\epsilon^{j\omega t}] = \mathcal{R}_e[\mathbf{E}_{2m}\epsilon^{j\omega t}]. \tag{2d}$$

If relations can be found which make the two sides of each of Eqs. 1d

and 2d identities, the problem is solved. To find these relations, the integrodifferential operators as defined in Eqs. 12 to 14 are substituted term by term in Eqs. 1d and 2d. The first term of the first equation becomes

$$a_{11}\mathcal{R}_e[\mathrm{I}_{1m}\epsilon^{j\omega t}] = \mathcal{R}_e\left[\mathrm{I}_{1m}\left(j\omega L_{11} + R_{11} + \frac{S_{11}}{j\omega}\right)\epsilon^{j\omega t}\right]. \qquad [24]$$

In a similar manner every derivative symbol d/dt becomes $j\omega$, while every integral symbol $\int dt$ becomes $1/j\omega$. Every term in Eqs. 1d and 2d involves the real part expression; hence the $\mathcal{R}_e$'s may be omitted, because, if the complex expressions are equal, their real parts must likewise be equal. At the same time, the I_m and E_m terms may all be replaced by the corresponding effective vector values, I and E, by dividing the two equations by $\sqrt{2}$, and the equations may also be divided by the common factor $\epsilon^{j\omega t}$. If all these changes are carried out, Eqs. 1d and 2d become

$$\left(R_{11} + j\omega L_{11} + \frac{S_{11}}{j\omega}\right)\mathrm{I}_1 + \left(R_{12} + j\omega L_{12} + \frac{S_{12}}{j\omega}\right)\mathrm{I}_2 = \mathrm{E}_1, \qquad [1e]$$

$$\left(R_{21} + j\omega L_{21} + \frac{S_{21}}{j\omega}\right)\mathrm{I}_1 + \left(R_{22} + j\omega L_{22} + \frac{S_{22}}{j\omega}\right)\mathrm{I}_2 = \mathrm{E}_2. \qquad [2e]$$

Equations 1e and 2e express the simultaneous relations between vector loop currents and vector-loop-voltage rises. A similarity is immediately evident between the terms in parentheses and the impedance of the series *RLS* circuit. This suggests that each of these groups of terms be called an impedance as follows:

$$\left.\begin{aligned}\mathrm{Z}_{11} &\equiv R_{11} + j\left(\omega L_{11} - \frac{S_{11}}{\omega}\right)\\ &= \text{self-impedance of loop 1,}\end{aligned}\right\} \qquad [25]$$

$$\left.\begin{aligned}\mathrm{Z}_{21} &= \mathrm{Z}_{12} \equiv R_{12} + j\left(\omega L_{12} - \frac{S_{12}}{\omega}\right)\\ &= \text{mutual impedance of loops 1 and 2,}\end{aligned}\right\} \qquad [26]$$

$$\left.\begin{aligned}\mathrm{Z}_{22} &\equiv R_{22} + j\left(\omega L_{22} - \frac{S_{22}}{\omega}\right)\\ &= \text{self-impedance of loop 2.}\end{aligned}\right\} \qquad [27]$$

By using Eqs. 25 to 27, Eqs. 1e and 2e can be written

$$\mathrm{Z}_{11}\mathrm{I}_1 + \mathrm{Z}_{12}\mathrm{I}_2 = \mathrm{E}_1, \qquad \blacktriangleright[1f]$$

$$\mathrm{Z}_{21}\mathrm{I}_1 + \mathrm{Z}_{22}\mathrm{I}_2 = \mathrm{E}_2, \qquad \blacktriangleright[2f]$$

from which I_1 and I_2 are obtained by simultaneous solution:

$$I_1 = \frac{Z_{22}E_1 - Z_{12}E_2}{Z_{11}Z_{22} - Z_{12}^2},$$

$$\mathrm{I}_2 = \frac{\mathrm{Z}_{11}\mathrm{E}_2 - \mathrm{Z}_{12}\mathrm{E}_1}{\mathrm{Z}_{11}\mathrm{Z}_{22} - \mathrm{Z}_{12}^2}. \qquad [29]$$

By means of Eqs. 22 and 23, Eqs. 28 and 29 yield values of I_1, I_2, ϕ_1, and ϕ_2 for which Eqs. 18 and 19 are solutions of the original circuit differential equations 1 and 2.

4. Direct formulation of steady-state vector loop equations

From the solution of Eqs. 1 and 2 it is possible to make certain generalizations that permit the rapid formulation of equations in vector form for any two-loop network. Since the network contains only constant parameters, the steady-state loop currents have only the one angular frequency — the common frequency of the various sources. Under this condition, a vector current I of angular frequency ω produces a vector-voltage drop across a series impedance Z equal to IZ, the voltage drop V having the same angular frequency ω. Since the vector-voltage drops around a closed loop all have the same angular frequency, their vector sum equals the

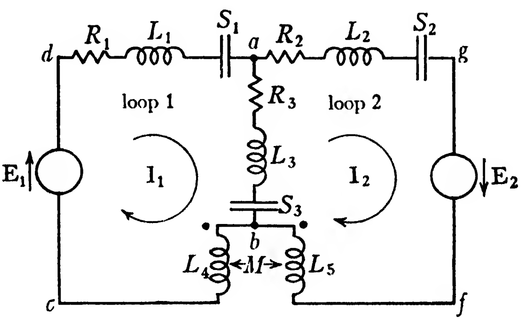


FIG. 3. Representation of loop currents and source voltages in the two-loop network by vector values.

total applied vector-voltage rise around the loop. Similarly the vector currents in parallel branches can be added to give the total current entering the parallel combination.

In Fig. 3, the same circuit as that shown in Fig. 1 is used, but electromotive forces and loop currents are replaced by the vectors E_1, E_2, I_1, and I_2, the arrows representing the positive directions used for reference.

The voltage drop around the path *dabc* caused by the current I_1 is $Z_{11}I_1$, where Z_{11} is the entire impedance on the boundary of the loop; that is,

$$Z_{11} \equiv R_1 + R_3 + j\omega(L_1 + L_3 + L_4) - \frac{j}{\omega}(S_1 + S_3). \qquad [30]$$

But I_2 also produces a voltage drop in loop 1 which is I_2Z_{12}, where

$$Z_{12} \equiv -\left[R_3 + j\left(\omega L_3 - \frac{S_3}{\omega}\right)\right] - j\omega M. \qquad [31]$$

The inductance L_3 common to the two loops and the mutual inductance M with the polarity indicated produce additive voltages in loop 1 because of a rate of change of current in loop 2; consequently, the M term has the same sign as the L_3 term in Eq. 31. If the polarities of the mutual inductance had been as shown in Fig. 4, a positive rate of change of i_2 would produce because of M a voltage *drop* in the arrow direction of loop 1, and because of L_3 it would produce a voltage *rise*. Hence the mutual-impedance term in Eq. 31 would be $+j\omega M$, and the L_3 voltage and the M voltage would be of opposite sign in loop 1. Since the same directions are used for vector currents as for instantaneous currents, the sign of the mutual-impedance term can be determined directly from the vector currents shown in the diagram.

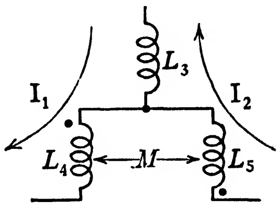


FIG. 4. Mutual inductance having relative polarities opposite to those of Fig. 3.

Adding the vector-voltage drops in loop 1 and equating to the vector electromotive force give

$$Z_{11}I_1 + Z_{12}I_2 = E_1. \qquad [1f]$$

In a similar way the vector voltages in loop 2 can be added. Thus because of I_2 there is a vector-voltage drop $Z_{22}I_2$, where

$$Z_{22} \equiv R_2 + R_3 + j\omega(L_2 + L_3 + L_5) - \frac{j}{\omega}(S_2 + S_3). \qquad [32]$$

The current I_1 causes a vector-voltage drop in the arrow direction in loop 2 of $Z_{21}I_1$, where

$$Z_{21} = Z_{12} \equiv -\left[R_3 + j\left(\omega L_3 - \frac{S_3}{\omega}\right)\right] - j\omega M. \qquad [31a]$$

Equating the vector-voltage drops to the vector electromotive force E_2 in loop 2 gives

$$Z_{21}I_1 + Z_{22}I_2 = E_2. \qquad [2f]$$

5. Certain special considerations in network problems

Through the preceding articles it is assumed that the meaning of polarity and direction of vector voltages and current is clearly understood. The student who is not confident of his knowledge of this important subject should carefully review Arts. 5 and 6 of Ch. IV. It should also be borne in mind that, once reference directions have been assigned to the unknown vector quantities such as loop currents, these directions must be strictly adhered to throughout the solution of any network problem. This procedure parallels exactly the methods of direct-current network analysis presented in Ch. II.

Another detail which requires careful attention is the sign of the mutual-impedance terms of the type of Z_{12}, Eq. 31. In that case a minus sign precedes the expression which represents the impedance Z_{ab} of the branch *ab* common to loops 1 and 2 of Fig. 3. This minus sign appears because the currents $\mathbf{I}_1$ and $\mathbf{I}_2$ are arbitrarily assigned opposite directions through the common branch *ab* so that a positive $\mathbf{I}_2$ in loop *2* causes a

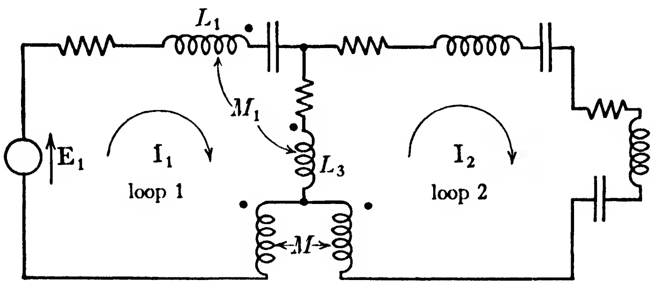


Fig. 5. A circuit in which the same loop current $\mathbf{I}_1$ is present in two coils having the mutual inductance M_1.

voltage rise or a negative voltage drop in the positive direction of $\mathbf{I}_1$ in loop 1. If the reference arrows for positive loop currents should be so assigned that both currents $\mathbf{I}_1$ and $\mathbf{I}_2$ were in the same direction in the common branch *ab*, there would not be a minus sign preceding the expression for the impedance Z_{ab}, as previously mentioned.

It frequently happens that the same loop current is present in two coils having mutual inductance. In this situation, the same type of reasoning regarding the mutual impedance as that developed in Art. 4 applies, but the effect on the impedance terms is different. The case represented in Fig. 5 differs from that in Fig. 3 only by the addition of the mutual inductance M_1. Under the polarity conditions indicated, the current $\mathbf{I}_1$ passing through the coil L_1 causes a voltage *rise* in the positive direction of $\mathbf{I}_1$ in coil L_3. Hence the Z_{11} term is increased by $-j\omega M_1$. The same current in L_3 causes a voltage rise in L_1, further increasing the

Z_{11} term by $-j\omega M_1$. Hence the self-impedance Z_{11} of loop 1 is increased by $-2j\omega M_1$. At the same time, the current I_1 in L_1 causes a voltage drop in the positive direction of I_2 in loop 2; so that Z_{21} and Z_{12} are each increased by $j\omega M_1$. Otherwise Z_{11}, Z_{12}, Z_{21}, and Z_{22} are as given in Art. 4.

6. Two-loop network with complicated branches

For steady-state analysis, a circuit which contains complicated series and parallel combinations of elements can be handled as a two-loop network if the method presented in the preceding article is used. Thus Z_{11} is the total impedance of loop 1, or the impedance that source E_1 sees when loop 2 is open-circuited; Z_{12} is the impedance representing the effect of current I_2 in loop 1; while Z_{22} is the total impedance seen by source E_2 when loop 1 is open-circuited. Hence Z_{11}, Z_{12}, and Z_{22} may contain any combination of elements for which the expressions for impedance can be written in accordance with Ch. IV.

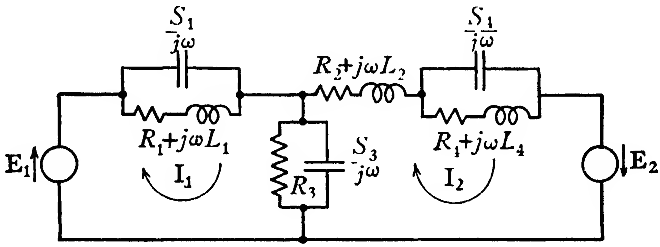


FIG. 6. Two-loop network some of whose impedances contain parallel branches.

As an example the network of Fig. 6 is considered. Since the impedance of two branches in parallel is the product of their individual impedances divided by the sum of their individual impedances, Z_{11} can be expressed

$$Z_{11} = \frac{-j\dfrac{S_1}{\omega}(R_1 + j\omega L_1)}{R_1 + j\left(\omega L_1 - \dfrac{S_1}{\omega}\right)} + \frac{-j\dfrac{S_3}{\omega}R_3}{R_3 - j\dfrac{S_3}{\omega}}, \tag{33}$$

$$Z_{12} = -\frac{-j\dfrac{S_3}{\omega}R_3}{R_3 - j\dfrac{S_3}{\omega}}, \tag{34}$$

$$Z_{22} = R_2 + j\omega L_2 + \frac{-j\dfrac{S_4}{\omega}(R_4 + j\omega L_4)}{R_4 + j\left(\omega L_4 - \dfrac{S_4}{\omega}\right)} + \frac{-j\dfrac{S_3}{\omega}R_3}{R_3 - j\dfrac{S_3}{\omega}}. \tag{35}$$

These values can be put directly in Eqs. 1f and 2f and the network then can be solved as a two-loop network. While Eqs. 33 to 35 look rather complicated, the solution by this method is considerably simpler than if the case were handled as a five-loop network, the other alternative. The complications involved in handling the circuit on the five-loop basis can be appreciated better after the analysis for the ℓ-loop case is developed. It should be remembered that this two-loop formulation of a more complicated network can be carried out by the methods developed here only for steady-state conditions. The method can, however, be extended to include transient conditions.

The two-loop scheme has the further advantage in numerical work that each calculation can be associated with a group of circuit elements, and the reasonableness of the result can be verified readily. In the ℓ-loop scheme the calculations have to be carried out by the evaluation of determinants or some equivalent method in which all sight is lost of the physical meaning of the terms, and the reasonableness of the calculations cannot be verified step by step. In practical work the possibility of associating the individual steps of a calculation with the physical aspects of a problem often becomes very valuable as a means of discovering numerical errors soon after they are made. Otherwise it may happen that days and even weeks of work are built up on an error that when finally discovered invalidates the work.

7. Input impedance of a two-loop network

From the general steady-state solution for a two-loop circuit presented in preceding articles, a number of useful relations can be developed for special cases. This article considers the effect that the second loop has on the impedance of the circuit as measured from the terminals of the first loop. This relation has numerous practical applications in both power and communications work.

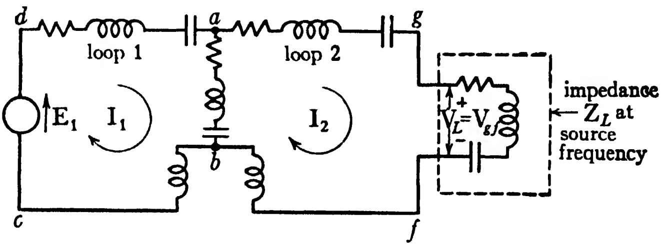


FIG. 7. Two-loop network for derivation of input impedance.

The voltage V_L or V_{gf} is the drop through a load impedance Z_L as shown in Fig. 7. The function of such a circuit is usually to transfer energy

from the source supplying E_1 to a load whose impedance is Z_L. The student may well inquire why Z_L is not connected directly to E_1 with much saving in circuit complication. If the source is put in one loop and the load in a second loop having a branch in common with the first in a communications application, the source can often be made to deliver much more power to the load. Also if E_1 contains several or many frequencies, such a two-loop network may enable the load to take power over only a narrow range of frequencies, as in radio circuits. Often, in fact, the source and load are separated by several or even many loops for the same reasons In a power application, long distances may separate the source from the load, so that the natural constants of the transmission line and associated apparatus form a network which can often be simplified to a two-loop circuit. In practical cases some of the parameters shown in the general two-loop circuit are usually omitted, but this omission in no way alters the usefulness of the final equations as some of the terms of the Z's are simply made equal to zero. Some of these special cases are treated subsequently.

The equations relating the applied electromotive force E_1 and the currents I_1 and I_2 are Eqs. 1f and 2f, which are rewritten here for convenience as

$$Z_{11}I_1 + Z_{12}I_2 = E_1, \tag{1f}$$

$$Z_{21}I_1 + Z_{22}I_2 = 0, \tag{2g}$$

where Z_{22} is the total impedance of loop 2 including the load impedance Z_L.

The *input impedance* of the circuit as viewed from the source terminals d and c of loop 1, also termed the ***apparent impedance*** of loop 1, is designated by Z_{1a} and defined by

$$Z_{1a} \equiv \frac{E_1}{I_1}. \qquad ▶[36]$$

Since E_2 is zero, Eq. 28 may be solved for I_1, as follows:

$$I_1 = E_1 \frac{Z_{22}}{Z_{11}Z_{22} - Z_{12}^2}, \tag{28a}$$

from which

$$Z_{1a} = \frac{E_1}{I_1} = \frac{Z_{11}Z_{22} - Z_{12}^2}{Z_{22}} = Z_{11} - \frac{Z_{12}^2}{Z_{22}}. \qquad ▶[36a]$$

Equation 36a expresses the important fact that the input impedance of loop 1 is its own total loop impedance Z_{11} reduced by the term Z_{12}^2/Z_{22}.

8. Current and voltage ratios in two-loop network

The ratios of the vector currents in the two loops and the ratio of the vector applied voltage E_1 to that appearing across the load impedance Z_L are readily obtained from Eqs. 1f and 2g. Thus from Eq. 2g for the *circuit of Fig. 7*

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = -\frac{Z_{22}}{Z_{12}}. \tag{37}$$

To determine the ratio of $\mathbf{V}_L$ to $\mathbf{E}_1$, $\mathbf{I}_1$ can be eliminated between Eqs. 1f and 2g, and $\mathbf{I}_2$ expressed in terms of $\mathbf{V}_L$ can be substituted in the result. Thus from Eq. 37

$$\mathbf{I}_1 = -\frac{Z_{22}}{Z_{12}}\mathbf{I}_2. \tag{37a}$$

Putting Eq. 37a into Eq. 1f gives

$$-\left(\frac{Z_{11}Z_{22}}{Z_{12}} - Z_{12}\right)\mathbf{I}_2 = \mathbf{E}_1. \tag{38}$$

But

$$\mathbf{V}_L = Z_L\mathbf{I}_2 \quad \text{or} \quad \mathbf{I}_2 = \frac{\mathbf{V}_L}{Z_L}. \tag{39}$$

Substituting Eqs. 39 and 37a in Eq. 1f and solving for $\mathbf{V}_L/\mathbf{E}_1$ give

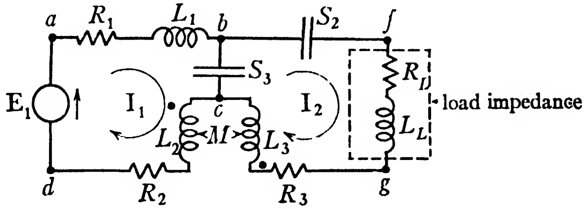
$$\frac{\mathbf{V}_L}{\mathbf{E}_1} = \frac{Z_L Z_{12}}{Z_{12}^2 - Z_{11}Z_{22}}. \tag{40}$$

By a similar use of Eqs. 1f, 2g, and 39, the ratio of either loop current to the terminal voltage of the other loop can readily be obtained. These ratios are not evaluated formally in the two-loop case but are derived and discussed in Art. 7, Ch. VIII, for the ℓ-loop case. The two-loop results can readily be obtained by the student.

9. Illustrative example of two-loop network

In order that the foregoing analysis may be applied to an actual circuit, the voltage and current ratios, the average power delivered to the load, and the efficiency of power transmission of the two-loop network of Fig. 8 are calculated. The circuit of Fig. 8 is somewhat less general than that formally analyzed in the preceding sections because some of the parameters included in Fig. 7 are omitted. However, all the essential processes involved in the solution of the general case are present in the example. The problem is to calculate the total loop impedances, the mutual impedance, the apparent impedance of loop 1, the currents in

the two loops, the current ratio, the power supplied by the source, the power absorbed by the load impedance, and the efficiency of power transmission In practice these calculations would probably be done at a number of frequencies to determine how the behavior of the network



$E_1 = 100$ v
$\omega = 2\pi 50{,}000$ radians/sec

$R_1 = 3.4$ ohms	$C_2 = \dfrac{1}{S_2} = 0.0421\ \mu f$
$R_2 = 5.1$ ohms	$C_3 = \dfrac{1}{S_3} = 0.00760\ \mu f$
$R_3 = 0.5$ ohm	
$L_1 = 55\ \mu h$	$R_L = 120$ ohms
$L_2 = 725\ \mu h$	$L_L = 150\ \mu h$
$L_3 = 106\ \mu h$	$M = 268\ \mu h$

FIG. 8. Two-loop network for example of Art. 9.

varies with the frequency of the source. Here the calculation for only one frequency is carried out, since the other frequencies involve exactly the same processes with different numbers.

Solution: The total or self-impedance Z_{11} of loop 1 is

$$R_{11} = R_1 + R_2 = 3.4 + 5.1 = 8.5 \text{ ohms,} \qquad [3a]$$

$$L_{11} = L_1 + L_2 = 55 + 725 = 780\ \mu h, \qquad [4a]$$

$$\omega L_{11} = 2 \times 3.14 \times 50{,}000 \times 780 \times 10^{-6} = 245 \text{ ohms,} \qquad [41]$$

$$S_{11} = S_3 = \frac{10^9}{7.60} \text{ darafs,} \qquad [5a]$$

$$-\frac{S_{11}}{\omega} = -419 \text{ ohms,} \qquad [42]$$

$$\omega L_{11} - \frac{S_{11}}{\omega} = -174 \text{ ohms,} \qquad [43]$$

$$Z_{11} = R_{11} + j\left(\omega L_{11} - \frac{S_{11}}{\omega}\right) = 8.5 - j174 \text{ ohms.} \qquad [25a]$$

The total or self-impedance Z_{22} of loop 2 is

$$R_{22} = R_3 + R_L = 0.5 + 120 = 120.5 \text{ ohms,} \quad [6a]$$

$$L_{22} = L_3 + L_L = 106 + 450 = 556 \ \mu\text{h,} \quad [7a]$$

$$\omega L_{22} = 175 \text{ ohms,} \quad [44]$$

$$\left.\begin{aligned} S_{22} &= S_2 + S_3 = 10^6\left(\frac{1}{0.0421} + \frac{1}{0.00760}\right) \\ &= 10^6(23.7 + 131) = 155 \times 10^6 \text{ darafs,} \end{aligned}\right\} \quad [8a]$$

$$-\frac{S_{22}}{\omega} = -495 \text{ ohms,} \quad [45]$$

$$\omega L_{22} - \frac{S_{22}}{\omega} = -320 \text{ ohms,} \quad [46]$$

$$\mathbf{Z}_{22} = 120.5 - j320 = 342\underline{/-69.4^\circ} \text{ ohms.} \quad [27a]$$

The mutual impedance $\mathbf{Z}_{12}$ includes the impedance $\mathbf{Z}_{bc}$ of the common branch bc and the mutual impedance resulting from the mutual inductance M. The current $\mathbf{I}_1$ causes a voltage drop $-jS_3\mathbf{I}_1/\omega$, in the direction of $\mathbf{I}_1$, which is a voltage drop of $+j(S_3\mathbf{I}_1/\omega)$ in the direction of $\mathbf{I}_2$. Hence the mutual impedance contributed to $\mathbf{Z}_{12}$ by branch bc is $j(S_3/\omega)$. A positive $\mathbf{I}_1$ causes a positive voltage drop in the direction of $\mathbf{I}_2$ through the mutual inductance M. Hence the contribution of this inductance to the mutual impedance $\mathbf{Z}_{12}$ is $j\omega M$. Therefore

$$\left.\begin{aligned} \mathbf{Z}_{12} &= j\left(\frac{S_3}{\omega} + \omega M\right) = j(419 + 2 \times 3.14 \times 50{,}000 \times 268 \times 10^{-6}) \\ &= j503 \text{ ohms.} \end{aligned}\right\} \quad [26a]$$

The load impedance $\mathbf{Z}_L$ is

$$\left.\begin{aligned} \mathbf{Z}_L = R_L + j\omega L_L &= 120 + j2 \times 3.14 \times 50{,}000 \times 450 \times 10^{-6} \\ &= 120 + j141 = 185\underline{/49.7^\circ} \text{ ohms.} \end{aligned}\right\} \quad [47]$$

Next the apparent impedance $\mathbf{Z}_{1a}$ which the source voltage sees at the terminals of loop 1 is calculated.

$$\mathbf{Z}_{12}^2 = -2.54 \times 10^5, \quad [48]$$

$$-\frac{\mathbf{Z}_{12}^2}{\mathbf{Z}_{22}} = \frac{2.54 \times 10^5}{342\underline{/-69.4^\circ}} = 743\underline{/69.4^\circ} = 262 + j696, \quad [49]$$

$$\mathbf{Z}_{1a} = \mathbf{Z}_{11} - \frac{\mathbf{Z}_{12}^2}{\mathbf{Z}_{22}} = 271 + j522 = 589\underline{/62.6^\circ}. \quad [36b]$$

By taking $\mathbf{E}_1$ along the axis of reals for convenience,

$$\mathbf{I}_1 = \frac{\mathbf{E}_1}{\mathbf{Z}_{1a}} = \frac{10.0\underline{/0^\circ}}{589\underline{/62.6^\circ}} = 0.0170\underline{/-62.6^\circ} \text{ amp.} \quad [36c]$$

The current ratio is calculated next

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = -\frac{\mathbf{Z}_{22}}{\mathbf{Z}_{12}} = -\frac{342\underline{/-69.4^\circ}}{503\underline{/90^\circ}} = 0.679\underline{/20.6^\circ}. \quad [37b]$$

Then

$$I_2 = \frac{0.0170\underline{/-62.6^\circ}}{0.679\underline{/20.6^\circ}} = 0.0251\underline{/-83.2^\circ} \text{ amp.} \qquad [37c]$$

The average power supplied by the source is readily calculated from

$$\text{Source power} = I_1^2\mathcal{R}_e[Z_{1a}] = 0.0170^2 \times 271 = 0.0784 \text{ w.} \qquad [50]$$

The average power absorbed by the load impedance Z_L is

$$\text{Load power} = I_2^2 R_L = 0.0251^2 \times 120 = 0.0756 \text{ w.} \qquad [51]$$

As a partial check on the calculations, the losses in the network itself are computed. These losses added to the load power should give the source power. In loops 1 and 2 the losses are, respectively,

$$I_1^2 R_{11} = 0.017^2 \times 8.5 = 24.6 \times 10^{-4} \text{ w,} \qquad [52]$$

and

$$I_2^2 R_3 = 0.0251^2 \times 0.5 = 3.2 \times 10^{-4} \text{ w,} \qquad [53]$$

$$\text{total losses} = 27.8 \times 10^{-4} \text{ w,} \qquad [54]$$

$$\text{output} = 756 \times 10^{-4} \text{ w,} \qquad [51]$$

$$\text{input} = 784 \times 10^{-4} \text{ w,} \qquad [50]$$

which checks the foregoing figure.

The efficiency is

$$\frac{\text{Load power}}{\text{Source power}} = \frac{756}{784} = 0.965, \text{ or } 96.5\%. \qquad [55]$$

10. Two magnetically coupled circuits

The special case of the two-loop circuit in which the coupling between the loops consists of mutual inductance only is sufficiently important to justify detailed analysis. In Fig. 9 is shown such a network, in which the source e_1 supplies power to a load impedance Z_L through the mutual inductance of the magnetically coupled coils. The differential equations are

$$R_1 i_1 + L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = e_1, \qquad [1g]$$

$$-M \frac{di_1}{dt} + (R_2 + R_L)i_2 + (L_2 + L_L) \frac{di_2}{dt} + S_L \int i_2 dt = 0. \qquad [2h]$$

By use of the relations developed in Art. 4 the vector equations can be written immediately as

$$(R_1 + j\omega L_1)I_1 - j\omega M I_2 = E_1, \qquad [1h]$$

$$-j\omega M I_1 + \left[R_2 + R_L + j\omega (L_2 + L_L) + \frac{S_L}{j\omega}\right] I_2 = 0, \qquad [2i]$$

or by using the simplified form of impedance notation,

$$Z_{11} = R_1 + j\omega L_1, \tag{25b}$$

$$Z_{21} = Z_{12} = -j\omega M, \tag{26b}$$

$$Z_{22} = R_2 + R_L + j\omega(L_2 + L_L) + \frac{S_L}{j\omega}. \tag{27b}$$

Equations 25b to 27b substituted in Eqs. 1h and 2i reduce them to forms of Eqs. 1f and 2g for a two-loop network. The solution of Eqs. 1f and 2g is Eq. 28a for I_1 and, for the load current,

$$I_2 = E_1 \frac{-Z_{12}}{Z_{11}Z_{22} - Z_{12}^2}. \tag{29a}$$

The results obtained in Arts. 7 and 8 apply directly to this case, as they must in any linear single-frequency two-loop circuit.

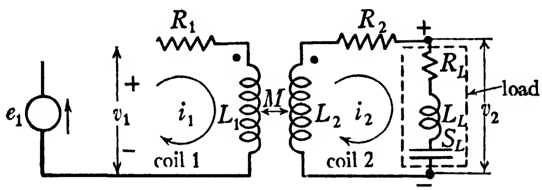


FIG. 9. Two loops magnetically coupled.

Two magnetically coupled coils constitute a *transformer*, one of the most important devices used in electric circuits. A transformer may behave as a linear or as a nonlinear circuit element, depending on its construction and the conditions of operation. In general, the air-core transformer is considered linear, and, for many aspects of circuit analysis, the iron-core transformer is also treated as a linear circuit element. The limitations on the linear point of view and the methods of studying the nonlinear characteristics are presented in this series in the volume on magnetic circuits and transformers.

11. THE IDEAL TRANSFORMER

Because of its utility in circuit analysis, a device known as an *ideal transformer* is of considerable importance. It can never be realized physically, although many iron-core transformers can with a fair degree of approximation be considered ideal in the sense used here. Since an actual transformer can be represented rather accurately, however, by a simple network plus an ideal transformer, the ideal transformer is not relegated to use in circuit diagrams representing purely fictional situations.

When two circuits, 1 and 2, carrying currents i_1 and i_2, respectively, are in proximity, four distinct components of magnetic flux may be considered to exist. The total fluxes of the respective circuits are ϕ_{11} and ϕ_{22}. Circuit 1 is linked by some flux $\phi_{1\ell}$, which does not link circuit 2, while circuit 2 is linked by some flux $\phi_{2\ell}$, which does not link circuit 1. In the study of transformers and rotating machines these fluxes are called the leakage fluxes of the two circuits. The third flux ϕ_{12} is that linking both circuits as a result of the current i_2 in circuit 2, while the fourth flux ϕ_{21} is that linking both circuits as a result of the current i_1 in circuit 1. If L_1 and L_2 are the self-inductances of circuits 1 and 2, respectively, and M is their mutual inductance, the constants k_1 and k_2 express the following ratios:

$$k_1 = \frac{\phi_{21}}{\phi_{11}}, \qquad [56]$$

$$k_2 = \frac{\phi_{12}}{\phi_{22}}. \qquad [57]$$

For concentrated coils

$$M = \frac{N_1\phi_{12}}{i_2} = \frac{N_2\phi_{21}}{i_1}, \qquad [58]$$

approximately, where N_1 and N_2 are the number of turns in circuits 1 and 2, respectively. Also,

$$L_1 = \frac{N_1\phi_{11}}{i_1} \qquad [59]$$

and

$$L_2 = \frac{N_2\phi_{22}}{i_2}, \qquad [60]$$

approximately. Hence

$$M = \frac{N_1k_2\phi_{22}}{i_2} = \frac{N_1k_2L_2}{N_2} \qquad [61]$$

and

$$M = \frac{N_2k_1\phi_{11}}{i_1} = \frac{N_2k_1L_1}{N_1}. \qquad [62]$$

Multiplying together Eqs. 61 and 62 and taking the square root give

$$M = \sqrt{k_1k_2}\sqrt{L_1L_2}. \qquad [63]$$

The quantity $\sqrt{k_1 k_2}$ is defined as the coefficient of coupling k of the two circuits. By this definition,

$$M = k\sqrt{L_1 L_2}. \tag{64}$$

The coefficient of coupling k is a measure of the magnetic proximity of the circuits. Often the magnetic coupling of two circuits is contributed almost entirely by coils included in them. When two such coils are wound very closely together, as can be done, for example, by winding two conductors, one for each coil, simultaneously on a form, particularly when they have a highly magnetic core, the coefficient of coupling k can approach unity, *expressing mathematically the fact that all but a very small fraction of the flux linking one coil also links the other, and vice versa. The same expression holds for coils whose windings are not concentrated. It can be* developed by giving k_1 and k_2 more general definitions in terms of flux linkages instead of in terms of fluxes.

The ideal transformer is defined as a pair of magnetically coupled coils which have the following properties:

(a) The coefficient of coupling is unity.

(b) There are no losses associated with the coils; that is, the coils have no ohmic resistance, and no energy is dissipated in the flux path in the form of hysteresis or eddy-current losses.

(c) The self-inductances L_1 and L_2 are infinite, so that the impedances of any circuit elements in series with L_1 and L_2, respectively, are negligible compared with ωL_1 and ωL_2, respectively.

It follows from property (a) and the definitions of self- and mutual inductance as flux linkages per unit current that, if

$$a \equiv \frac{N_1}{N_2} \equiv \frac{\text{turns in coil 1}}{\text{turns in coil 2}}, \tag{65}$$

$$a = \frac{L_1}{M} = \frac{M}{L_2} = \sqrt{\frac{L_1}{L_2}}. \tag{66}$$

The calculation of the current and voltage ratios and the apparent impedance Z_{1a} at the source terminals of the ideal transformer may be readily accomplished by using the results of Art. 8, all properties of the transformer having their limiting values. The current ratio is

$$\frac{I_1}{I_2} = -\frac{Z_{22}}{Z_{12}}. \tag{37}$$

Limits may be taken in Eq. 37 as the actual circuit approaches that of the

ideal transformer. Property (b) of this transformer here affects only R_2;

$$\lim_{R_2\to 0}\frac{I_1}{I_2} = \lim_{R_2\to 0} -\left(\frac{R_2 + j\omega L_2 + Z_L}{Z_{12}}\right) = -\frac{j\omega L_2 + Z_L}{Z_{12}}. \qquad [37d]$$

Using properties (c) gives

$$Z_L \ll \omega L_2. \qquad [67]$$

This, together with Eqs. 26b and 66 used in Eq. 37d, gives

$$\frac{I_1}{I_2} = \frac{j\omega L_2}{j\omega M} = \frac{1}{a} \qquad [37e]$$

for the ideal transformer.

The voltage ratio may be found by applying Eq. 40 as follows:

$$\frac{V_1}{V_2} = \frac{Z_{12}^2 - Z_{11}Z_{22}}{Z_L Z_{12}}, \qquad [40a]$$

from which, by substituting detailed values and dropping the R_1 and R_2 terms in Z_{11} and Z_{22},

$$\frac{V_1}{V_2} = \frac{(-j\omega M)^2 - j\omega L_1(j\omega L_2 + Z_L)}{-j\omega M Z_L} \qquad [40b]$$

or

$$\frac{V_1}{V_2} = \frac{j\omega L_1 Z_L + (j\omega)^2(L_1L_2 - M^2)}{j\omega M Z_L}. \qquad [40c]$$

But from the property (a) the $L_1L_2 - M^2$ term is zero; hence,

$$\frac{V_1}{V_2} = \frac{j\omega L_1 Z_L}{j\omega M Z_L} = a \qquad [40d]$$

for the ideal transformer. The input impedance Z_{1a} at the source terminal is

$$Z_{1a} = \frac{Z_{11}Z_{22} - Z_{12}^2}{Z_{22}}. \qquad [36d]$$

The numerator of Eq. 36d is the negative of the numerator of Eq. 40a; hence, if R_2 is omitted from the denominator, Eq. 36d becomes

$$Z_{1a} = \frac{j\omega L_1 Z_L}{j\omega L_2 + Z_L}. \qquad [36e]$$

Dropping Z_L by comparison with ωL_2 and using Eq. 66 give

$$Z_{1a} = a^2 Z_L \qquad [36f]$$

for the ideal transformer.

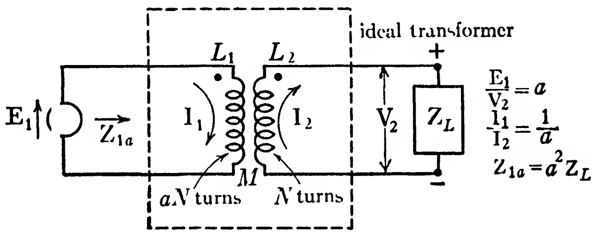
By expressing the results of Eqs. 37e, 40d, and 36f in descriptive form, the following important relations are obtained:

▶ *In an ideal transformer, the ratio of currents in the two windings is* the reciprocal of the ratio of the number of turns of the respective windings; the ratio of the two winding voltages is directly proportional to the number of turns; and the impedance as seen by the source is the load impedance multiplied by the square of the turns ratio.◀

An interpretation of Eq. 36f furnishes a set of qualities alternative to (a), (b), and (c) that are often more useful for defining the ideal transformer practically. Thus from Eq. 36f an ideal transformer is one that has:

(a) no losses,
(b) infinite input impedance when the load impedance is infinite,
(c) zero input impedance when the load impedance is zero.

In Fig. 10 an ideal transformer connecting a source E_1 and a load impedance Z_L is shown together with a tabulation of its properties.



$\omega L_1 \gg$ other impedance in loop 1
$\omega L_2 \gg Z_L$ plus other impedance in loop 2
$M = \sqrt{L_1 L_2}$
$k = 1$

FIG. 10 Properties of ideal transformer.

As stated at the beginning of this article, the behavior of an actual transformer can often be approximately described by that of an ideal transformer. Moreover, the fact that the actual transformer possesses approximately the properties of the ideal is the chief cause of its being such a valuable piece of electrical apparatus, as can be seen from a brief consideration of some of its uses.

In power transmission the principal energy losses are the I^2R losses which can be reduced by decreasing the current. Losses in insulation which vary in general with V^2 are usually small in comparison. Therefore power is usually transmitted at high voltage and low current. However, because of safety and insulation problems in devices for using power, a relatively low voltage is desirable for them. Both requirements can be satisfied by transmitting the power at high voltage with the accompany-

ing low losses, and by stepping down the voltage and stepping up the current at the load with a transformer. For example, in a large power development electrical energy may be generated at 13,000 volts, stepped up to 220,000 volts for transmission, stepped down to 33,000 volts for large-area distribution and eventually stepped down again to 2,300 volts for industrial loads and to 230 and 115 volts for small power and domestic use. For each transformation a power transformer has a power loss of a fraction of one per cent of its full-load rating for large units to a few per cent for small units.

In communications work the transformer is not operated as an efficient device for changing the operating voltage of the circuit; it is employed as a means of changing the apparent impedance of a load to fit the particular requirements of the circuit. A typical problem of this sort involves the delivery of audio-frequency power from a vacuum-tube source having an effective internal resistance of perhaps 50,000 ohms or more to a loud-speaker coil having an effective resistance of perhaps 50 ohms. If the source voltage is 100 volts, the load receives about 2×10^{-4} watt when directly connected to the source. If, on the other hand, an ideal transformer of ratio $\sqrt{50{,}000/50}$ or $\sqrt{1{,}000}$ is used to make the 50 ohms appear as $a^2 50$, or 50,000 ohms, to the source, the load receives about 0.05 watt or about 250 times as much power as when direct connected. An actual transformer delivers perhaps 90 per cent of the power calculated for an ideal transformer in this case. The great value of the transformer is at once apparent. In comparing an actual transformer with its ideal in considering a communication problem such as that above, not only the question of efficiency but also that of fidelity of reproduction of the applied force as a function of time is, of course, important. This question cannot be discussed adequately at this time because it depends fundamentally upon the consideration of transient response, which has not yet been treated for this circuit. However, it may be stated merely for the sake of completeness that the ideal transformer as introduced here is ideal also from the standpoint of fidelity requirement, and that actual transformers may be made to yield sufficiently good results for practical purposes, the comparison being made on the basis of frequency response characteristics, about which more is said in this series in the volume on magnetic circuits and transformers. Certain second-order effects that arise in the use of actual transformers and various special considerations that apply in certain cases are also discussed in that volume.

12. Equivalent current and voltage sources

In the preceding articles of this chapter, a network is analyzed on the loop basis by the formulation of loop-voltage equations. In Art. 13 the

alternative method is developed, namely, a formulation in which the equilibrium conditions are expressed by summing the currents at a node to zero.

In the loop method it is assumed that the sources are voltage sources because this type of source is convenient to use in the equations. In the node method, current sources are the natural ones to use. Fortunately a source can be represented equally well on either the current or the voltage basis and can thereby be adapted to either method of network formulation. Of course one can write the loop equations in terms of current sources, and the node equations in terms of voltage sources, but it is probably less confusing to change the sources to fit the equations as written.

In Ch. II, it is demonstrated that a constant-voltage source can usually be represented by an electromotive force which is independent of the current, in series with a resistance. It is further shown that such a voltage source may, for purposes of convenient use in the node method of circuit solution, be transformed to a mathematically equivalent current

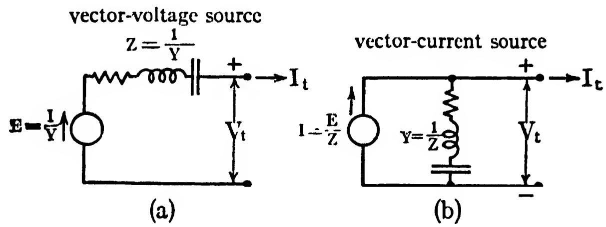


FIG. 11. Equivalent vector-voltage and vector-current sources.

source in parallel with a conductance. This transformation is perfectly general in the case of a direct-current circuit, applying to both transient and steady-state conditions, if only resistance is involved in the source.

In the case of a sinusoidal source, a similar equivalence applying only to steady-state conditions is useful in the solution of two-loop circuits by the node method. Figure 11 illustrates the equivalent vector-voltage and vector-current sources. These equivalent sources have identical steady-state terminal characteristics when connected to an external circuit, but it should be remembered that the internal power losses are not identical. The proof of the equivalence of the two sources of Fig. 11 is analogous to the direct-current case, Art. 4, Ch. II.

In applying the equivalent current source to the node method of circuit analysis, the shunt admittance Y is often converted to three parallel elements, containing conductance, capacitance, and reciprocal inductance, by the conversion methods outlined in Art. 17, Ch. IV.

13. Differential equations of the two-node network

At the beginning of this chapter it is pointed out that there are two general methods of formulating the equations for a linear passive network, one in terms of loop voltages, the other in terms of node currents. Paralleling the presentation of the loop method of analysis, the two-node network is now treated by the node method. Because of the similarity between the forms of the loop and the node equations mathematically, however, it is unnecessary to carry through the solution of the node equations in detail; they are the duals of the loop-network solutions and can be written immediately by analogy.

In Fig. 12 is shown a two-node network of nearly general form, with two current sources. All the possible parameters are included in Fig. 12, except magnetic coupling between the reciprocal inductances. While this can be included in a node formulation, it introduces a slight complication, so that the analysis of a network including magnetic coupling is better treated on the loop basis. This preference is reasonable since the effect of magnetic coupling is essentially that of a series voltage which enters better into loop-voltage equations than into node-current equations.

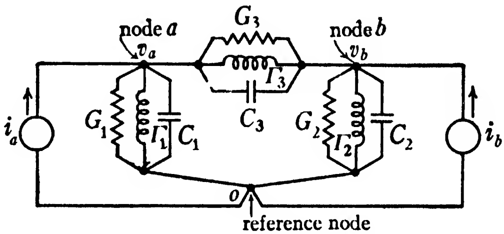


Fig. 12. Circuit diagram of a two-node network.

Node equations are an expression of Kirchhoff's current law, that the sum of the currents entering or leaving any junction is zero, the currents being expressed in terms of the node voltages and the circuit parameters. The node voltages are determined with respect to one arbitrarily selected reference node. The current between any two nodes is equal to the voltage drop between these nodes multiplied by the admittance of the path in which the current exists.

Before the differential equations are written, the subject of signs should be clearly understood. Arrows indicate the positive direction for source currents. The node potentials v_a and v_b are expressed as the amount by which the potentials of nodes a and b, respectively, are greater than that of the reference node. Thus v_a and v_b are the voltage drops from nodes a and b, respectively, to the reference node.

The equation for the sum of the currents directed away from node a is written first, using the principle of superposition. Because v_a acts alone, v_b being zero, the current directed away from node a toward the reference

node (through G_1, Γ_1, and C_1) is

$$i'_a = G_1 v_a + C_1 \frac{dv_a}{dt} + \Gamma_1 \int v_a dt. \tag{68}$$

The current directed away from node a toward node b is

$$i'_{ab} = G_3 v_a + C_3 \frac{dv_a}{dt} + \Gamma_3 \int v_a dt. \tag{69}$$

If v_b acts alone and v_a is zero, the current directed away from node a toward node b is

$$i''_{ab} = -G_3 v_b - C_3 \frac{dv_b}{dt} - \Gamma_3 \int v_b dt. \tag{70}$$

Equating these expressions to i_a, which is directed toward node a, gives

$$\left.\begin{aligned} G_1 v_a + C_1 \frac{dv_a}{dt} + \Gamma_1 \int v_a dt + G_3 v_a + C_3 \frac{dv_a}{dt} + \Gamma_3 \int v_a dt \\ - G_3 v_b - C_3 \frac{dv_b}{dt} - \Gamma_3 \int v_b dt = i_a. \end{aligned}\right\} \tag{71}$$

By the same process of reasoning, the corresponding equation for the total current directed away from node b is determined to be:

$$\left.\begin{aligned} -G_3 v_a - C_3 \frac{dv_a}{dt} - \Gamma_3 \int v_a dt + G_2 v_b + C_2 \frac{dv_b}{dt} + \Gamma_2 \int v_b dt \\ + G_3 v_b + C_3 \frac{dv_b}{dt} + \Gamma_3 \int v_b dt = i_b. \end{aligned}\right\} \tag{72}$$

Already the similarity of form between the node equations for the two-node network and the loop equations for the two-loop network is apparent if Eqs. 71 and 72 are compared with Eqs. 1 and 2, respectively. It is seen that with the understanding that common capacitance is the dual of both common self-inductance and mutual inductance — that is, that capacitance between two nodes is the dual of inductance common to two loops – the equations are duals in form, term for term. This being the case, Eqs. 71 and 72 can be treated by exactly the same methods as those used for the solutions of Eqs. 1 and 2. The coefficients of like terms in the v's are collected, letting

$$G_{aa} \equiv G_1 + G_3 = \text{total or self-conductance of node } a, \tag{73}$$

$$C_{aa} \equiv C_1 + C_3 = \text{total or self-capacitance of node } a, \tag{74}$$

$$\Gamma_{aa} \equiv \Gamma_1 + \Gamma_3 = \text{total or self-reciprocal inductance of node } a, \tag{75}$$

$$G_{bb} \equiv G_2 + G_3 = \text{total or self-conductance of node } b, \tag{76}$$

$$C_{bb} \equiv C_2 + C_3 = \text{total or self-capacitance of node } b, \tag{77}$$

$$\Gamma_{bb} \equiv \Gamma_2 + \Gamma_3 = \text{total or self-reciprocal inductance of node } a, \tag{78}$$

$$G_{ab} \equiv G_{ba} = -G_3 = \text{mutual conductance of nodes } a \text{ and } b, \tag{79}$$

$$C_{ab} \equiv C_{ba} = -C_3 = \text{mutual capacitance of nodes } a \text{ and } b, \tag{80}$$

$$\Gamma_{ab} \equiv \Gamma_{ba} = -\Gamma_3 = \text{mutual reciprocal inductance of nodes } a \text{ and } b. \tag{81}$$

Substituting Eqs. 73 to 81 in Eqs. 71 and 72 gives as the equations upon which the solution is based:

$$\left.\begin{aligned} G_{aa}v_a + C_{aa}\frac{dv_a}{dt} + \Gamma_{aa}\int v_a dt + G_{ab}v_b + C_{ab}\frac{dv_b}{dt} \\ + \Gamma_{ab}\int v_b dt = i_a, \end{aligned}\right\} \tag{71a}$$

$$\left.\begin{aligned} G_{ba}v_a + C_{ba}\frac{dv_a}{dt} + \Gamma_{ba}\int v_a dt + G_{bb}v_b + C_{bb}\frac{dv_b}{dt} \\ + \Gamma_{bb}\int v_b dt = i_b. \end{aligned}\right\} \tag{72a}$$

Equations 71a and 72a are the duals of Eqs. 1b and 2b, respectively.

14. Steady-state solution of two-node network

If the currents impressed on the two-node network have the forms

$$i_a = I_{am}\cos(\omega t + \phi_1), \tag{82}$$

$$i_b = I_{bm}\cos(\omega t + \phi_2), \tag{83}$$

the resulting node voltages have the forms

$$v_a = V_{am}\cos(\omega t + \psi_1), \tag{84}$$

$$v_b = V_{bm}\cos(\omega t + \psi_2), \tag{85}$$

in which the V_m's and ψ's must be determined from the differential equations 71a and 72a and the known I's and ϕ's. By following steps entirely analogous to those used in the two-loop case and by letting

$$\mathbf{Y}_{aa} = G_{aa} + j\left(\omega C_{aa} - \frac{\Gamma_{aa}}{\omega}\right), \tag{86}$$

$$\mathbf{Y}_{ba} = \mathbf{Y}_{ab} = G_{ab} + j\left(\omega C_{ab} - \frac{\Gamma_{ab}}{\omega}\right), \tag{87}$$

$$\mathbf{Y}_{bb} = G_{bb} + j\left(\omega C_{bb} - \frac{\Gamma_{bb}}{\omega}\right), \tag{88}$$

the differential equations 71a and 72a can be replaced by

$$Y_{aa}V_a + Y_{ab}V_b = I_a, \qquad \blacktriangleright[71b]$$

$$\mathbf{Y}_{ba}\mathbf{V}_a + \mathbf{Y}_{bb}\mathbf{V}_b = \mathbf{I}_b, \qquad \blacktriangleright[72b]$$

in which the V's and I's have these relations:

$$\mathbf{I}_a = \frac{I_{am}}{\sqrt{2}}\epsilon^{j\phi_1}, \qquad [89]$$

$$\mathbf{I}_b = \frac{I_{bm}}{\sqrt{2}}\epsilon^{j\phi_2}, \qquad [90]$$

$$\mathbf{V}_a = \frac{V_{am}}{\sqrt{2}}\epsilon^{j\psi_1}, \qquad [91]$$

$$\mathbf{V}_b = \frac{V_{bm}}{\sqrt{2}}\epsilon^{j\psi_2}. \qquad [92]$$

From Eqs. 71b and 72b the values of the node vector voltages are found to be

$$\mathbf{V}_a = \frac{\mathbf{Y}_{bb}\mathbf{I}_a - \mathbf{Y}_{ab}\mathbf{I}_b}{\mathbf{Y}_{aa}\mathbf{Y}_{bb} - \mathbf{Y}_{ab}^2}, \qquad [93]$$

$$\mathbf{V}_b = \frac{-\mathbf{Y}_{ab}\mathbf{I}_a + \mathbf{Y}_{aa}\mathbf{I}_b}{\mathbf{Y}_{aa}\mathbf{Y}_{bb} - \mathbf{Y}_{ab}^2}. \qquad [94]$$

In the two-node case the apparent admittance $\mathbf{Y}_{1a}$ of the network viewed from the node-a current source when the node-b current source is replaced by a load admittance $\mathbf{Y}_L$ (which is included in the node-b self-admittance $\mathbf{Y}_{bb}$) is

$$\mathbf{Y}_{1a} = \mathbf{Y}_{aa} - \frac{\mathbf{Y}_{ab}^2}{\mathbf{Y}_{bb}}. \qquad \blacktriangleright[95]$$

In this same network, with the i_b source replaced by an admittance $\mathbf{Y}_L$, the ratio of the node voltages is

$$\frac{\mathbf{V}_a}{\mathbf{V}_b} = -\frac{\mathbf{Y}_{bb}}{\mathbf{Y}_{ab}}. \qquad \blacktriangleright[96]$$

The ratio of the current in the load admittance to the source current is

$$\frac{\mathbf{I}_L}{\mathbf{I}_a} = \frac{-\mathbf{Y}_L\mathbf{Y}_{ab}}{\mathbf{Y}_{aa}\mathbf{Y}_{bb} - \mathbf{Y}_{ab}^2}. \qquad \blacktriangleright[97]$$

The polarities and arrow directions are so assumed in Fig. 12 that the signs in Eqs. 95, 96, and 97 are the same as in the corresponding equations for the two-loop network, Eqs. 36a, 37, and 40, respectively.

15. Direct formulation of the steady-state vector node equations

If only the steady-state solution of a network is desired, it is usually more convenient to write equations of the form of Eqs. 71b and 72b directly by inspection of the circuit diagram rather than to obtain them by way of the differential equations. Another advantage of writing the steady-state equations directly is that by this means the number of nodes, and therefore the number of equations, can often be reduced, as is shown in an illustrative circuit such as Fig. 13. If the differential equations are written, this network has four independent nodes, a, b, c, and d, because a differential expression for the current in the G_3, Γ_3, C_3 branch cannot be written in terms of the potentials of nodes a and b alone. More generally it can be said that, in order to write the differential equations by the node method as developed here, every connection between elements must be made a node. For the circuit of Fig. 13 these equations, for currents directed away from the nodes, become

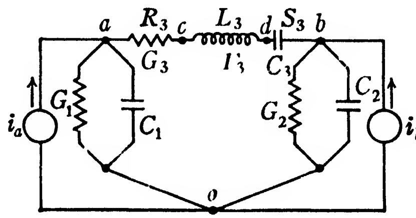


FIG. 13. Four-node network solved as two-node network.

$$(G_1 + G_3)v_a + C_1 \frac{dv_a}{dt} - G_3 v_c = i_a, \tag{98}$$

$$-G_3 v_a + G_3 v_c + \Gamma_3 \int v_c dt - \Gamma_3 \int v_d dt = 0, \tag{99}$$

$$-\Gamma_3 \int v_c dt + \Gamma_3 \int v_d dt + C_3 \frac{dv_d}{dt} - C_3 \frac{dv_b}{dt} = 0, \tag{100}$$

$$G_2 v_b + (C_2 + C_3) \frac{dv_b}{dt} - C_3 \frac{dv_d}{dt} = i_b. \tag{101}$$

This is a rather complicated formulation, considering the simplicity of the circuit

For steady-state analysis, however, as can be seen from Art. 17, Ch. IV, only those points where more than two elements join need be considered, provided the meaning of the admittances Y_{aa}, Y_{ab}, and Y_{bb} of Eqs. 71b and 72b is correctly understood. The total vector current directed away from node a is $Y_{aa}V_a$ when the vector potential of this node is V_a, and the vector potential V_b is zero. Also $Y_{ab}V_b$ is the vector current directed away from node a towards node b when V_a is zero. Similar interpretations apply to the other terms of Eqs. 71b and 72b. If these ideas are applied

to the circuit of Fig. 13, two nodes, a and b, suffice for the solution. The Y's are expressed as follows:

$$Y_{aa} = G_1 + j\omega C_1 + \frac{1}{R_3 + j\left(\omega L_3 - \dfrac{S_3}{\omega}\right)}, \qquad [102]$$

$$Y_{ab} = \frac{-1}{R_3 + j\left(\omega L_3 - \dfrac{S_3}{\omega}\right)}, \qquad [103]$$

$$Y_{bb} = G_2 + j\omega C_2 + \frac{1}{R_3 + j\left(\omega L_3 - \dfrac{S_3}{\omega}\right)}. \qquad [104]$$

In Fig. 13, the elements in branch a–b are labeled in both possible ways, R_3 and G_3, L_3 and Γ_3, and S_3 and C_3, for convenience in writing the equations, it being understood that the paired symbols are reciprocals. It is easily seen that for actual steady-state calculations, Eqs. 71b and 72b with the Y's as given by Eqs. 102 to 104 offer a much more direct method for obtaining the steady-state solution than do Eqs. 98 to 101. Both, of course, yield the same result.

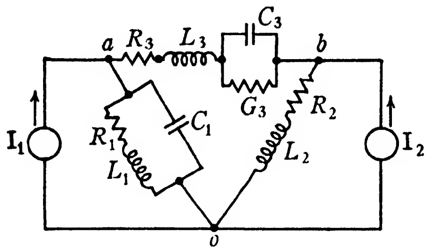


FIG. 14. Six-node network.

This idea of combining elements in steady-state calculations can be carried further. Any one of the three branches a–o, a–b, and b–o can be a series-parallel combination of elements and still the alternating-current steady-state equilibrium conditions can be expressed by Eqs. 71b and 72b. For example, the circuit of Fig. 14 can be analyzed in the alternating-current steady state as a two-node network by Eqs. 71b and 72b if Y_{aa} is called the admittance of the complicated branches a–o and a–b in parallel, $-Y_{ab}$ the admittance of branch a–b, and Y_{bb} the combined admittance of branches a–b and b–o in parallel. Thus for this circuit,

$$Y_{aa} = j\omega C_1 + \frac{1}{R_1 + j\omega L_1} - Y_{ab}, \qquad [102a]$$

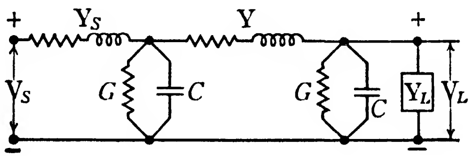
$$Y_{ab} = \frac{-1}{R_3 + j\omega L_3 + \dfrac{1}{G_3 + j\omega C_3}}, \qquad [103a]$$

$$Y_{bb} = \frac{1}{R_2 + j\omega L_2} - Y_{ab}. \qquad [104a]$$

While Eqs. 102a to 104a look complex, it is probably easier to analyze this circuit as a two-node circuit with complicated branches, than as a six-node circuit with single elements as branches. This advantage becomes more evident in connection with the analysis of the n-node case in Ch. VIII.

16. Illustrative example of two-node network

For the circuit of Fig. 15, the receiving-end voltage V_L is desired. The constants Y, G, C approximate those of a circuit equivalent to the



$$
\begin{aligned}
V_S &= 150 \text{ v}, 100 \sim \\
\mathrm{Y}_S &= 2{,}000 - j200 \ \mu \text{ mhos} \\
\mathrm{Y} &= 1{,}000 - j100 \ \mu \text{ mhos} \\
G &= 100 \ \mu \text{ mhos} \\
C &= 5.0 \ \mu\text{f} \\
\mathrm{Y}_L &= 0.002 - j0.0015 \text{ mho}
\end{aligned}
$$

FIG. 15. Equivalent circuit of submarine cable.

submarine cable laid in 1924 between Aldeburgh, Suffolk, England, and Dombert, Walcheren, on the Dutch coast, a cable length of 86 nautical miles.

Solution: The sending-end admittance Y_s and the sending-end voltage V_s are replaced by a current source (Fig. 16) in which

$$\mathrm{I}_s = 150(0.002 - j0.0002) \text{ amp.} \qquad [105]$$

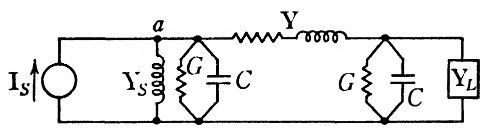


FIG. 16. Circuit of Fig. 15 but with current source.

The node equations are,

$$\mathrm{V}_a(\mathrm{Y}_s + G + j\omega C + \mathrm{Y}) - \mathrm{V}_L\mathrm{Y} = \mathrm{I}_s, \qquad [71c]$$

$$-\mathrm{V}_a\mathrm{Y} + \mathrm{V}_L(\mathrm{Y} + G + j\omega C + \mathrm{Y}_L) = 0, \qquad [72c]$$

$$\left.\begin{aligned}\mathrm{Y}_s + G + j\omega C + \mathrm{Y} &= 0.002 - j0.0002 + 0.0001 + j0.0031 \\ &\quad + 0.001 - j0.0001 = 0.0031 + j0.0028.\end{aligned}\right\} \qquad [102b]$$

$$\left.\begin{aligned}\mathrm{Y} + G + j\omega C + \mathrm{Y}_L &= 0.001 - j0.0001 + 0.0001 + j0.0031 \\ &\quad + 0.002 - j0.0015 = 0.0031 + j0.0015.\end{aligned}\right\} \qquad [104b]$$

From Eq. 72c

$$\mathbf{V}_a = \mathbf{V}_L \frac{0.0031 + j0.0015}{0.001 - j0.0001} = \mathbf{V}_L(2.92 + j1.18). \qquad [106]$$

Hence, by using Eq. 71c,

$$\left.\begin{aligned} 150(0.002 - j0.0002) &= \mathbf{V}_L[(2.92 + j1.18)(0.0031 + j0.0028) - 0.001 \\ &+ j0.0001] = \mathbf{V}_L(0.0057 + j0.0120) \end{aligned}\right\} \qquad [71d]$$

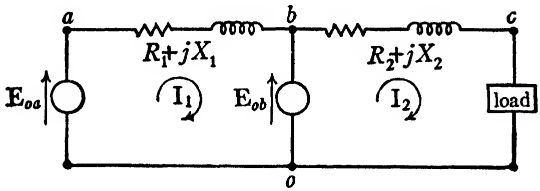
whence

$$\mathbf{V}_L = 7.6 - j21.3 \text{ v}, \qquad [107]$$

$$V_L = 22.6 \text{ v}. \qquad [107a]$$

17. Illustrative example of graphical solution of two-loop network

This is a situation in which currents and voltages are known in magnitude only. The power taken by the load and the power delivered by each of the two sources (Fig. 17) are desired.



$E_{oa} = 130$ v $\qquad I_{ab} = 30.7$ amp
$E_{ob} = 110$ v $\qquad I_{bc} = 20.0$ amp
$V_{co} = 120$ v

$R_1 + jX_1 = 0.7 + j0.26$ ohm
$R_2 + jX_2 = 1.5 + j3.0$ ohms

FIG. 17. Two-loop network used in Art. 17.

Solution: The steps in constructing the vector diagram (Fig. 18) are:

a. The load voltage vector $\mathbf{V}_{co}$ is taken as $120\underline{/0°}$ v.
b. With O as a center, an arc of radius E_{ob}, or 110 v, is described.
c. The voltage $\mathbf{V}_{bc}$ is calculated:

$$\mathbf{V}_{bc} = 20\underline{/\delta}(1.5 + j3.0) = (30 + j60)\underline{/\delta}, \qquad [108]$$

$$V_{bc} = 67 \text{ v}. \qquad [108a]$$

d. With the tip P, of the $\mathbf{V}_{co}$ vector, as a center, an arc of radius 67 v is described. The intersection of this arc with the arc of (b) fixes the $\mathbf{E}_{ob}$ and the $\mathbf{I}_2\mathbf{Z}_2$ vectors.
e. With the $\mathbf{I}_2\mathbf{Z}_2$ vector as hypotenuse, the right triangle of legs I_2R_2, or 30 v, and I_2X_2, or 60 v, is constructed.
f. The current vector $\mathbf{I}_2$ is constructed parallel to $\mathbf{I}_2R_2$, of length 20 amp.
g. With O as a center, an arc of radius E_{oa}, or 130 v, is described.

h. The voltage V_{ab} is calculated

$$\mathrm{V}_{ab} = 30.7\underline{/\gamma}(0.7 + j0.26) = (21.5 + j7.8)\underline{/\gamma}, \quad [109]$$

$$V_{ab} = 22.9 \text{ v} \quad [109a]$$

i. With the tip Q of the E_{ob} vector as a center, an arc of radius 22.9 v is described. The intersection of this arc with the arc of (g) fixes the E_{oa} and the $\mathrm{I}_1\mathrm{Z}_1$ vectors.
j. With the $\mathrm{I}_1\mathrm{Z}_1$ vector as hypotenuse, the right triangle of legs I_1R_1, or 21.5 v, and I_1X_1, or 7.8 v, is constructed.
k. The current vector I_1 is constructed parallel to I_1R_1, of length 30.7 amp.
l. The current vector

$$\mathrm{I}_{ob} = \mathrm{I}_2 - \mathrm{I}_1 \quad [110]$$

is constructed by vector subtraction.

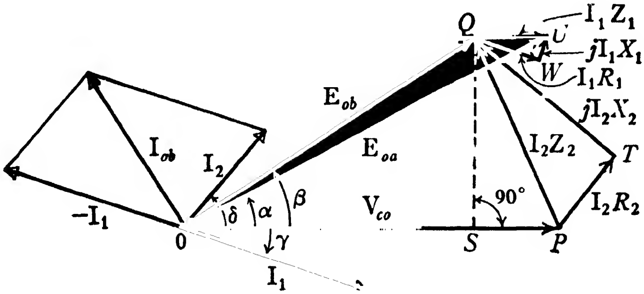


FIG. 18. Vector diagram of relations in circuit of Fig. 17.

The above steps may all be done roughly, and the problem may then be analyzed in the following manner. The three sides of triangle OQP are known. Therefore, from the cosine law,

$$\cos\beta = \frac{120^2 + 110^2 - 67^2}{2(110)(120)} = \frac{22{,}000}{26{,}400} = 0.833, \quad [111]$$

$$\beta = 33.5°, \quad [112]$$

$$SP = 120 - 110\cos\beta = 120 - 91.5 = 28.5 \text{ v}, \quad [113]$$

$$\cos\underline{/SPQ} = \frac{SP}{PQ} = \frac{28.5}{67} = 0.425, \quad [114]$$

$$\underline{/SPQ} = 64.9°. \quad [115]$$

Since I_2 is parallel to PT,

$$\delta = 180° - (\underline{/SPQ} + \underline{/QPT}) = 180° - (64.9° + 63.4°) = 51.7°. \quad [116]$$

In triangle OQU the three sides are known. Hence

$$\cos(\beta - \alpha) = \frac{110^2 + 130^2 - 22.9^2}{2(110)(130)} = \frac{28{,}500}{28{,}600} = 0.996, \quad [117]$$

$$\beta - \alpha = 5.1°, \quad [118]$$

$$\alpha = \beta - 5.1° = 28.4°. \quad [119]$$

There are two possible positions of the vector E_{oa}, and two corresponding values of α, but, for simplicity, only one is used here.

$$\cos \underline{/QUO} = \frac{22.9^2 + 130^2 - 110^2}{2(22.9)(130)} = \frac{5{,}320}{5{,}950} = 0.89, \qquad [120]$$

$$\underline{/QUO} = 26.6°. \qquad [121]$$

Therefore the line QU makes an angle of $+1.8°$ with the reference axis. The angle UQW is $\tan^{-1}(7.8/21.5)$ or $20.1°$. Therefore QW, representing the vector $\mathbf{I}_1 R_1$, makes an angle of $-18.3°$ with the reference axis. Since the vector $\mathbf{I}_1$ is parallel to the vector $\mathbf{I}_1 R_1$, the angle γ is $-18.3°$.

As all the phase angles α, β, γ, and δ are now known, one solution of the problem to determine source and load powers may be completed readily. It should be noted that for completeness, all possible intersections of the various arcs used in constructing the diagram must be considered in the light of circuit conditions. Additional data may be needed to fix one solution definitely as the only possible one.

PROBLEMS

1. For the circuit of Fig. 19:
(a) What is the potential difference across the open switch?
(b) What is the current in the generator?
(c) If the switch is closed, what is the current in the generator?
(d) What is the current in the closed switch?
(e) Vector diagrams are to be drawn showing all currents and voltages for each switch position.

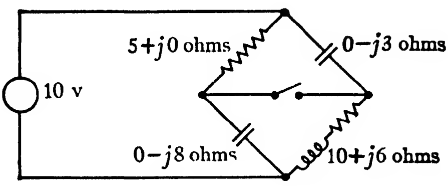


FIG. 19. Two-loop network for Prob. 1.

2. For what value of E is the current 4 amp in the condenser of Fig. 20?

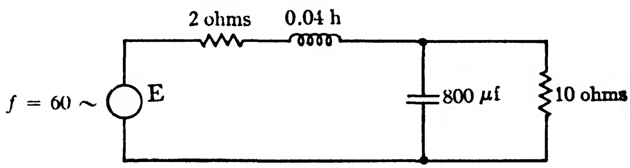


FIG. 20 Two-loop circuit for Prob. 2.

3. If the voltage across the condenser is 500 v, what is the value of E in Fig. 21?

4. A power-transmission line and terminal transformers are represented approximately by the equivalent T-circuit of Fig. 22. What is the load voltage?

5. Two generating stations supply a common load through transmission lines as indicated in Fig. 23.

(a) What is the load current?
(b) What power does each station supply?
(c) What power does the load receive?

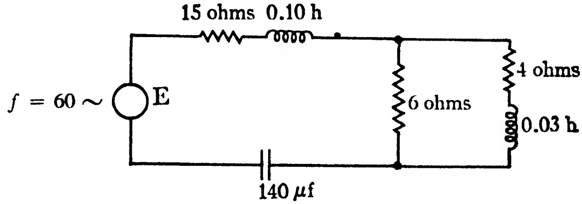
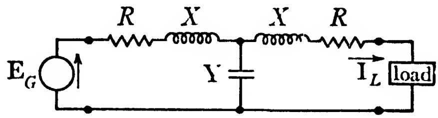
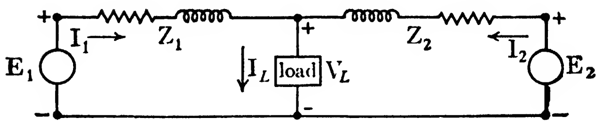


FIG. 21. Two-loop circuit for Prob. 3.



$R = 10$ ohms, $X = 100$ ohms
$Y = j10^{-3}$ mho $E_G = 100{,}000$ v,
$I_L = 100$ amp
power factor of load = 0.80 lagging

FIG. 22. Equivalent circuit of transmission line, Prob. 4.



$E_1 = 1{,}910$ v, $V_L = 2{,}000$ v, $E_2 = 2{,}150$ v
$I_1 = 50$ amp, $I_2 = 86.6$ amp,
$Z_1 = 2.0 + j3.46$ ohms, $Z_2 = 1.0 + j1.73$ ohms

FIG. 23. Simplified power network, Prob. 5.

6. The circuit shown in Fig 24 represents a transformer for which the inductance parameters may be assumed essentially constant All losses are neglected. For the assumed reference arrows, the following equilibrium equations may be written.

$$j\omega L_1 \mathrm{I}_1 + j\omega M \mathrm{I}_2 = \mathrm{E}_1, \qquad [1i]$$

$$j\omega M \mathrm{I}_1 + j\omega L_2 \mathrm{I}_2 = \mathrm{E}_2. \qquad [2k]$$

(a) By introducing any factor m in the following way,

$$j\omega L_1 \mathrm{I}_1 + j\omega \frac{M}{m}(m\mathrm{I}_2) = \mathrm{E}_1, \qquad [1k]$$

$$j\omega \frac{M}{m}\mathrm{I}_1 + j\omega \frac{L_2}{m^2}(m\mathrm{I}_2) = \frac{\mathrm{E}_2}{m}, \qquad [2m]$$

can the corresponding circuit be represented as shown in Fig. 24a, or, alternatively as shown in Fig. 24b?

(b) What is the value for m which leads to the equivalent circuit of Fig. 24c, and what are the values of the inductance parameters L_a and L_b? The necessary m value is to be expressed in terms of the nominal transformer ratio

$$a = \sqrt{\frac{L_1}{L_2}} \qquad [66a]$$

and the coupling coefficient

$$k = \frac{M}{\sqrt{L_1 L_2}}. \qquad [64a]$$

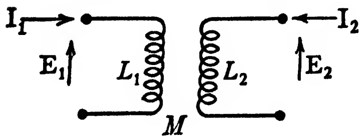


Fig. 24. Transformer circuit for Prob. 6.

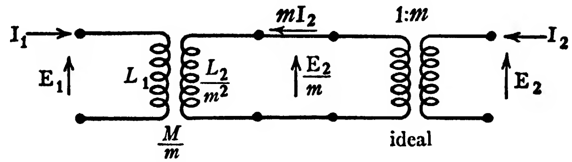


Fig. 24a. First alternative circuit for Prob. 6.

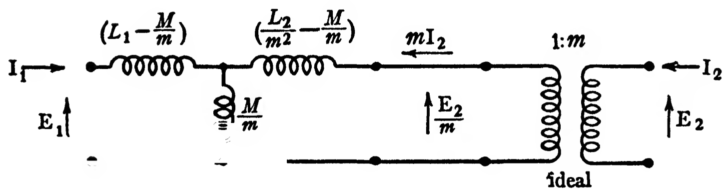


Fig. 24b. Second alternative circuit for Prob. 6.

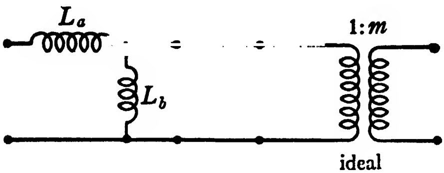


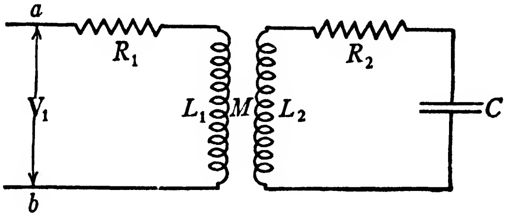
Fig. 24c. Another equivalent circuit for Prob. 6.

7. Figure 25 is the schematic diagram of a resonating air-core transformer circuit. What are the input impedances of this transformer, that is, the apparent impedances measured across the terminals ab, at 997, 1,000, 1,003, and 1,006 kc/sec?

8. A vacuum-tube oscillator has an internal resistance of 800 ohms, negligible internal reactance, and an effective internal voltage of 5.0 v at 1,000 ~. Power is to be supplied from this source to a pure resistance load of 20 ohms, either by connecting the load directly to the oscillator or through an iron-core transformer having self-inductances of 1.2 h and 0.030 h and coefficient of coupling 95%. For the purpose of the problem the inductances may be considered constant and the transformer resistances and core loss may be neglected.

(a) With the apparatus available, how should the load be connected in order that it may receive the most power? What is that power?

(b) A vector diagram is to be sketched for (a) showing the internal voltage of the source, the terminal voltage of the source, the voltage across the load, the current in the source, and the current in the load. The voltages and currents are to be indicated on a circuit diagram and the direction in which each is considered is to be made clear.



$R_1 = 10.0$ ohms	$R_2 = 25.0$ ohms
$L_1 = 64.0\ \mu h$	$L_2 = 150\ \mu h$
$M = 49.0\ \mu h$	$C = 169\ \mu\mu f$

FIG. 25. Resonating air-core transformer circuit, Prob. 7.

9. A transformer has the following parameter values:

L_1	L_2	$\omega L_1/R_1$	$\omega L_2/R_2$	k
1.0 h	0.20 h	200	200	0.995

A voltage source having an internal resistance of 500 ohms and an angular frequency of 5,000 radians/sec is connected to the primary, and a load resistance of 100 ohms is connected to the secondary.

(a) What is the power delivered to the load for a 1-v source? What is the power delivered for source voltages 1/10th, 10 times, 100 times as great?

(b) What is the power for unit source voltage if the transformer is ideal and has the same inductance ratio?

(c) What is the power for unit source voltage with no transformer between the source and load?

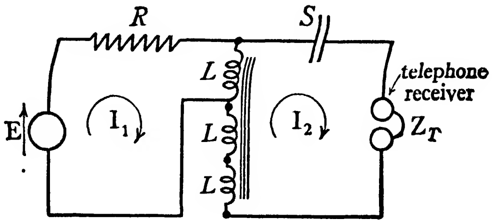
10. A telephone receiving circuit is to be coupled to a transmitter through a transformer. The impedance of the receiving circuit is $260\epsilon^{j(\pi/4)}$ ohms at 1,000 ~. The constants of the transformer are:

R_1	R_2	L_1	L_2	M
43 ohms	43 ohms	3.6 h	3.6 h	3.5 h

If R_1, R_2, L_1, L_2, and M are assumed to be constant, what is the apparent impedance of the receiving circuit plus transformer as viewed from the primary terminals of the transformer? The core loss of the transformer may be neglected as it has negligible effect under the conditions of operation.

11. For the circuit of Fig. 26, the transformer resistance and core loss may be neglected, and the coupling coefficient may be considered essentially unity. The windings are connected series aiding. How much power is delivered to the telephone receiver?

12. The successive steps in the conversion of the circuits shown in Fig. 27 are to be justified by applying the method of converting a voltage source in series with a circuit element into an equivalent current source in parallel with that element. In particular, for the frequency $1/2\pi\sqrt{LC}$ the circuit becomes that of Fig. 27c. On this basis the circuit to the left of the dotted line in Fig. 27a is to be designed so that it may represent a current source of 10 amp when E is 110 v and the frequency is 60 ~.



E = 20.0 v
ω = 2,000 radians/sec
R = 200 ohms
S = 36.0×10^6 darafs
Z_T = $1{,}790 + j17{,}900$ ohms
L = 1.0 h (each of three coils on common core)
M = 1.0 h (between any pair of coils)

FIG. 26. Telephone receiver connected through transformer, Prob. 11.

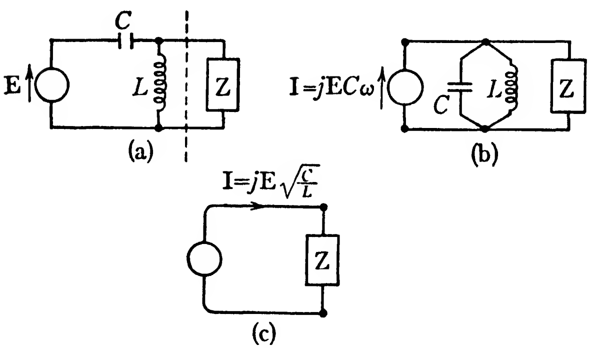


FIG. 27. Conversion of voltage and current sources, Prob. 12.

13. If the resistance of the coil in Prob. 12 is taken into account, the source circuit looks like Fig. 28a.

(a) Can this be replaced by the equivalent circuit of Fig. 28b in which

$$r = R(1 + Q^2), \tag{122}$$

$$L' = L(1 + Q^{-2}), \tag{123}$$

where

$$Q = \frac{\omega L}{R}, \tag{124}$$

so that for reasonably large values of Q as used in practice the approximate equivalent current source becomes that of Fig. 28c?

(b) If the Q of the coil is 200, the terminal voltage is 110 v at full load, the source operates at 10 amp, and the frequency of 60 ~ still is antiresonant, what are:

1. Full load terminal current?
2. Loss in the coil at full load?
3. Full-load power deliverable?
4. Efficiency at full load?

} with L and C as in Prob. 12, and a resistance load.

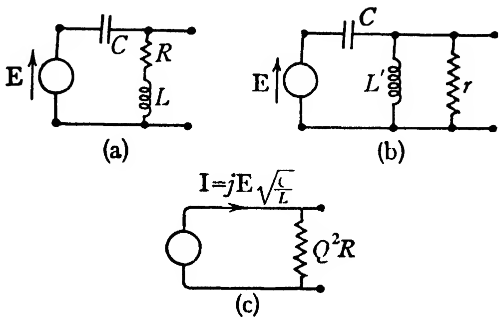


FIG. 28. Circuits of Fig. 27 modified to take into account resistance of coil, Prob. 13.

14. What is the expression for E_1/V_3 for the circuit of Fig. 29 in terms of the circuit parameters and ω?

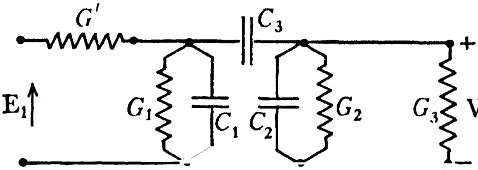


FIG. 29. Three-node network for Prob. 14.

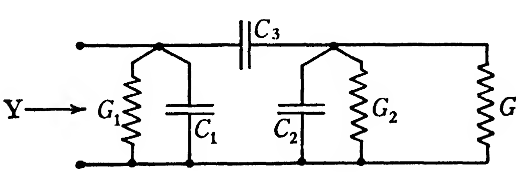


FIG. 30. Dual circuit of that of Prob. 9, for use in Prob. 15.

15. For the circuit of Fig. 30 the parameter values in farads and mhos are such that

$$C_1 + C_3 = 10^{-6}, \qquad [125]$$

$$C_2 + C_3 = \frac{10^{-6}}{5}, \qquad [126]$$

$$\frac{C_3}{\sqrt{(C_1 + C_3)(C_2 + C_3)}} = 0.995, \tag{127}$$

$$\frac{(C_1 + C_3)\omega}{G_1} = \frac{(C_2 + C_3)\omega}{G_2} = 200. \tag{128}$$

Also,

$$\omega = 5{,}000 \text{ radians/sec} \tag{129}$$

and

$$G = 10^{-4} \text{ mho.} \tag{130}$$

What is the input admittance Y? Is the circuit physically realizable? This is the dual of the circuit in Prob. 9.

CHAPTER VII

Transient Analysis of Alternating-Current Circuits Involving Two Unknowns

1. Transient and steady-state components

In Ch. VI the differential equations for alternating-current circuits involving two unknowns are developed and solved for steady-state conditions, by the use of both the loop method and the node method. For networks having two or more loops, just as for the simple single-loop circuits, the steady state is substantially reached only after an interval of time following any change in the circuit or the forces impressed upon it. During this interval the currents and charges are gradually adjusting themselves from values which they had at the instant that the change was made, to their steady-state values. In general, for constant or periodic impressed forces, it may be assumed that each current and charge is made up of two components, a steady-state component and a transient component.

The steady-state *component* can be considered as becoming established immediately following any change in circuit or source conditions. This concept may be difficult to accept at first, but it arises logically from the differential equations, and the solutions obtained on this assumption can always be verified experimentally as to *resultant* or *total* values of current and charge. Such experiments justify this method of solution in which the steady-state component is assumed to be established immediately. The steady-state components can be calculated readily by the methods previously discussed, when the electromotive forces are sinusoidal or constant voltages.

The idea of a transient component serves the same purpose in the multi-branch network as it does in the single-loop circuit. The current in a coil and the charge on a condenser cannot change instantaneously, or cannot be discontinuous functions, in any physical circuit. Since at the instant that a circuit or source change occurs, the current in each inductance and the charge in each capacitance are, in general, different from their new steady-state values, there must be another component of current or charge in each, which when added to the new steady-state value, makes the resultant equal to the actual existing current or charge. These components arise solely from the discrepancy between the existing and the new steady-state values and die out exactly as they would if they alone were present in the circuit, all sources of energy other than that stored in

the inductances and condensers being removed. Because of their transitory nature they are called transient components of charge and current. This explanation is a duplication of what is given in Chs. III and V, but the ideas involved underlie all linear-circuit behavior and can scarcely be overemphasized. Once these ideas are understood clearly, the theory of transient analysis in multibranch networks is quite simple and straightforward. Unfortunately, the arithmetical labor involved in making a complete transient analysis of any but the simplest of networks is extremely arduous. Two arithmetical tasks are involved: one the location of the roots of an algebraic equation, usually of the third or higher degree; the other the solution of a set of simultaneous linear equations having, in general, complex coefficients and unknowns.

The differential equations 1 and 2, p. 368, describe the behavior of the two-loop network of Fig. 1, p. 366, under all conditions. The transient components of currents are those values of current which satisfy the force-free network, in which the e's are made zero. The differential equations of the two loops then become

$$R_{11}i_1 + L_{11}\frac{di_1}{dt} + S_{11}\int i_1 dt + R_{12}i_2 + L_{12}\frac{di_2}{dt} + S_{12}\int i_2 dt = 0, \qquad [1]$$

$$R_{21}i_1 + L_{21}\frac{di_1}{dt} + S_{21}\int i_1 dt + R_{22}i_2 + L_{22}\frac{di_2}{dt} + S_{22}\int i_2 dt = 0. \qquad [2]$$

As in previous differential equations written with i rather than q for the dependent variable, the integrals of the currents in Eqs. 1 and 2 must be so interpreted as to include the total charges on the respective elastances at the instant under consideration.

If values of i can be found that satisfy Eqs. 1 and 2, those values can be added to the steady-state i's that satisfy Eqs. 1a and 2a, p. 369, and the sums also satisfy Eqs. 1a and 2a, p. 369. Mathematically, this procedure is that regularly used for the solution of simultaneous linear differential equations with constant coefficients, the solutions of Eqs. 1 and 2 being called the complementary functions, and the solutions of Eqs. 1a and 2a, p. 369, previously found in Eqs. 18 and 19, p. 372, being called the particular integrals. Physically, the solutions of Eqs. 1 and 2 provide the values of transient components of current, i_t, to add to the steady-state values for the complete solution in cases where transients may exist. From previous work it is known that exponential functions satisfy linear differential equations with constant coefficients. In addition, it may be assumed for trial, subject to verification, that the same exponent applies to the solutions of both loops, giving the assumed solutions

$$i_{1t} = A\epsilon^{pt} \qquad [3]$$

and

$$i_{2t} = B\epsilon^{pt} \qquad [4]$$

where A, B, and p are to be determined. In this treatment the coefficients A and B are not identical with the same symbols as used in Chs. III and V; they are more general and may turn out to be either real or complex. Substituting Eqs. 3 and 4 in Eqs. 1 and 2 and performing the indicated operations give

$$\left(R_{11} + pL_{11} + \frac{S_{11}}{p}\right)A\epsilon^{pt} + \left(R_{12} + pL_{12} + \frac{S_{12}}{p}\right)B\epsilon^{pt} = 0, \qquad [1a]$$

$$\left(R_{21} + pL_{21} + \frac{S_{21}}{p}\right)A\epsilon^{pt} + \left(R_{22} + pL_{22} + \frac{S_{22}}{p}\right)B\epsilon^{pt} = 0. \qquad [2a]$$

The expressions in parentheses are very similar to those for the steady state in Eqs. 1e and 2e, p. 373; hence, they are called transient impedances as follows, the functional notation Z(p) being used to distinguish them from the corresponding steady-state impedances $Z(j\omega)$, (which are abbreviated to merely Z):

$$Z_{11}(p) \equiv R_{11} + pL_{11} + \frac{S_{11}}{p}, \qquad [5]$$

$$Z_{12}(p) \equiv Z_{21}(p) \equiv R_{12} + pL_{12} + \frac{S_{12}}{p}, \qquad [6]$$

$$Z_{22}(p) \equiv R_{22} + pL_{22} + \frac{S_{22}}{p}. \qquad [7]$$

Putting Eqs. 5 to 7 in Eqs. 1a and 2a and dividing out the common term ϵ^{pt} give

$$Z_{11}(p)A + Z_{12}(p)B = 0, \qquad [1b]$$

$$Z_{12}(p)A + Z_{22}(p)B = 0. \qquad [2b]$$

Equations 1b and 2b have as solutions

$$A = \frac{0}{Z_{11}(p)Z_{22}(p) - [Z_{12}(p)]^2} = B, \qquad [8]$$

which are trivial if the denominator is nonvanishing. That is, zero currents are a mathematically correct solution of Eqs. 1b and 2b but one which is of no importance. If, on the other hand, the denominator vanishes, the indeterminate forms

$$A = \frac{0}{0} \qquad [9]$$

and

$$B = \frac{0}{0} \qquad [10]$$

result. These are in reality precisely what are wanted, since A and B must be fixed by the particular initial conditions of the problem and not by the basic differential equation if the transient is to serve its purpose as a buffer between the initial and steady-state conditions. Consequently the possibility of the denominator being zero is investigated.

2. The characteristic equation

The denominator of Eq. 8 is indicated by D(p), which is a polynomial in p as can be seen by expanding it. Thus

$$\left.\begin{aligned} D(p) &= Z_{11}(p)Z_{22}(p) - [Z_{12}(p)]^2 = 0 \\ &= p^2(L_{11}L_{22} - L_{12}^2) + p(R_{11}L_{22} + R_{22}L_{11} - 2R_{12}L_{12}) \\ &\quad + (R_{11}R_{22} - R_{12}^2 + L_{11}S_{22} + L_{22}S_{11} - 2L_{12}S_{12}) \\ &\quad + \frac{1}{p}(R_{11}S_{22} + R_{22}S_{11} - 2R_{12}S_{12}) \\ &\quad + \frac{1}{p^2}(S_{11}S_{22} - S_{12}^2) = 0. \end{aligned}\right\} \qquad [11]$$

Equation 11 is called the *characteristic* or *determinantal equation* of this network. Since, in general, p equal to zero is not expected to be a root of Eq. 11, it can be multiplied by p^2 to obtain

$$a_1p^4 + a_2p^3 + a_3p^2 + a_4p + a_5 = 0, \qquad [11a]$$

in which the coefficients are indicated by a's merely for ease in writing. These a's are quite unrelated to those used in Eqs. 1c and 2c, p. 371. From the theory of equations it is known that Eq. 11a has four roots. These may be real, imaginary, or complex. The imaginary and complex roots always come in conjugate pairs. Thus if one root p_1 is $-\alpha + j\beta$, there is always a second root p_2, or $-\alpha - j\beta$, conjugate to p_1.

Before the important properties of these roots are examined further, the fact that there are four values of p satisfying Eq. 11a should be compared with the original assumptions, Eqs. 3 and 4. Apparently i_{1t} and i_{2t} each contains four terms, a fact which leads to four A's and four B's. These eight terms are not independent, however, since either Eq. 1b or 2b relates A to B for any given value of p. In the light of these findings, the assumptions of Eqs. 3 and 4 may be expanded to include the four

values p_1, p_2, p_3, and p_4 of p. First the assumptions are made that

$$i_{1t} = A_1\epsilon^{p_1 t} + A_2\epsilon^{p_2 t} + A_3\epsilon^{p_3 t} + A_4\epsilon^{p_4 t}, \qquad \blacktriangleright[3a]$$

$$i_{2t} = B_1\epsilon^{p_1 t} + B_2\epsilon^{p_2 t} + B_3\epsilon^{p_3 t} + B_4\epsilon^{p_4 t}. \qquad \blacktriangleright[4a]$$

In Eqs. 3a and 4a the subscripts to A and B show the value of p with which they are associated.

An example of the physical meaning of the terms of Eqs. 3a and 4a can be developed from an examination of any one, such as the term $B_3\epsilon^{p_3 t}$. This term states that loop 2 contains a transient component of current varying with time according to the factor $\epsilon^{p_3 t}$ and that the amplitude of the transient component initially is the constant coefficient B_3. Similarly in loop 1 there is a transient component of current $A_3\epsilon^{p_3 t}$ varying in the same manner with time but having the different initial magnitude A_3. Equations 1b and 2b state that for a current varying in the manner $\epsilon^{p_3 t}$, the loop 1 and loop 2 magnitudes are related by

$$\frac{A_3}{B_3} = \frac{-Z_{12}(p_3)}{Z_{11}(p_3)} = \frac{-Z_{22}(p_3)}{Z_{12}(p_3)}. \qquad [12]$$

That the two ratios of Z's in Eq. 12 should be equal is evident when the elements are cross-multiplied and transposed, with the result that

$$Z_{11}(p_3)Z_{22}(p_3) - [Z_{12}(p_3)]^2 = 0, \qquad [11b]$$

for p_3 is one of the values of p that satisfies this very Eq. 11. A similar interpretation applies to each of the other pairs of terms containing a given value of p. Thus the complete transient in each loop consists of four exponential components.

If both L and S are present in the network — as they are in the case under discussion — and if the resistances are sufficiently small, two or all four of the values of p are complex quantities occurring in conjugate pairs as just pointed out. If, for example, p_1 and p_2 are conjugate complex roots, it is found that the corresponding A's and B's for each loop are conjugate complex numbers. The two terms combine to form a term as follows, for example, for loop 2:

$$\left.\begin{aligned} B_1\epsilon^{(-\alpha+j\omega_d)t} + B_2\epsilon^{(-\alpha-j\omega_d)t} &= 2B_1\epsilon^{-\alpha t}\cos(\omega_d t + \delta) \\ = 2\mathcal{R}_e[B_1\epsilon^{(-\alpha+j\omega_d)t}] &= 2\mathcal{R}_e[B_2\epsilon^{(-\alpha-j\omega_d)t}], \end{aligned}\right\} \qquad [13]$$

$$p_1 = -\alpha + j\omega_d, \qquad [14a]$$

$$p_2 = -\alpha - j\omega_d, \qquad [14b]$$

$$B_1 = B_1\underline{/\delta} = \overline{B}_2. \qquad [15]$$

One further property of the p's is important. From physical considera-

tions it can be seen that the real part of p (which may be all of p) must be negative. If it were positive, transient components would increase without limit with time, an absurdity from the physical point of view, especially since this discussion concerns the force-free solutions.

A few words are appropriate here concerning the possibility of p equal to zero. When Eq. 11 is changed to the polynomial Eq. 11a containing only positive powers of p, the statement is made that in general p equal to zero is not a root of Eq. 11. This is always true for a physically realizable network, which necessarily contains resistances. For if zero were a root, there would be a term containing ϵ^0 as an indefinitely persisting part of the transient current, which is not in accord with the law of conservation of energy for a system containing dissipative elements. The student may encounter this case if he idealizes a circuit by omitting resistances, as is sometimes done for simplification. A treatment of this somewhat special situation is given in more complete works on network analysis.[1]

3. Evaluation of the constants of integration

The transient solutions for currents in the two-loop network is now in the tentative form of Eqs. 3a and 4a, some of whose properties have been investigated. Each of the solutions contains four undetermined constants: a real A or B for each term of i_1 or i_2, respectively, for which the p is real; and the two components of a complex A or B for each pair of complex p's. If this number of undetermined constants is equal to the number of initial conditions that can be given for the network arbitrarily, and if these constants can be determined in terms of the initial conditions, Eqs. 3a and 4a are the solution of the problem. These equations satisfy the differential equations 1 and 2, p. 368, but to be a correct solution they must also be unique, for a given network always reacts the same way to a given set of impressed conditions. Equations 3a and 4a can be unique only if the constants can be uniquely determined from the initial currents in inductances and from the initial charges in condensers.

An examination of the circuit of Fig. 1, p. 366, discloses the number of initial conditions that can be fixed arbitrarily. Loops 1 and 2 both contain inductance; hence their currents are mathematically continuous through any instantaneous change in the circuit. This fact means that if the value of each current is known for an instant immediately preceding any change in the circuit or source, it is known also for the instant immediately following the change, for the two differ only by the infinitesimal amount that a quantity which changes at a finite rate can change in an infinitesimal time. The common branch also contains inductance, but this

[1] E. A. Guillemin, *Communications Networks* (New York: John Wiley & Sons, 1931), Vol. I, Ch. v., paragraph 10.

introduces no additional information since the current in it is merely the difference of the two loop currents. Initial current values of the two loop currents then serve for two of the initial conditions.

There are independent capacitances in both loops, each in series with resistance; hence, the integral of either loop current, representing the charge displaced around either loop, cannot be changed instantaneously. Therefore the values of the loop-current integrals immediately following any instantaneous change in the circuit or the applied forces are the same as those immediately preceding the change. This fact provides two more initial conditions, making four in all, which is exactly the number required.

From the foregoing argument:

$$i_1(0) = i_{1s}(0) + i_{1t}(0), \tag{16}$$

$$i_2(0) = i_{2s}(0) + i_{2t}(0), \tag{17}$$

$$q_1(0) = q_{1s}(0) + q_{1t}(0), \tag{18}$$

$$q_2(0) = q_{2s}(0) + q_{2t}(0) \tag{19}$$

But just after the change of circuit conditions, the transient currents are, from Eqs. 3a, 4a, and 12,

$$i_{1t}(0) = i_1(0) - i_{1s}(0) = \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 + \mathrm{A}_4, \tag{20}$$

$$i_{2t}(0) = i_2(0) - i_{2s}(0) = \mathrm{B}_1 + \mathrm{B}_2 + \mathrm{B}_3 + \mathrm{B}_4, \tag{21}$$

$$\left.\begin{aligned} i_{2t}(0) = \mathrm{A}_1 \frac{\mathrm{Z}_{11}(\mathrm{p}_1)}{-\mathrm{Z}_{12}(\mathrm{p}_1)} + \mathrm{A}_2 \frac{\mathrm{Z}_{11}(\mathrm{p}_2)}{-\mathrm{Z}_{12}(\mathrm{p}_2)} \\ + \mathrm{A}_3 \frac{\mathrm{Z}_{11}(\mathrm{p}_3)}{-\mathrm{Z}_{12}(\mathrm{p}_3)} + \mathrm{A}_4 \frac{\mathrm{Z}_{11}(\mathrm{p}_4)}{-\mathrm{Z}_{12}(\mathrm{p}_4)}. \end{aligned}\right\} \tag{21a}$$

If p_1 and p_2 are conjugate complex roots and p_3 and p_4 are real, then the A and B terms can be reduced to

$$i_{1t}(0) = 2\mathcal{R}_e[\mathrm{A}_1] + A_3 + A_4, \tag{20a}$$

$$i_{2t}(0) = 2\mathcal{R}_e[\mathrm{B}_1] + B_3 + B_4, \tag{21b}$$

$$\left.\begin{aligned} i_{2t}(0) = 2\mathcal{R}_e\left[\mathrm{A}_1 \frac{\mathrm{Z}_{11}(-\alpha + j\omega_d)}{-\mathrm{Z}_{12}(-\alpha + j\omega_d)}\right] + A_3 \frac{Z_{11}(p_3)}{-Z_{12}(p_3)} \\ + A_4 \frac{Z_{11}(p_4)}{-Z_{12}(p_4)}. \end{aligned}\right\} \tag{21c}$$

If all four roots are complex, then the p_3 and p_4 terms take the form of the term for p_1 and p_2 in Eq. 21c.

The transient loop charges are found by integrating Eqs. 3a and 4a.

Putting t equal to zero in the resulting expressions and using Eqs. 18 and 19 give

$$q_{1t}(0) = q_1(0) - q_{1s}(0) = \frac{A_1}{p_1} + \frac{A_2}{p_2} + \frac{A_3}{p_3} + \frac{A_4}{p_4}, \qquad [22]$$

$$q_{2t}(0) = q_2(0) - q_{2s}(0) = \frac{B_1}{p_1} + \frac{B_2}{p_2} + \frac{B_3}{p_3} + \frac{B_4}{p_4}, \qquad [23]$$

$$\left. \begin{aligned} q_{2t}(0) = \frac{A_1}{p_1}\frac{Z_{11}(p_1)}{[-Z_{12}(p_1)]} + \frac{A_2}{p_2}\frac{Z_{11}(p_2)}{[-Z_{12}(p_2)]} + \frac{A_3}{p_3}\frac{Z_{11}(p_3)}{[-Z_{12}(p_3)]} \\ + \frac{A_4}{p_4}\frac{Z_{11}(p_4)}{-[Z_{12}(p_4)]}. \end{aligned} \right\} \qquad [23a]$$

If p_1 and p_2 are conjugate complex roots, and p_3 and p_4 are real, the expressions reduce to

$$q_{1t}(0) = 2\mathcal{R}_e\left[\frac{A_1}{-\alpha + j\omega_d}\right] + \frac{A_3}{p_3} + \frac{A_4}{p_4}, \qquad [22a]$$

$$q_{2t}(0) = 2\mathcal{R}_e\left[\frac{B_1}{-\alpha + j\omega_d}\right] + \frac{B_3}{p_3} + \frac{B_4}{p_4}, \qquad [23b]$$

$$\left. \begin{aligned} q_{2t}(0) = 2\mathcal{R}_e\left[\frac{A_1}{-\alpha + j\omega_d}\,\frac{Z_{11}(-\alpha + j\omega_d)}{\{-Z_{12}(-\alpha + j\omega_d)\}}\right] \\ + \frac{A_3}{p_3}\frac{Z_{11}(p_3)}{[-Z_{12}(p_3)]} + \frac{A_4}{p_4}\frac{Z_{11}(p_4)}{[-Z_{12}(p_4)]}. \end{aligned} \right\} \qquad [23c]$$

By solving Eqs. 20, 21a, 22, and 23a simultaneously, the values of the A's are definitely determined from the initial conditions. If one pair of conjugate roots occurs, these equations can be replaced by Eqs. 20a, 21c, 22a, and 23c. If there are two pairs of conjugate roots, the terms for the p_3 and p_4 roots merely repeat the form of the terms for the first pair of conjugate roots.

The four constants having been obtained, the final transient solution for loop 1 can be written as Eqs. 3a and 4a with numerical values inserted, or perhaps more conveniently with terms of the cosine form of Eq. 13 substituted for each pair of conjugate complex roots that occur. The coefficients for loop 2 are calculated from the A's by the process of Eq. 12, after which the complete expression for the transient current in loop 2 can be written.

4. Summary of procedure for transient solution

The method of solution for transient currents in a two-loop network whose initial charges and currents are known may be summarized briefly in the following suggested procedure:

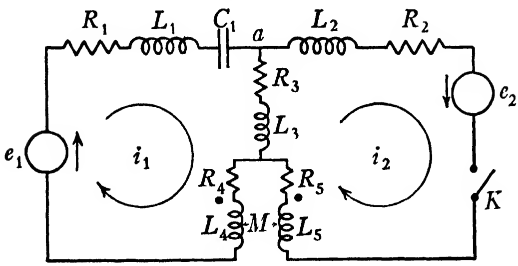
(a) The loop-voltage force-free equations are written, Eqs. 1 and 2.
(b) Solutions of the forms Eqs. 3 and 4 are assumed.
(c) Equations 3 and 4 are substituted into Eqs. 1 and 2 to obtain Eqs. 1a and 2a.
(d) The characteristic equation 11 is formed and is simplified to Eq. 11a.
(e) Equation 11a is solved for all values of p.
(f) The equations for transient currents are written in the forms of Eqs. 3a and 4a.
(g) The steady-state components of initial charge and current are computed.
(h) Equations 20 and 21 (or forms similar to Eq. 20a and 21b) are written.
(i) Equation 12 is substituted in Eq. 21 (or Eq. 21b) to obtain Eq. 21a (or Eq. 21c).
(j) Equations 3a and 4a are integrated to obtain the charge equations.
(k) By use of the charge equations with t equal to zero, Eqs. 22 and 23 (or forms similar to Eqs 22a and 23b) are obtained.
(l) Equation 12 is substituted in Eq. 23 (or Eq. 23b) to obtain Eq. 23a (or Eq. 23c).
(m) The four values of A are obtained by solving the four corresponding equations for initial charge and current.
(n) The four values of B are obtained by use of Eq. 12.
(o) The values of A and B are substituted in Eqs. 3a and 4a.

It is readily seen that to carry through this complete process numerically for even the two-loop network containing all the parameters involves many hours of tedious calculation. Often, however, complete solutions are not necessary and very useful results can be obtained with comparative ease. For example, a knowledge of the nature of the roots, whether real or complex, and of their approximate numerical values yields a considerable amount of information. Thus the reciprocal of the real part of a root gives the time constant of that component of the transient, or the time required for its amplitude to decrease to $1/\epsilon$ of its initial value. One can rather readily determine in this way how long an appreciable transient lasts, information which may be very useful. Whether or not the transient is oscillatory is also evident from the nature of the roots of Eq. 11a. A complex root yields an oscillatory transient. The ratio of ω_d to α shows immediately whether the oscillations are slightly or highly damped, for $\omega_d/(2\pi\alpha)$ is the number of cycles occurring in the interval during which the amplitudes of the oscillations decay to $1/\epsilon$ of their initial value. If the circuit is highly oscillatory, approximate methods of locating the roots may be great timesavers. The student who is interested in pursuing the subject further can find suitable material in the literature.[2]

[2] Footnote p. 418, and bibliography p. 761.

5. Illustrative Example of Complete Solution of Two-Loop Circuit

Figure 1 shows a two-loop circuit for which a complete steady-state and transient analysis is made in the following numerical example. The circuit is somewhat simpler than the general two-loop circuit of Fig. 1, p. 366, in order that the determinantal equation shall be of the third, rather than the fourth, degree. Thus the actual arithmetical work is made less tedious to follow without sacrificing any real generality in the methods required for the solution of the circuit.



$E_1 = 75$ v, $f = 60 \sim$,	$E_2 = 115$ v
$L_1 = 6.20$ mh	$R_1 = 0.54$ ohm
$L_2 = 306.$ mh	$R_2 = 7.68$ ohms
$L_3 = 32\ 0$ mh	$R_3 = 0.74$ ohm
$L_4 = 59.3$ mh	$R_4 = 3.31$ ohms
$L_5 = 47.5$ mh	$R_5 = 2.61$ ohms
$M = 39.7$ mh	$C_1 = 30\ 0\ \mu$f

FIG. 1. Two-loop network solved for transients, Art. 4.

The problem solved can be stated as follows: In Fig. 1, e_1 has been impressed on the circuit long enough for steady-state conditions to have been reached, switch K being open. Switch K is closed at the instant when e_1 and e_2 are zero, e_1 having a negative and e_2 a positive derivative.

The desired results are complete expressions for i_1, i_2, and q_1 after closure of switch K.

Solution: In order to make the procedure clear, the steps in the solution of the problem are lettered in accordance with the outline of Art. 4, additional detailed explanations being made as required.

(a). The loop-voltage force-free equations are written first from inspection of the circuit diagram:

$$R_{11}i_1 + L_{11}\frac{di_1}{dt} + S_{11}\int i_1 dt + R_{12}i_2 + L_{12}\frac{di_2}{dt} = 0, \qquad [1c]$$

$$R_{12}i_1 + L_{12}\frac{di_1}{dt} \qquad\qquad + R_{22}i_2 + L_{22}\frac{di_2}{dt} = 0, \qquad [2c]$$

in which

$$R_{11} = R_1 + R_3 + R_4 = 0.54 + 0.74 + 3.31 = 4.59 \text{ ohms}, \qquad [24]$$

$$L_{11} = L_1 + L_3 + L_4 = (6.2 + 32.0 + 59.3)10^{-3} = 97.5 \times 10^{-3} \text{ h}, \qquad [25]$$

$$S_{11} = \frac{10^6}{30} = 3.33 \times 10^4 \text{ darafs.} \qquad [26]$$

$$R_{22} = R_2 + R_3 + R_5 = 7.68 + 0.74 + 2.61 = 11.03 \text{ ohms,} \qquad [27]$$

$$L_{22} = L_2 + L_3 + L_5 = (306 + 32.0 + 47.5)10^{-3} = 0.3855 \text{ h,} \qquad [28]$$

$$R_{12} = -R_3 = -0.74 \text{ ohm,} \qquad [29]$$

$$L_{12} = -(L_3 + M) = -(32.0 + 39.7)10^{-3} = -71.7 \times 10^{-3} \text{ h.} \qquad [30]$$

(b) and (c). Assuming that the transient solutions have the form of Eq. 3, these equations, similar to Eqs. 1a and 2a, are obtained:

$$\left(4.59 + 0.0975\text{p} + \frac{33{,}300}{\text{p}}\right)\text{A}\epsilon^{\text{p}t} + (-0.74 - 0.0717\text{p})\text{B}\epsilon^{\text{p}t} = 0, \qquad [1\text{d}]$$

$$(-0.74 - 0.0717\text{p})\text{A}\epsilon^{\text{p}t} + (11.03 + 0.3855\text{p})\text{B}\epsilon^{\text{p}t} = 0. \qquad [2\text{d}]$$

The common term $\epsilon^{\text{p}t}$ is now divided out, giving equations of the form

$$\left(4.59 + 0.0975\text{p} + \frac{33{,}300}{\text{p}}\right)\text{A} + (-0.74 - 0.0717\text{p})\text{B} = 0, \qquad [1\text{e}]$$

$$(-0.74 - 0.0717\text{p})\text{A} + (11.03 + 0.3855\text{p})\text{B} = 0. \qquad [2\text{e}]$$

(d) and (e). The characteristic equation is now written and simplified as follows:

$$\left.\begin{aligned} \text{D(p)} &= \left(4.59 + 0.0975\text{p} + \frac{33{,}300}{\text{p}}\right)(11.03 + 0.3855\text{p}) \\ &\quad - (-0.74 - 0.0717\text{p})^2 \\ &= \text{p}^2[97.5 \times 385.5 - 71.7^2]10^{-6} \\ &\quad + \text{p}[4.59 \times 385.5 + 11.03 \times 97.5 - 2 \times 0.74 \times 71.7]10^{-3} \\ &\quad + [4.59 \times 11.03 - 0.74^2 + 0 + 0.3855 \times 3.33 \times 10^4 + 0] \\ &\quad + \frac{1}{\text{p}}[0 + 11.03 \times 3.33 \times 10^4] = 0. \end{aligned}\right\} \qquad [11\text{c}]$$

Multiplying by p and evaluating the coefficients give

$$\text{D(p)} = 0.0324\text{p}^3 + 2.74\text{p}^2 + 12{,}900\text{p} + 3.68 \times 10^5 = 0, \qquad [11\text{d}]$$

which is satisfied by the three roots

$$\text{p}_1 = -27.9 + j629, \qquad [14\text{c}]$$

$$\text{p}_2 = -27.9 - j629, \qquad [14\text{d}]$$

$$p_3 = -28.6. \qquad [31]$$

In this problem, owing to the absence of a condenser in one of the loops, the characteristic equation has only three roots, though the equation for the general two-loop circuit has four. The location of the roots of a higher-degree algebraic equation is an arithmetical task of some length. For a single simple case such as the present example, possibly the quickest procedure is to locate, by trial and error, the real root, in this case p_3. The term $(\text{p} - p_3)$ is then a known factor which can be divided out of the

equation, leaving a quadratic that can be solved by the usual method. For more difficult problems, various systematized methods are useful.[3]

(f). The numerical values of the roots given are now substituted into the general forms assumed for transient current:

$$i_{1t} = \mathrm{A}_1\epsilon^{(-27.9+j629)t} + \mathrm{A}_2\epsilon^{(-27.9-j629)t} + A_3\epsilon^{-28.6t}, \qquad [3b]$$

$$i_{2t} = \mathrm{B}_1\epsilon^{(-27.9+j629)t} + \mathrm{B}_2\epsilon^{(-27.9-j629)t} + B_3\epsilon^{-28.6t}. \qquad [4b]$$

(g). Before proceeding with the evaluation of the complex coefficients in Eqs. 3b and 4b, it is necessary to determine the steady-state components of initial charge and currents. These can be found from the vector values of current and charge. The self- and mutual impedances are evaluated for steady-state conditions, the notation of Art. 3, Ch. VI, being followed thus:

$$\left.\begin{aligned} \mathrm{Z}_{11} &= R_{11} + j\left(\omega L_{11} - \frac{S_{11}}{\omega}\right) \\ &= 4.59 + j\left(2\pi 60 \times 97.5 \times 10^{-3} - \frac{3.33 \times 10^4}{2\pi 60}\right) \\ &= 4.59 - j51.6 = 51.8\underline{/-84.9^\circ} \text{ ohms,} \end{aligned}\right\} \qquad [32]$$

$$\left.\begin{aligned} \mathrm{Z}_{22} &= R_{22} + j\omega L_{22} = 11.03 + j2\pi 60 \times 0.3855 \\ &= 11.03 + j145.2 = 145.7\underline{/85.7^\circ} \text{ ohms,} \end{aligned}\right\} \qquad [33]$$

$$\left.\begin{aligned} \mathrm{Z}_{12} &= R_{12} + j\omega L_{12} = -0.74 - j2\pi 60 \times 71.7 \times 10^{-3} \\ &= -0.74 - j27.0 = 27.0\underline{/-91.6^\circ} \text{ ohms.} \end{aligned}\right\} \qquad [34]$$

These values together with the vector E's are used in Eqs. 28 and 29, p. 374, to obtain the steady-state I's subsequent to the switching. Thus,

$$\left.\begin{aligned} \mathrm{I}_{1s} &= \frac{145.7\underline{/85.7^\circ} \times 75.0\underline{/90.0^\circ} - 27.0\underline{/-91.6^\circ} \times 115\underline{/90.0^\circ}}{51.8\underline{/-84.9^\circ} \times 145.7\underline{/85.7^\circ} - (27.0\underline{/-91.6^\circ})^2} \\ &= 0.946\underline{/174.1^\circ} \text{ amp,} \end{aligned}\right\} \qquad [35]$$

$$\left.\begin{aligned} \mathrm{I}_{2s} &= \frac{51.8\underline{/-84.9^\circ} \times 115\underline{/90.0^\circ} - 27.0\underline{/-91.6^\circ} \times 75.0\underline{/90.0^\circ}}{51.8\underline{/-84.9^\circ} \times 145.7\underline{/85.7^\circ} - (27.0\underline{/-91.6^\circ})^2} \\ &= 0.963\underline{/-176.9^\circ} \text{ amp,} \end{aligned}\right\} \qquad [36]$$

$$\mathrm{Q}_{1s} = \frac{\mathrm{I}_{1s}}{j\omega} = \frac{0.946/174.1^\circ}{2\pi 60\underline{/90^\circ}} = 2.51 \times 10^{-3}\underline{/84.1^\circ} \text{ coulomb.} \qquad [37]$$

Prior to the instant of switching e_1 alone acts, e_2 being zero; hence for this period the current I_1' and charge Q_1' are given by

$$\mathrm{I}_1' = \frac{\mathrm{E}_1}{Z_{11}} = \frac{75.0/90.0^\circ}{51.8\underline{/-84.9^\circ}} = 1.448/174.9^\circ \text{ amp,} \qquad [35a]$$

[3] W. V. Lyon, "Note on a Method of Evaluating the Complex Roots of a Quartic Equation," *J. Math. Phys.*, III (Apr., 1924), 188–190; L. F. Woodruff, "Note on a Method of Evaluating the Complex Roots of Sixth- and Higher-Order Equations," *J. Math. Phys.*, IV (May, 1925), 164–166; Y. H. Ku, "Note on a Method of Evaluating the Complex Roots of a Quartic Equation," *J. Math. Phys.*, V (Feb., 1926), 125–128.

$$Q_1' = \frac{I_1'}{j\omega} = \frac{1.448\underline{/174.9^\circ}}{2\pi 60\underline{/90.0^\circ}} = 3.84 \times 10^{-3}\underline{/84.9^\circ} \text{ coulomb.} \qquad [37a]$$

The instantaneous values of the above currents and charges are determined at the initial instant, and are the real parts of the corresponding vector quantities, multiplied by $\sqrt{2}$. Thus

$$i_{1s}(0+) = \mathcal{R}_e[0.946\sqrt{2}\underline{/174.1^\circ}] = -1.33 \text{ amp.} \qquad [38]$$

$$i_{2s}(0+) = \mathcal{R}_e[0.963\sqrt{2}\underline{/176.9^\circ}] = -1.36 \text{ amp,} \qquad [39]$$

$$q_{1s}(0+) = \mathcal{R}_e[2.51 \times 10^{-3}\sqrt{2}\underline{/84.1^\circ}] = 3.65 \times 10^{-4} \text{ coulomb,} \qquad [40]$$

$$i_{1s}(0-) = \mathcal{R}_e[1.448\sqrt{2}\underline{/174.9^\circ}] = -2.04 \text{ amp,} \qquad [41]$$

$$q_{1s}(0-) = \mathcal{R}_e[3.84 \times 10^{-3}\sqrt{2}\underline{/84.9^\circ}] = 4.83 \times 10^{-4} \text{ coulomb.} \qquad [42]$$

Also

$$i_{2s}(0-) = 0. \qquad [43]$$

(h) and (i). It is known that $i_{1s}(0-)$, $i_{2s}(0-)$, and $q_{1s}(0-)$ are the actual currents and charge at the instant preceding the closing of switch K. But they are also the actual currents and charge immediately after the closing of switch K, for none of the three can change instantaneously. However, the steady-state components of the two currents and the charge existing subsequent to the switching have instantaneous values $i_{1s}(0+)$, $i_{2s}(0+)$, and $q_{1s}(0+)$ immediately after the switching. Consequently there must be transient components of current and charge having values just after switching such that when they are added to the new steady-state values, the sums are the actual values. Thus, if the transient components are $i_{1t}(0+)$, $i_{2t}(0+)$, and $q_{1t}(0+)$, respectively,

$$i_{1s}(0+) + i_{1t}(0+) = i_{1s}(0-), \qquad [44]$$

$$i_{2s}(0+) + i_{2t}(0+) = i_{2s}(0-), \qquad [45]$$

$$q_{1s}(0+) + q_{1t}(0+) = q_{1s}(0-). \qquad [46]$$

Hence the initial values of the transient components are

$$i_{1t}(0+) = i_{1s}(0-) - i_{1s}(0+) = -2.04 + 1.33 = -0.71 \text{ amp,} \qquad [44a]$$

$$i_{2t}(0+) = i_{2s}(0-) - i_{2s}(0+) = 0 + 1.36 = 1.36 \text{ amp,} \qquad [45a]$$

$$q_{1t}(0+) = q_{1s}(0-) - q_{1s}(0+) = (4.83 - 3.65)10^{-4} = 1.18 \times 10^{-4} \text{ coulomb.} \qquad [46a]$$

These values of transient components are now substituted into Eqs. 20 and 21a, giving

$$-0.710 = A_1 + A_2 + A_3, \qquad [20b]$$

$$1.36 = A_1 \frac{Z_{11}(p_1)}{-Z_{12}(p_1)} + A_2 \frac{Z_{11}(p_2)}{-Z_{12}(p_2)} + A_3 \frac{Z_{11}(p_3)}{-Z_{12}(p_3)}. \qquad [21d]$$

Since p_1 and p_2 are conjugate complex roots, Eqs. 20a and 21c may be used:

$$i_{1t}(0+) = -0.710 = 2\mathcal{R}_e[A_1] + A_3, \qquad [20c]$$

$$i_{2t}(0+) = 1.36 = 2\mathcal{R}_e\left[A_1 \frac{Z_{11}(p_1)}{-Z_{12}(p_1)}\right] + A_3 \frac{Z_{11}(p_3)}{-Z_{12}(p_3)}. \qquad [21e]$$

The coefficients of the A's in Eq. 21e are now evaluated:

$$\left.\begin{aligned}\frac{Z_{11}(p_1)}{-Z_{12}(p_1)} &= \frac{Z_{11}(-27.9+j629)}{-Z_{12}(-27.9+j629)}\\ &= \frac{4.59+(-27.9+j629)97.5\times10^{-3}+\dfrac{3.33\times10^4}{-27.9+j629}}{0.74+(-27.9+j629)71.7\times10^{-3}}\\ &= (188+j5.4)10^{-3} = 188\times10^{-3}\underline{/1.7^\circ},\end{aligned}\right\} \qquad [47]$$

$$\left.\begin{aligned}\frac{Z_{11}(p_3)}{-Z_{12}(p_3)} &= \frac{Z_{11}(-28.6)}{-Z_{12}(-28.6)} = \frac{4.59+(-28.6)97.5\times10^{-3}+\dfrac{3.33\times10^4}{-28.6}}{0.74+(-28.6)71.7\times10^{-3}}\\ &= 888.\end{aligned}\right\} \qquad [48]$$

Substituting these values into Eq. 21e,

$$-0.710 = 2\mathcal{R}_e[A_1] + A_3, \qquad [20d]$$

$$1.36 = 2\mathcal{R}_e[A_1\times188\times10^{-3}\underline{/1.7^\circ}] + 888A_3. \qquad [21f]$$

(j) and (k). Equation 4b is integrated to obtain q_{1t} with the result

$$q_{1t} = 2\mathcal{R}_e\left[\frac{A_1\epsilon^{p_1t}}{p_1}\right] + \frac{A_3\epsilon^{p_3t}}{p_3}, \qquad [49]$$

$$q_{1t}(0+) = 2\mathcal{R}_e\left[\frac{A_1}{p_1}\right] + \frac{A_3}{p_3}, \qquad [22b]$$

$$\left.\begin{aligned}1.18\times10^{-4} &= 2\mathcal{R}_e\left[\frac{A_1}{-27.9+j629}\right] + \frac{A_3}{-28.6}\\ &= 2\mathcal{R}_e[(-0.0704-j1.59)10^{-3}A_1] - 0.0350A_3.\end{aligned}\right\} \qquad [22c]$$

Equations 20d, 21f, and 22c are three equations involving the complex constant A_1 and the real constant A_3. For convenience in solving these equations simultaneously, the complex constant A_1 is written as

$$A_1 = A_r + jA_j \qquad [50]$$

where A_r and A_j are constants to be determined. Equations 20d, 21f, and 22c then become

$$-0.71 = 2A_r + 0 + A_3, \qquad [20e]$$

$$1.36 = 2(188A_r - 5.4A_j)10^{-3} + 888A_3, \qquad [21g]$$

$$1.18\times10^{-4} = 2(-0.0704A_r + 1.59A_j)10^{-3} - 0.0350A_3. \qquad [22d]$$

(m) and (n). The simultaneous solution of these three equations may be accomplished in any suitable way, such as the use of determinants or the elimination by multiplication and subtraction. One method that is self-proving and particularly useful in higher-degree sets of equations is known by statisticians as the Doolittle method.[4] This can be laid out so that any computing-machine operator is capable of

[4] F. C. Mills, *Statistical Methods* (New York: Henry Holt and Company, 1924), pp. 577–581.

doing the work. The result of the solution in this case is

$$A_r = -0.356, \tag{51}$$

$$A_i = 39.9 \times 10^{-3}, \tag{52}$$

$$A_3 = 1.68 \times 10^{-3}. \tag{53}$$

From these and coefficients found previously,

$$\mathbf{A}_1 = -0.356 + j39.9 \times 10^{-3} = 0.358\underline{/173.6^\circ}, \tag{54}$$

$$\left.\begin{aligned} \mathbf{B}_1 &= \mathbf{A}_1 \frac{\mathbf{Z}_{11}(-\alpha + j\beta)}{-\mathbf{Z}_{12}(-\alpha + j\beta)} = 0.358\underline{/173.6^\circ} \times 188 \times 10^{-3}\underline{/1.7^\circ} \\ &= 67.3 \times 10^{-3}\underline{/175.3^\circ}, \end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned} B_3 &= A_3 \frac{Z_{11}(p_3)}{-Z_{12}(p_3)} = (1.68 \times 10^{-3})(888) \\ &= 1.49, \end{aligned}\right\} \tag{56}$$

$$\frac{\mathbf{A}_1}{-\alpha + j\beta} = \frac{0.358\underline{/173.6^\circ}}{629\underline{/92.5^\circ}} = 5.69 \times 10^{-4}\underline{/81.1^\circ}, \tag{57}$$

$$\frac{A_3}{p_3} = \frac{1.68 \times 10^{-3}}{-28.6} = 5.88 \times 10^{-5}. \tag{58}$$

(o). All the coefficients have now been evaluated, and as the final step the expressions for the currents and charge are written, by substituting the coefficients in Eqs. 3b, 4b, and 49 and adding the steady-state terms:

$$\begin{aligned} i_1(t) &= 1.34 \cos (2\pi 60t + 174.1^\circ) \\ &\quad + \mathcal{R}_e[0.716\underline{/173.6^\circ}\epsilon^{(-27.9+j629)t}] + 1.68 \times 10^{-3}\epsilon^{-28.6t}, \end{aligned} \tag{59}$$

$$\begin{aligned} i_2(t) &= 1.36 \cos (2\pi 60t - 176.9^\circ) \\ &\quad + \mathcal{R}_e[0.135\underline{/175.3^\circ}\epsilon^{(-27.9+j629)t}] + 1.49\epsilon^{-28.6t}, \end{aligned} \tag{60}$$

$$\begin{aligned} q_1(t) &= 3.55 \times 10^{-3} \cos (2\pi 60t + 84.1^\circ) \\ &\quad + \mathcal{R}_e[1.14 \times 10^{-3}\underline{/81.1^\circ}\epsilon^{(-27.9+j629)t}] - 5.88 \times 10^{-5}\epsilon^{-28.6t}. \end{aligned} \tag{61}$$

In Eqs. 59, 60, and 61 the first term on the right side is the steady-state component; the remaining terms comprise the transient component.

The arithmetic is partly checked by substituting t equal to zero in Eqs. 59, 60, and 61, giving the following results:

	From equation	*Given initial value*
$i_1(0)$	-2.04	-2.04
$i_2(0)$	-0.003	0.000
$q_1(0)$	4.83×10^{-4}	4.83×10^{-4}

By looking at the solution as a whole it is readily seen that a considerable amount of labor is involved. In fact, the details of two of the most laborious computations have been omitted, namely, the location of the roots of the characteristic equation, and the solution of the set of simultaneous equations for the coefficients. It can be appreciated that the complete solution of a circuit giving rise to a characteristic equation of

much higher degree than the third is not a task to be undertaken lightly, or without first having satisfied oneself that the importance of the result justifies the necessary labor. Certain methods involving more advanced mathematics can lighten the labor somewhat, the proportionate reduction in labor increasing with the complexity of the problem.

6. Transient solution of two-node network

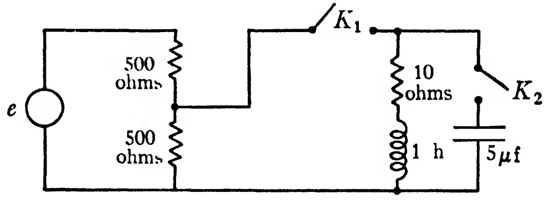
For the transient treatment of a two-node network, the circuit of Fig. 12, p. 391, and the differential Eqs. 71a and 72a, p. 393, are used. As the transient is a force-free behavior the impressed force or i terms drop out of Eqs. 71a and 72a, p. 393, to give the differential equations which the transient node voltages satisfy. The resulting equations are the duals of the corresponding Eqs. 1 and 2 of this chapter for the two-loop network. Enough of the parallelism between the differential equations of the two-loop and two-node networks and their steady-state solution is shown in preceding articles so that it seems unnecessary to repeat the rather lengthy process of solving for the transient in the two-node case. The procedure is analogous in every step to that carried out for the two-loop network; so the student who understands that case should be able, with a little thought, to apply the same ideas to the transient in the two-node network.

PROBLEMS

1. In Prob. 9, Ch. VI, the source has an amplitude of $\sqrt{2}$ v and is impressed when the voltage is a positive maximum. What is the equation of current in the secondary?

2. Two coils are so closely wound on a common magnetic core of reluctance $\mathcal{R}$ that the coefficient of coupling can be taken as unity for the purpose of the problem. Coil 1 has N_1 turns of resistance R_1; coil 2 has N_2 turns of resistance R_2. A voltage $V_m \cos \omega t$ is impressed on coil 1 when t is 0. Coil 2 is short-circuited. What is the equation of flux in the core?

3. In Prob 15, Ch. VI, a sinusoidal current source of amplitude $\sqrt{2}$ amp having shunt conductance of 500×10^{-6} mho is impressed at the left-hand terminals when the current is a positive maximum. What is the equation of the voltage across G?



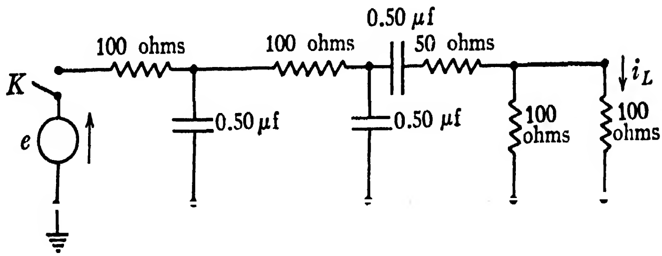
$e = 170 \cos 377t$ v, where t is in sec

Fig. 2. Circuit fed from dropwire, Prob. 4.

4. In the circuit of Fig. 2, with both switches open there is no initial charge on the condenser.

(a) What is the current in the inductance as a function of time after switch K_1 is closed, if it is closed when e is 0 and decreasing?

(b) What is the current in the inductance as a function of time after switch K_2 is closed, if it is closed a long time after switch K_1 is closed, and when e is a positive maximum?



$e = 14.1 \cos 5{,}000\, t$ v, where t is in sec

FIG. 3. Circuit for Prob 5

5. In Fig. 3, if switch K is closed when t is 0, what is the equation of i_L? The circuit is initially at rest.

CHAPTER VIII

Multibranch Alternating-Current Networks

1. Introduction

The method of analysis developed for solution of two-loop or two-node networks is, in more elaborate form, applicable to the analysis of linear networks having a larger number of loops or nodes. Such extension to more general cases involves certain additional features and theorems widely useful in network analysis, but as the complete analysis of the general network is beyond the scope of this text, this discussion is limited to those features which are likely to be most useful to electrical engineers.

The study of lumped-constant electric circuits has two main aspects: the analysis of the behavior of a given network under all conditions and the design or synthesis of a network to perform given functions. The problem of design has the greater potential importance practically and is at the same time much the more difficult and complex of the two aspects. Certain phases of network synthesis, such as simple filter theory or the design of networks to exercise a selective effect in the steady state with regard to bands of frequencies, are considered in this series in the volume on electronics. The more general problem of designing a network that shall have arbitrarily specified characteristics in either the transient or steady state is an advanced phase of the subject which is now in the stage of active development.

Although the analysis of a given network can be carried out as a unit by a more advanced procedure, it can, if it is separated into its steady-state and transient aspects, be handled by the use of methods already developed. Since the labor involved in transient analysis is formidable when advanced methods are not used, and since the steady-state aspect is especially important to everyone dealing with circuits, only the steady-state solution of the general lumped linear network is presented here. The following four considerations are discussed:

(a) the nature of the components constituting a network;
(b) relations existing between network geometry and the number of equations required;
(c) particular networks involving essentially all the features encountered in the general case;
(d) the equations and their steady-state solution for a general ℓ-loop or n-node network.

2. Network components

The simplest components into which a lumped linear network can be resolved are sources and resistance, inductance, and elastance elements or their reciprocals.* In the most general case of formulation, each of these components is considered to be a separate branch having its own two terminals. These components can be grouped variously for the different methods of writing the network equations, thus reducing the number of branches that have to be considered. If each of the elementary components is considered as a separate branch, the potential difference across a branch (excepting sources) can be written as a constant times the current in the branch or its derivative or integral. Or, alternatively, the current in the branch (again excepting sources) can be written as a constant times the potential difference across the branch or its derivative or integral.† For source branches, the electromotive force of a pure voltage source or the current of a pure current source may be a known quantity. Then, the current in a pure voltage source or the potential difference across a pure current source is an unknown and must be so considered in finding the number of equations required to fix a unique solution. It should be recalled that a physical source can be represented by either a pure voltage source or a pure current source in combination with an appropriate element in series or parallel.‡

3. Network geometry and the number of equations

The circuit diagram of a network comprises a definite number of branches connected in a manner indicated by the geometry of the diagram. The study of the properties of this diagram is called *network geometry*. If the physical sources of a network are represented as voltage sources, the geometry of the network is slightly different from that for the same network when the physical sources are considered as current sources. In general, voltage sources are convenient for solution by the loop method and current sources for solution by the node method, but this correspondence is not essential. If circuit data are known in terms of impedances, the loop method of solution may be more convenient, while the node method may be more convenient when circuit admittances are known, irrespective of the number of equations required.

3a. Branch method

Figure 1 is the geometrical diagram of a somewhat complicated network. The branch adjacent to a source branch contains the series element associated with the source. In each of the branches is a current whose

* Article 3, Ch. I.

† Article 20, Ch. I.

‡ Article 12, Ch. VI.

assumed direction is indicated by an arrow and whose magnitude is given by an i with literal subscripts. Across each branch exists a voltage. Except for the sources, both the branch currents and the branch voltages are unknowns. For the source branches either the current or voltage is known, the other being an unknown quantity. There are 14 branches of which 3 are sources; hence there are 22 nonsource-branch unknown cur-

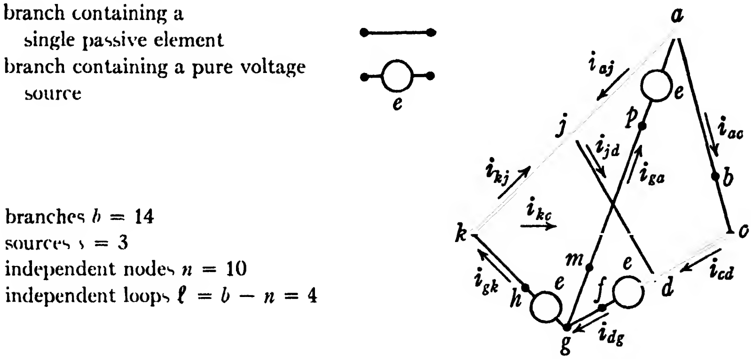


FIG. 1. Simplified diagram of nodes and branches constituting a multibranch network.

rents and voltages and 3 source-branch unknown currents or voltages, making 25 unknown quantities in all. For a general network containing b branches of which s are sources, there are $2b - s$ unknowns. Consequently 25, or in general $2b - s$, independent equations are required to fix them uniquely. Of these the relations between branch current and branch voltage supply 11, or in general $b - s$, leaving 14 or b others to be found.

As the laws for individual elements are used in determining the 11 relations, the laws for networks may be investigated as a possible source of the remaining 14 relations which are needed.* Kirchhoff's current law can be applied to yield an independent equation at every node, or junction of branches, but one. There are 11 nodes in the example, yielding 10 independent equations. This network is therefore called a 10-node network. Only 10 equations can be obtained, since the equation for the eleventh contains only the information already contained in the other 10. These 10 equations, together with the 11 already found, supply 21 of the required 25, leaving 4 to be found. If the general b-branch network contains n independent nodes, $(b - s) + n$ of the required $2b - s$ equations are now known and ℓ, or $b - n$, additional equations are still required. By the use of Kirchhoff's voltage law, loop-voltage equations can be written, but only $b - n$, or 4, of these loop-voltage equations are independent. These ℓ equations are written for various closed paths or loops

* Article 3, Ch. II, should be reviewed.

with the stipulation that every branch must be included in at least one loop.

In the analysis of a network having b branches and n independent nodes, it is therefore necessary to employ n node-current equations and ℓ loop-voltage equations in addition to the $b - s$ branch equations.

3b. Loop method

For very simple networks where b is small, the foregoing method is sometimes the most direct way of obtaining the desired branch voltages or currents. When b is greater than three or four, however, it becomes desirable to reduce the number of simultaneous equations to be solved. In the loop method this reduction is effected by choosing new current variables which identically satisfy the n node-current equations.*

The first step in this reduction process is to select ℓ independent loops, in general choosing the shortest paths for convenience. In complicated circuits it is well to check with the expression $b - n$ the number ℓ of loops required from a count of the branches and nodes. In each of these loops a current is assumed to exist in an arbitrarily designated direction. Because such a loop current leaves every node that it enters, the n node-current equations are automatically satisfied, leaving only ℓ loop-voltage equations to be written and solved for the ℓ loop currents which are now the unknowns. The method of writing these equations in terms of the loop currents and solving them is carried out in detail for the two-loop case in Ch. VI. By the loop method, then, ℓ equations are obtained to solve for ℓ unknown loop currents. From these the branch currents are readily obtained by algebraic addition.

3c. Node method

In Fig. 2, each pure voltage source e of Fig. 1 with its associated series resistance is replaced by a pure current source in parallel with the same elements. As has been emphasized previously, this source conversion, while not essential to the solution of the problem by the node method, is likely to be convenient in a complicated network. It renders the node equations identical in mathematical form to the loop equations. Figure 2 contains the same number of branches as Fig. 1, namely, 14, or b in general. It also contains the same number of sources and therefore has the same number of unknowns, $2(14 - 3) + 3$, or 25. Of the 25, or $2b - s$, equations required to fix these unknowns uniquely, 11 or $b - s$ can be supplied by the current-voltage relations for individual elements as for the branch method. The remaining 14, or b, can be supplied by 7

* Articles 3 and 5, Ch. II, should be reviewed.

or n node-current equations and $14 - 7$, or $b - n$, loop-voltage equations. As compared with the formulation for Fig. 1 there are 3 more loop and 3 fewer node equations. This results from the removal of 3 nodes and the addition of 3 loops in modifying Fig. 1 to Fig. 2, as the representation of the sources is changed.

Whereas in the loop method the n node-current equations are eliminated through the choosing of new current variables that satisfy these

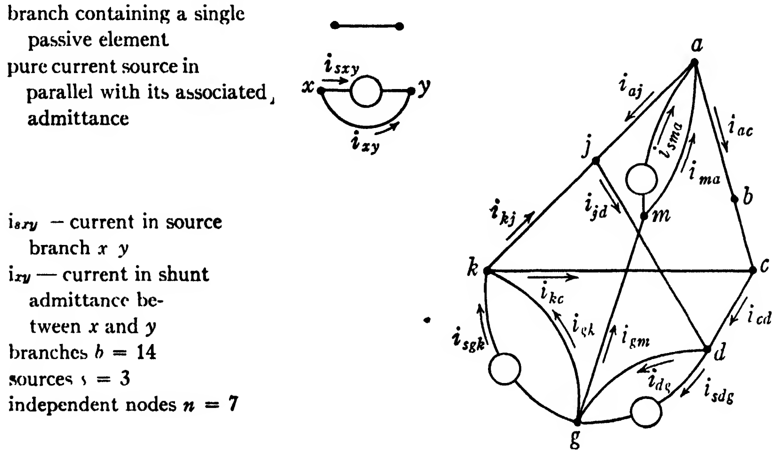


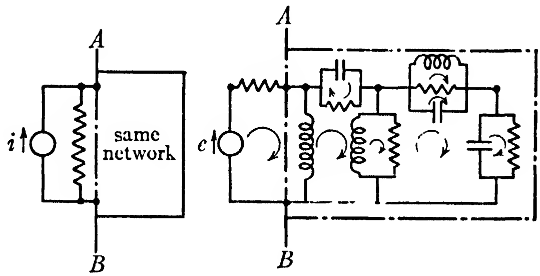
FIG. 2. Multibranch network of circuit of Fig. 1 with voltage sources changed to current sources.

equations identically, in the node method the ℓ loop equations are satisfied identically by replacement of the branch-potential-difference unknowns with unknown node potentials. More specifically, one node to which numerous branches converge is designated to serve as a reference node. Potentials $v_1, v_2 \cdots v_n$ are then assigned to all other nodes, each node potential being the amount by which the potential of the particular node exceeds that of the reference node. These n node potentials are the new unknown voltages replacing the branch-potential differences. It is easily seen that the branch voltages, if wanted, can be obtained as differences of node potentials. The important property of the node potentials, however, is that they satisfy the Kirchhoff loop-voltage law identically. This can be seen if each of the branch voltages around any closed loop is expressed as the difference between the potentials of its two terminal nodes, and the branch voltages so expressed are added. The potential of each node on the loop appears twice in the sum — once with a positive, and once with a negative, sign — and the sum becomes identically zero as required by Kirchhoff's law. This fact shows that the ℓ loop-voltage equations are eliminated and only the n node-current equations

need be considered. These node-current equations can be written in terms of the node potentials and the branch parameters, as shown later, and the resulting set of n simultaneous equations can be solved for the n node potentials. From these any current or voltage in the network can be calculated readily by explicit arithmetical operations, and the network is therefore considered as solved mathematically.

3d. Comparison of loop and node methods

In any actual problem there is a choice between the loop and node methods of solution. One is frequently much shorter than the other, as



Node method with current source

branches $b = 12$

independent nodes $n = 3$

3 equations by node method

Loop method with voltage source

branches $b = 12$

independent nodes $n = 4$

independent loops $l = b - n = 8$

8 equations by loop method

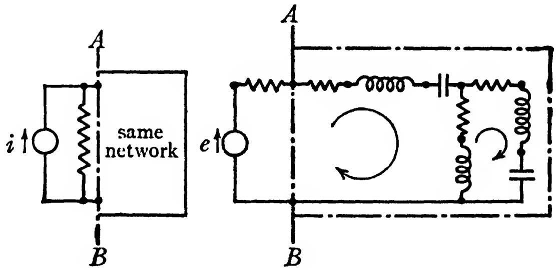
FIG. 3. Circuit in which solution by the node method involves fewer equations than the solution by the loop method.

can be seen from a comparison of the two methods to the circuits of Figs. 3 and 4. The choice of method is itself an engineering problem of determining how to obtain the required results with least time and effort. For steady-state solutions the contrast between the two methods is usually less marked than that indicated by the examples given, because of the possibility of combining elements. Even here, however, one method may lead much more directly to the desired result than the other.

The discussion in this section is purposely made general in order that the main thread of the argument may stand out. In the actual solution the details are sufficiently numerous to obscure the main outline while the actual mathematical work is being done.

In order to remove any possible confusion in the reader's mind, it

should be noted that the alternative loop and node methods of formulating the equations for a given network have nothing to do with duality.



Node method with current source
branches $b = 10$
independent nodes $n = 7$
7 equations by node method

Loop method with voltage source
branches $b = 10$
independent nodes $n = 8$
independent loops $l = b - n = 2$
2 equations by loop method

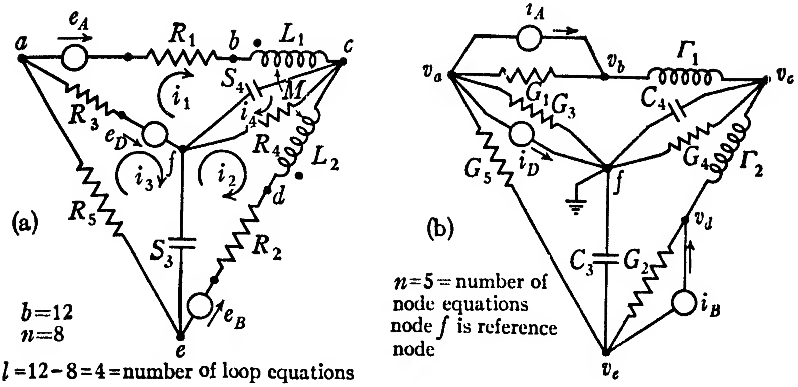
FIG. 4. Circuit in which the solution by the loop method involves fewer equations than the solution by the node method.

4. Formulation and steady-state solution of differential equations for multibranch networks

Figure 5a represents an actual circuit whose sources are represented as pure voltage sources in series with resistance. This circuit is adapted to solution by the loop method. In Fig. 5b these voltage sources are replaced by current sources shunted by conductance to facilitate solution by the node method. One difference is made in the actual network: The magnetic coupling represented by M in Fig. 5a is omitted from Fig. 5b. While magnetic coupling can be treated on the node basis, it is treated somewhat more simply by the loop method. Consequently M is assumed to be zero for the circuit to be analyzed by the node method.

4a. Loop method

In Fig. 5a there are twelve branches and eight nodes; so 12 − 8, or 4, loop equations are required for finding four loop currents. In this example this number is also easily found by inspection. The selected loops are indicated by the arrows associated with the loop currents i_1, i_2, i_3, and i_4. By observation of the polarities for source voltages and the mutual inductance, and the assumed current directions, the four necessary loop-



$$R_1G_1 = 1 \qquad R_3G_3 = 1 \qquad S_4C_4 = 1$$
$$L_1\Gamma_1 = 1 \qquad R_4G_4 = 1 \qquad e_1 = i_1R_1$$
$$R_2G_2 = 1 \qquad R_5G_5 = 1 \qquad e_2 = i_2R_2$$
$$L_2\Gamma_2 = 1 \qquad S_3C_3 = 1 \qquad e_3 = i_3R_3$$

FIG. 5. Circuits used in Arts. 4a and 4b for formulation of equations by both the loop and the node methods.

voltage equations can be written as follows:

$$\text{loop 1: } R_1 i_1 + L_1 \frac{di_1}{dt} + S_4 \int i_1 dt + R_3 i_1 - M \frac{di_2}{dt} \quad R_3 i_3 \\ - S_4 \int i_4 dt = e_A - e_D; \tag{1}$$

$$\text{loop 2: } -M \frac{di_1}{dt} + R_2 i_2 + L_2 \frac{di_2}{dt} + S_3 \int i_2 dt + R_4 i_2 \\ -S_3 \int i_3 dt - R_4 i_4 = -e_B; \tag{2}$$

$$\text{loop 3: } -R_3 i_1 - S_3 \int i_2 dt + R_3 i_3 + R_5 i_3 + S_3 \int i_3 dt = e_D; \tag{3}$$

$$\text{loop 4: } -S_4 \int i_1 dt - R_4 i_2 + R_4 i_4 + S_4 \int i_4 dt = 0. \tag{4}$$

In Eqs. 1 and 2, M is a positive number; the sign of the term takes into account the direction of the induced voltage.

Equations 1 to 4 are four simultaneous equations in the four unknown loop currents. Their complete solution contains both transient and steady-state terms. For reasons stated previously, only the steady-state solution for constant-frequency, constant-amplitude sinusoidal impressed voltages is considered here, all three source voltages having the same

angular frequency ω. If other frequencies are present in the sources, the solution carried out below can be applied independently for each frequency and the results can be added. This very convenient procedure results from the linear character of the network, which makes the principle of superposition applicable.

In mathematical terms the source voltages are described by

$$e_A = E_{Am} \cos(\omega t + \psi_A) = \mathcal{R}_e[\mathbf{E}_{Am}\epsilon^{j\omega t}], \tag{5}$$

$$e_B = E_{Bm} \cos(\omega t + \psi_B) = \mathcal{R}_e[\mathbf{E}_{Bm}\epsilon^{j\omega t}], \tag{6}$$

$$e_D = E_{Dm} \cos(\omega t + \psi_D) = \mathcal{R}_e[\mathbf{E}_{Dm}\epsilon^{j\omega t}]. \tag{7}$$

From the theory of simpler linear networks it can be reasonably assumed that Eqs. 1 to 4 have as solutions the four currents

$$i_k = I_{km} \cos(\omega t + \phi_k) = \mathcal{R}_e[\mathbf{I}_{km}\epsilon^{j\omega t}], \tag{8}$$

$$k = 1, 2, 3, 4. \tag{9}$$

Through the reasoning and procedure given in Art. 3, Ch. VI, Eqs. 1 to 4 can be rewritten in steady-state terms as

$$\left.\begin{aligned} &\left[R_1 + R_3 + j\left(\omega L_1 - \frac{S_4}{\omega}\right)\right]\mathbf{I}_1 - j\omega M\mathbf{I}_2 - R_3\mathbf{I}_3 \\ &\qquad + j\frac{S_4}{\omega}\mathbf{I}_4 = \mathbf{E}_A \quad \mathbf{E}_D, \end{aligned}\right. \tag{1a}$$

$$\left.\begin{aligned} &-j\omega M\mathbf{I}_1 + \left[R_2 + R_4 + j\left(\omega L_2 - \frac{S_3}{\omega}\right)\right]\mathbf{I}_2 \\ &\qquad + j\frac{S_3}{\omega}\mathbf{I}_3 - R_4\mathbf{I}_4 = -\mathbf{E}_B, \end{aligned}\right\} \tag{2a}$$

$$-R_3\mathbf{I}_1 + j\frac{S_3}{\omega}\mathbf{I}_2 + \left[R_3 + R_5 - j\frac{S_3}{\omega}\right]\mathbf{I}_3 = \mathbf{E}_D, \tag{3a}$$

$$+j\frac{S_4}{\omega}\mathbf{I}_1 - R_4\mathbf{I}_2 + \left[R_4 - j\frac{S_4}{\omega}\right]\mathbf{I}_4 = 0. \tag{4a}$$

Equations 1 to 4 can be rewritten in terms of impedances Z and total loop voltages as

$$\mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2 + \mathbf{Z}_{13}\mathbf{I}_3 + \mathbf{Z}_{14}\mathbf{I}_4 = \mathbf{E}_1, \tag*{$\blacktriangleright$[1b]}$$

$$\mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 + \mathbf{Z}_{23}\mathbf{I}_3 + \mathbf{Z}_{24}\mathbf{I}_4 = \mathbf{E}_2, \tag*{$\blacktriangleright$[2b]}$$

$$\mathbf{Z}_{31}\mathbf{I}_1 + \mathbf{Z}_{32}\mathbf{I}_2 + \mathbf{Z}_{33}\mathbf{I}_3 + 0 \qquad = \mathbf{E}_3, \tag*{$\blacktriangleright$[3b]}$$

$$\mathbf{Z}_{41}\mathbf{I}_1 + \mathbf{Z}_{42}\mathbf{I}_2 + 0 \qquad + \mathbf{Z}_{44}\mathbf{I}_4 = 0, \tag*{$\blacktriangleright$[4b]}$$

in which in the arrow directions

$$\mathbf{E}_1 = \mathbf{E}_A - \mathbf{E}_D = \text{total source vector-potential rise in loop 1,} \qquad [10]$$

$$\mathbf{E}_2 = -\mathbf{E}_B = \text{total source vector-potential rise in loop 2,} \qquad [11]$$

$$\mathbf{E}_3 = \mathbf{E}_D = \text{total source vector-potential rise in loop 3,} \qquad [12]$$

and the Z's are identified with the coefficients in Eqs. 1 to 4, as, for example,

$$\mathbf{Z}_{11} = R_1 + R_3 + j\left(\omega L_1 - \frac{S_4}{\omega}\right) = \text{total self-impedance in loop 1,} \qquad [13]$$

$$\mathbf{Z}_{33} = R_3 + R_5 - j\frac{S_3}{\omega} = \text{total self-impedance in loop 3,} \qquad [14]$$

$$\mathbf{Z}_{21} = \mathbf{Z}_{12} = -j\omega M = \text{impedance common to loops 1 and 2,} \qquad [15]$$

$$\mathbf{Z}_{41} = \mathbf{Z}_{14} = j\frac{S_4}{\omega} = \text{impedance common to loops 1 and 4,} \qquad [16]$$

$$\mathbf{Z}_{42} = \mathbf{Z}_{24} = -R_4 = \text{impedance common to loops 2 and 4.} \qquad [17]$$

It may be pointed out again what self- and mutual impedances are. Impedance means vector volts per vector ampere. If the current and voltage concerned are both in the same loop, the impedance is called self-impedance. The self-impedance of a loop is all the impedance associated with that loop. Mutual impedance, on the other hand, is impedance shared by, or common to, two loops. It is the vector volts in one loop caused by a unit vector current in the other loop. Thus $\mathbf{Z}_{24}$ is the vector-voltage drop appearing in the arrow direction in loop 2 when there is one vector ampere in the arrow direction in loop 4, all other currents being zero.

The circuit of Fig. 5a is by no means the most general four-loop network that can be drawn. In fact, the number of elements is purposely kept low to avoid undue complexity. However, all the types of difficulty that are likely to be found in any network formulation are included in this circuit. A more complicated four-loop network, or p-loop network for that matter, would merely contain more terms of the same kinds. Hence, the way in which each term of the differential equations 1 to 4 is obtained from the circuit diagram, including signs, as well as the formulation of the impedances, should be clearly understood from the differential equations. Equations 1b to 4b can, as is shown in Art. 4, Ch. VI, for the simpler two-loop network, be written directly from the circuit diagram.

It should be mentioned that when only steady-state results are wanted, a considerable saving in labor may result from combining several ele-

ments into a single branch. For example, the circuit of Fig. 5a can be analyzed in the steady state as a three-loop network, provided R_4 and S_4 are combined into a single branch, common to loops 1 and 2, and having an impedance

$$\mathrm{Z}_4 = \frac{-j\dfrac{S_4}{\omega}R_4}{R_4 - j\dfrac{S_4}{\omega}}. \qquad [18]$$

In this way loop 4 is entirely eliminated. The total impedance common to loops 1 and 2 then becomes Z_4 plus the effect of M, or

$$\mathrm{Z}_{12} = \mathrm{Z}_{21} = -\mathrm{Z}_4 - j\omega M = \frac{j\dfrac{S_4}{\omega}R_4}{R_4 - j\dfrac{S_4}{\omega}} - j\omega M. \qquad [19]$$

Equations 1b to 4b are linear and can be solved for the I's by elimination or by determinants, using Cramer's rule,* which expresses the solutions as

$$\mathrm{I}_1 = \frac{\mathrm{M}_{11}}{\mathrm{D}_Z}\mathrm{E}_1 + \frac{\mathrm{M}_{21}}{\mathrm{D}_Z}\mathrm{E}_2 + \frac{\mathrm{M}_{31}}{\mathrm{D}_Z}\mathrm{E}_3, \qquad \blacktriangleright[20]$$

$$\mathrm{I}_2 = \frac{\mathrm{M}_{12}}{\mathrm{D}_Z}\mathrm{E}_1 + \frac{\mathrm{M}_{22}}{\mathrm{D}_Z}\mathrm{E}_2 + \frac{\mathrm{M}_{32}}{\mathrm{D}_Z}\mathrm{E}_3, \qquad \blacktriangleright[21]$$

$$\mathrm{I}_3 = \frac{\mathrm{M}_{13}}{\mathrm{D}_Z}\mathrm{E}_1 + \frac{\mathrm{M}_{23}}{\mathrm{D}_Z}\mathrm{E}_2 + \frac{\mathrm{M}_{33}}{\mathrm{D}_Z}\mathrm{E}_3, \qquad \blacktriangleright[22]$$

$$\mathrm{I}_4 = \frac{\mathrm{M}_{14}}{\mathrm{D}_Z}\mathrm{E}_1 + \frac{\mathrm{M}_{24}}{\mathrm{D}_Z}\mathrm{E}_2 + \frac{\mathrm{M}_{34}}{\mathrm{D}_Z}\mathrm{E}_3, \qquad \blacktriangleright[23]$$

in which, as shown in Art. 7, Ch. II,

$\mathrm{M}_{sk} \equiv$ cofactor of sth row and kth column of D_Z including the term $(-1)^{s+k}$, [24]

$\mathrm{D}_Z \equiv$ determinant of the Z's of Eqs. 1b to 4b. [25]

It should be emphasized that all the Z's, M's, and D's are, in general, functions of ω. The functional notation to indicate this fact becomes rather cumbersome in the lengthy expressions, and so is omitted for ease in reading and writing.

Numerous interesting deductions can be made from Eqs. 20 to 23, but

* Appendix B.

these are deferred until the general ℓ-loop network is considered. In this present discussion the technique of setting up and solving for the steady state the differential equations for actual circuits is of greatest interest. The generalities in linear-network theory are considered when the manner of obtaining solutions in terms of specific connections and elements is understood.

4b. Node method

The network of Fig. 5b, as stated before, is the same as that of Fig. 5a except for two features. The first difference is that the voltage representation of physical sources of Fig. 5a is replaced in Fig. 5b by the current representation of the same physical sources. In the nonsource branches, therefore, the values of currents and voltages obtained by solving either circuit are identical except for the second difference. This second difference is that M of Fig. 5a is assumed to be zero in Fig. 5b; M is included in the former to increase its generality; it is omitted from Fig 5b because of the awkwardness of handling mutual inductance in the node method.

Inspection of Fig. 5b shows that five node equations are required. Node f is used as reference node, since it has as many entering branches as any node and therefore eliminates as many terms as possible from the equations. To the remaining nodes are assigned the potentials v_a, $v_b \cdots v_e$, which are the amounts by which these nodes are positive, respectively, with regard to the reference node f. These node voltages become the unknowns in terms of which the equations are written.

Kirchhoff's current law states that the sum of currents directed away from a node is equal to the sum of the currents directed toward the node. This law is applied to node c as an illustration. The current directed away from node c toward node b is i_{cb}, given by

$$i_{cb} = \Gamma_1 \int (v_c - v_b)dt = \Gamma_1 \int v_c dt - \Gamma_1 \int v_b dt. \qquad [26]$$

A negative numerical value is interpreted as usual as a current directed oppositely to the stated direction. The right-hand form of Eq. 26 can be obtained by superposition, which is the most convenient way of formulating the equations. Thus, if all nodes except c are put at zero potential, the current from c to b is $\Gamma_1 \int v_c dt$. If all nodes but b are put at zero potential, the current from c to b is $-\Gamma_1 \int v_b dt$. The actual current with normal node potentials is the sum of these as given by Eq. 26. Since node f is at zero potential by definition, the current i_{cf} from c toward f is

$$i_{cf} = G_4 v_c + C_4 \frac{dv_c}{dt}. \qquad [27]$$

The current directed from c toward d is

$$i_{cd} = \Gamma_2 \int v_c dt - \Gamma_2 \int v_d dt. \qquad [28]$$

Adding Eqs. 26, 27, and 28 and equating the sum to zero, since there are no sources connected to node c, give the total current directed away from node c.

$$-\Gamma_1 \int v_b dt + G_4 v_c + C_4 \frac{dv_c}{dt} + (\Gamma_1 + \Gamma_2) \int v_c dt - \Gamma_2 \int v_d dt = 0. \qquad [29]$$

When the foregoing procedure is carried out for each of the five independent nodes, the following equations are obtained:

$$\text{node } a\text{:} \quad (G_1 + G_3 + G_5)v_a - G_1 v_b - G_5 v_e = -i_A - i_D, \qquad [30]$$

$$\text{node } b\text{:} \quad -G_1 v_a + G_1 v_b + \Gamma_1 \int v_b dt - \Gamma_1 \int v_c dt = i_A, \qquad [31]$$

$$\begin{aligned} \text{node } c\text{:} \quad & -\Gamma_1 \int v_b dt + G_4 v_c + C_4 \frac{dv_c}{dt} \\ & + (\Gamma_1 + \Gamma_2) \int v_c dt - \Gamma_2 \int v_d dt = 0, \end{aligned} \qquad [32]$$

$$\text{node } d\text{:} \quad -\Gamma_2 \int v_c dt + G_2 v_d + \Gamma_2 \int v_d dt - G_2 v_e = i_B, \qquad [33]$$

$$\text{node } e\text{:} \quad -G_5 v_a - G_2 v_d + (G_2 + G_5)v_e + C_3 \frac{dv_e}{dt} = -i_B. \qquad [34]$$

From these equations the entire behavior of the network can be determined. However, only the solution for steady-state conditions with sinusoidal sources having the same angular frequency ω is desired. By the method used in previous cases, the source currents are written as

$$i_A = I_{Am} \cos (\omega t + \phi_A) = \mathcal{R}_e[\mathrm{I}_{Am} \epsilon^{j\omega t}], \qquad [35]$$

$$i_B = I_{Bm} \cos (\omega t + \phi_B) = \mathcal{R}_e[\mathrm{I}_{Bm} \epsilon^{j\omega t}], \qquad [36]$$

$$i_D = I_{Dm} \cos (\omega t + \phi_D) = \mathcal{R}_e[\mathrm{I}_{Dm} \epsilon^{j\omega t}], \qquad [37]$$

in which the I's and ϕ's are given. It is certain that the five unknown node voltages have the form

$$v_k = V_{km} \cos (\omega t + \psi_k) = \mathcal{R}_e[\mathrm{V}_{km} \epsilon^{j\omega t}], \qquad [38]$$

$$k = a, b, c, d, e, \qquad [39]$$

in which the V_{km} and the ψ_k are to be determined. Substituting the exponential forms of Eqs. 35 to 38 into Eqs. 30 to 34, performing the

indicated differentiations and integrations, and dividing out the common factor $\epsilon^{j\omega t}$, which is equivalent to replacing $\int dt$ by $1/(j\omega)$ or $-j(1/\omega)$ and $d/(dt)$ by $j\omega$, give the following vector equations:

$$(G_1 + G_3 + G_5)\mathrm{V}_a - G_1\mathrm{V}_b - G_5\mathrm{V}_e = -\mathrm{I}_A - \mathrm{I}_D, \qquad [30a]$$

$$-G_1\mathrm{V}_a + \left(G_1 - j\frac{\Gamma_1}{\omega}\right)\mathrm{V}_b + j\frac{\Gamma_1}{\omega}\mathrm{V}_c = \mathrm{I}_A, \qquad [31a]$$

$$j\frac{\Gamma_1}{\omega}\mathrm{V}_b + \left[G_4 + j\left(\omega C_4 - \frac{\Gamma_1 + \Gamma_2}{\omega}\right)\right]\mathrm{V}_c + j\frac{\Gamma_2}{\omega}\mathrm{V}_d = 0, \qquad [32a]$$

$$j\frac{\Gamma_2}{\omega}\mathrm{V}_c + \left(G_2 - j\frac{\Gamma_2}{\omega}\right)\mathrm{V}_d - G_2\mathrm{V}_e = \mathrm{I}_B, \qquad [33a]$$

$$-G_5\mathrm{V}_a - G_2\mathrm{V}_d + (G_2 + G_5 + j\omega C_3)\mathrm{V}_e = -\mathrm{I}_B. \qquad [34a]$$

In Eqs. 30a to 34a the coefficients of the V's have the form of admittances. These equations may be rewritten in the more convenient form

$$\mathrm{Y}_{aa}\mathrm{V}_a + \mathrm{Y}_{ab}\mathrm{V}_b + 0 \quad + 0 \quad + \mathrm{Y}_{ae}\mathrm{V}_e = \mathrm{I}_a, \qquad [30b]$$

$$\mathrm{Y}_{ba}\mathrm{V}_a + \mathrm{Y}_{bb}\mathrm{V}_b + \mathrm{Y}_{bc}\mathrm{V}_c + 0 \quad + 0 \quad = \mathrm{I}_b, \qquad [31b]$$

$$0 + \mathrm{Y}_{cb}\mathrm{V}_b + \mathrm{Y}_{cc}\mathrm{V}_c + \mathrm{Y}_{cd}\mathrm{V}_d + 0 \quad = 0, \qquad [32b]$$

$$0 + 0 \quad + \mathrm{Y}_{dc}\mathrm{V}_c + \mathrm{Y}_{dd}\mathrm{V}_d + \mathrm{Y}_{de}\mathrm{V}_e = \mathrm{I}_d, \qquad [33b]$$

$$\mathrm{Y}_{ea}\mathrm{V}_a + 0 \quad + 0 \quad + \mathrm{Y}_{ed}\mathrm{V}_d + \mathrm{Y}_{ee}\mathrm{V}_e = \mathrm{I}_e, \qquad [34b]$$

in which

$\mathrm{I}_a = -\mathrm{I}_A - \mathrm{I}_D$ = total source current directed toward node a, [40]

$\mathrm{I}_b = \mathrm{I}_A$ = total source current directed toward node b, [41]

$\mathrm{I}_d = \mathrm{I}_B$ = total source current directed toward node d, [42]

$\mathrm{I}_e = -\mathrm{I}_B$ = total source current directed toward node e, [43]

and, for example,

$\mathrm{Y}_{aa} = G_1 + G_3 + G_5$ = total or self-admittance of node a, [44]

$\mathrm{Y}_{bb} = G_1 - j\dfrac{\Gamma_1}{\omega}$ = total or self-admittance of node b, [45]

$$\left.\begin{aligned}\mathrm{Y}_{cc} &= G_4 + j\left(\omega C_4 - \frac{\Gamma_1 + \Gamma_2}{\omega}\right)\\ &= \text{total or self-admittance of node } c,\end{aligned}\right\} \qquad [46]$$

$\mathrm{Y}_{ab} \equiv -G_1$ = admittance common to nodes a and b, [47]

$\mathrm{Y}_{cb} \equiv j\dfrac{\Gamma_1}{\omega}$ = admittance common to nodes c and b, [48]

$\mathrm{Y}_{cd} \equiv -G_2$ = admittance common to nodes c and d. [49]

The self-admittance of a node is thus seen to be the sum of all admittances connecting it with adjacent nodes, whereas the admittance common to two nodes is the negative of the sum of the admittances between them. The fact that the mutual admittance may be composed of more than one term is not illustrated in this example when node f is chosen as the reference node, but it can be easily seen that had node e been selected as reference node, the mutual admittance Y_{cf} would have been $-G_4 - j\omega C_4$.

Equations 30b to 34b can be, and in practice usually are, written immediately from the circuit diagram by inspection, by the use of the foregoing conceptions of self- and mutual admittance. In fact, for steady-state work, nodes can often be suppressed and the resulting number of simultaneous equations decreased if a combination of branches is considered as a single branch. Thus if the combination of elements involving four independent nodes shown in Fig. 6 is a portion of a network, nodes d and e can be suppressed from the steady-state equations of the network, and the number of equations can be decreased by two if

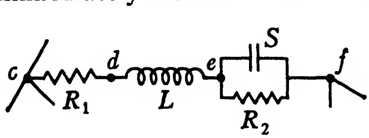


FIG. 6. Series-parallel combination of circuit elements.

$$Y_{cf} = \frac{-1}{R_1 + j\omega L - \dfrac{j\dfrac{S}{\omega} R_2}{R_2 - j\dfrac{S}{\omega}}} \qquad [50]$$

is taken as the mutual admittance of nodes c and f. The potentials of nodes d and e, should they be desired, are readily found from the potentials of nodes c and f and the parameters of the combination branch.

Since Eqs. 30b to 34b are obtained in terms of the unknown V's, their values can be stated explicitly in terms of the source I's and the Y's by the application of Cramer's rule. This yields for the voltage V_k of any node k

$$V_k = \frac{M_{ak}}{D_Y} I_a + \frac{M_{bk}}{D_Y} I_b + \frac{M_{dk}}{D_Y} I_d + \frac{M_{ek}}{D_Y} I_e. \qquad [51]$$

Since there is no current source I_c connected to node c, the term in I_c is missing from Eq. 51. In Eq. 51 as in Eqs. 20 to 23

$$M_{sk} = \text{cofactor of the } s\text{th row and } k\text{th column of } D_Y \text{ including the sign factor } (-1)^{s+k}, \qquad [52]$$

$$D_Y = \text{determinant of the Y's in Eqs. 30b to 34b.} \qquad [53]$$

For the determinants of either Eq. 51 or Eqs. 20 to 23 the labor involved in calculating a complete numerical solution is considerable. Fortunately, in many problems complete solutions are not required and only a few of the coefficients of the form occurring in Eq. 51 need be evaluated.

5. Steady-state loop equations for the general multibranch network

In the preceding article the formulation and steady-state solution of a particular, relatively complicated, network are carried out on both the node and loop bases. By following the methods there illustrated, the student should encounter little difficulty in writing the differential equations of any passive linear network on either basis and in writing from these the corresponding steady-state equations, such as Eqs. 1b to 4b for the loop basis or Eqs. 30b to 34b for the node basis. He should also be able to write either of these sets of steady-state vector equations directly from an inspection of the appropriate circuit diagram. In the development from this point on, this ability is presupposed and further analysis is based directly upon equations having a form such as that of Eqs. 1b to 4b and 30b to 34b. If, therefore, the physical and mathematical considerations leading to such sets of equations are not entirely clear in the student's mind, they should be reviewed or restudied.

For a network having ℓ loops, evidently ℓ loop-voltage equations can be written, which in their steady-state form are as follows:

$$Z_{11}I_1 + Z_{12}I_2 + \cdots + Z_{1k}I_k + \cdots + Z_{1\ell}I_\ell = E_1, \qquad \blacktriangleright[54]$$

$$Z_{21}I_1 + Z_{22}I_2 + \cdots + Z_{2k}I_k + \cdots + Z_{2\ell}I_\ell = E_2, \qquad \blacktriangleright[55]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Z_{j1}I_1 + Z_{j2}I_2 + \cdots + Z_{jk}I_k + \cdots + Z_{j\ell}I_\ell = E_j, \qquad \blacktriangleright[56]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Z_{\ell 1}I_1 + Z_{\ell 2}I_2 + \cdots + Z_{\ell k}I_k + \cdots + Z_{\ell\ell}I_\ell = E_\ell. \qquad \blacktriangleright[57]$$

Some of the terms may be, and usually are, zero in any particular case.

In these equations the one containing E_j, which is written for the jth loop, may be thought of as typical of all of them, and the quantities may therefore be defined for this loop. The quantity E_j is the total source potential rise in the arrow direction in loop j, expressed as a vector; I_j is the vector current circulating in this loop; Z_{jj} is the total or self-impedance of loop j; k designates any other loop k, in which the complex or vector current I_k exists; Z_{jk} or Z_{kj} is the mutual impedance of, or the impedance common to, loops j and k. Either j or k independently can be any number of the series 1, 2, $3 \cdots \ell$.

Equations 54 to 57, like Eqs. 1b to 4b, can be solved for the I's by

Cramer's rule, yielding the following equations:

$$\mathrm{I}_1 = \frac{\mathrm{M}_{11}}{\mathrm{D}_Z}\mathrm{E}_1 + \frac{\mathrm{M}_{21}}{\mathrm{D}_Z}\mathrm{E}_2 + \cdots + \frac{\mathrm{M}_{k1}}{\mathrm{D}_Z}\mathrm{E}_k + \cdots + \frac{\mathrm{M}_{l1}}{\mathrm{D}_Z}\mathrm{E}_l, \tag{58}$$

$$\mathrm{I}_2 = \frac{\mathrm{M}_{12}}{\mathrm{D}_Z}\mathrm{E}_1 + \frac{\mathrm{M}_{22}}{\mathrm{D}_Z}\mathrm{E}_2 + \cdots + \frac{\mathrm{M}_{k2}}{\mathrm{D}_Z}\mathrm{E}_k + \cdots + \frac{\mathrm{M}_{l2}}{\mathrm{D}_Z}\mathrm{E}_l, \tag{59}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathrm{I}_j = \frac{\mathrm{M}_{1j}}{\mathrm{D}_Z}\mathrm{E}_1 + \frac{\mathrm{M}_{2j}}{\mathrm{D}_Z}\mathrm{E}_2 + \cdots + \frac{\mathrm{M}_{kj}}{\mathrm{D}_Z}\mathrm{E}_k + \cdots + \frac{\mathrm{M}_{lj}}{\mathrm{D}_Z}\mathrm{E}_l, \tag{60}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathrm{I}_l = \frac{\mathrm{M}_{1l}}{\mathrm{D}_Z}\mathrm{E}_1 + \frac{\mathrm{M}_{2l}}{\mathrm{D}_Z}\mathrm{E}_2 + \cdots + \frac{\mathrm{M}_{kl}}{\mathrm{D}_Z}\mathrm{E}_k + \cdots + \frac{\mathrm{M}_{ll}}{\mathrm{D}_Z}\mathrm{E}_l, \tag{61}$$

in which M_{kj} is the cofactor of the kth row and jth column, including the sign factor $(-1)^{j+k}$, and D_Z is the determinant of the Z's of Eqs. 54 to 57.

In Eqs. 58 to 61 every E term has a coefficient which is the quotient of the appropriate cofactor and the determinant of the Z's. Each of these coefficients is representable as an admittance.* In general

$$\mathrm{y}_{jk} = \frac{\mathrm{M}_{kj}}{\mathrm{D}_Z} \tag{62}$$

is the vector current in loop j resulting from a unit source vector voltage in loop k when all other loop source voltages are zero. It is called the *short-circuit transfer admittance* between loops j and k. The term *short circuit* refers to the replacement of all sources by short circuits except the sources in loop k.† The term *transfer* may be associated with the fact that y_{jk} is a measure of the effect of E_k of loop k in producing current in loop j. When source voltage and current are in the same loop so that the admittance has the form

$$\mathrm{y}_{jj} = \frac{\mathrm{M}_{jj}}{\mathrm{D}_Z}, \tag{63}$$

the latter is called the *short-circuit self-admittance*, since it is the vector current in loop j per vector applied in loop j when the source voltages in all other loops are replaced by short circuits.†

* This case is analogous to the direct-current case for resistances, Arts. 7 and 8, Ch. II.

† If a source is common to two loops, it must be replaced by a short circuit with respect to the loop in which the source is not desired, and inserted in the loop where it is desired so as to be solely in that loop. For example, if in Fig. 5a e_D is to be replaced by a short circuit with respect to loop 3, but retained with respect to loop 1, it should be moved temporarily to be adjacent to e_A.

6. The reciprocity theorem

An important deduction can be made here from a consideration of Eqs. 58 to 61 and the nature of the coefficients of the E's in them. This is the *theorem of reciprocity*, which is used frequently in subsequent work. This theorem states:

▶In any passive linear network the current in any loop k caused by a given voltage applied in any other loop j is identical with the current in loop j caused by the same voltage applied in loop k, all other source voltages being zero.◀

In other words, in any linear passive network, a pure voltage source and an ideal current-measuring device can be interchanged without altering the indications of the measuring device. While this important theorem is demonstrated here only for steady-state alternating-current conditions, it is true in general, and for transient as well as steady-state conditions. In the transient case, however, the network must be initially at rest. This theorem is stated for direct-current resistance networks in Art. 8, Ch. II.

For the case under consideration the theorem is proved by proof that M_{kj} and M_{jk} are equal. This is the same thing as saying that if E_j alone acts, the current I_k is the same as I_j when E_k acts alone, provided that E_j and E_k are equal. By review of the way in which the Z's of Eqs. 1b to 4b are obtained from the network, it can be seen that Z_{jk} and Z_{kj} are equal and D_Z can be written

$$D_Z = \begin{vmatrix} Z_{11} & Z_{12} & \cdots & Z_{1j} & \cdots & Z_{1t} \\ Z_{12} & Z_{22} & \cdots & Z_{2j} & \cdots & Z_{2t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Z_{1j} & Z_{2j} & \cdots & Z_{jj} & \cdots & Z_{jt} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Z_{1t} & Z_{2t} & \cdots & Z_{jt} & \cdots & Z_{tt} \end{vmatrix} \qquad [64]$$

in which, to make the symmetry more evident, the jth column is written instead of the kth column as in Eqs. 54 to 57. The cofactor of, for example, the second column and jth row obtained by expanding along rows can be seen to be identical with the cofactor of the second row and jth column obtained by expanding along columns. This identity becomes evident when the two expansions are actually carried out for any simple case. The same result holds for the cofactors M_{jk} and M_{kj}. Thus M_{jk} and M_{kj} are equal and the reciprocity theorem is proved for steady-state conditions.

7. Current ratios

If, when a voltage source is present in only one loop, the ratio of two currents is wanted, this ratio can be calculated by evaluation of only two cofactors. If, for example, the ratio I_j/I_k is wanted when E_3 is the only source voltage, then

$$I_j = \frac{M_{3j}}{D_Z} E_3, \tag{65}$$

$$I_k = \frac{M_{3k}}{D_Z} E_3, \tag{66}$$

and the ratio of the two currents is

$$\frac{I_j}{I_k} = \frac{M_{3j}}{M_{3k}} \tag{67}$$

or, in general, when E_n is the only source voltage,

$$\frac{I_j}{I_k} = \frac{M_{nj}}{M_{nk}}, \tag{67a}$$

which can be obtained without evaluating D_Z or any but the two cofactors M_{nj} and M_{nk}.

It is readily seen that an evaluation of all the coefficients of the E's in a set of equations, such as Eqs. 58 to 61, is a laborious task, especially if ℓ exceeds three or four. Sometimes, however, the evaluation must be made. For example, in studies of power systems, particularly when the transient conditions following lightning, short circuits, switching, or other disturbances are being investigated, this evaluation or the equivalent may have to be made several times.[1] In such situations it is often more economical to do the work experimentally by actually reproducing the network at a small convenient scale on a device designed especially for the purpose.[2] Thus the general case of the ℓ-loop network containing sources in several or even all the loops is one of considerable practical importance. In practical situations, however, the solution of the network is made less laborious than the solution for the general case by the absence of many of the sources and coupling parameters indicated for the general

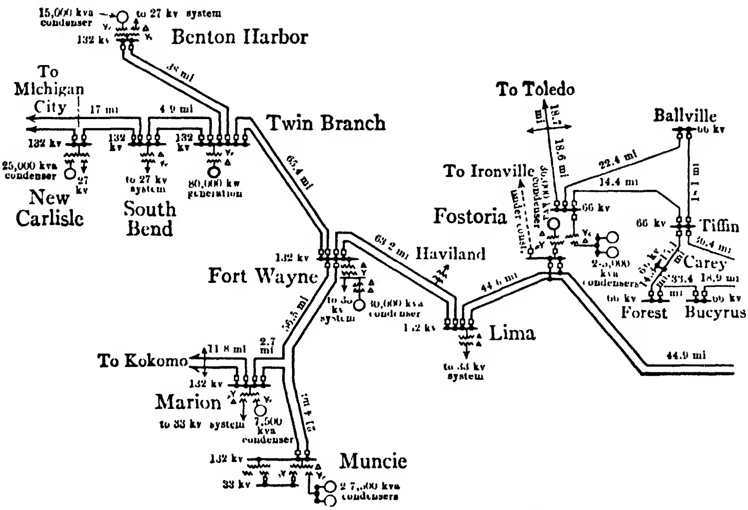
[1] C. F. Wagner and R. D. Evans, "Static Stability and the Intermediate Condenser Station," *A.I.E.E. Trans.*, XLVII (1928), 94–123; I. H. Summers and J. B. McClure, "Progress in the Study of System Stability," *ibid.*, XLIX (1930), 132–161; F. R. Longley, "The Calculation of Alternator Swing Curves," *id.*, 1129–1151, Edith Clarke and R. G. Lorraine, "Power Limit of Synchronous Machines," *E.E.*, LII (1933), 780–787.

[2] H. L. Hazen, O. R. Schurig, and M. F. Gardner, "M.I.T. Network Analyzer; Design and Application to Power System Problems," *A.I.E.E. Trans.*, XLIX (1930), 1102–1113; H. A. Travers and W. W. Parker, "An Alternating-Current Calculating Board," *Elec. J.*, XXVII (1930), 266–270.

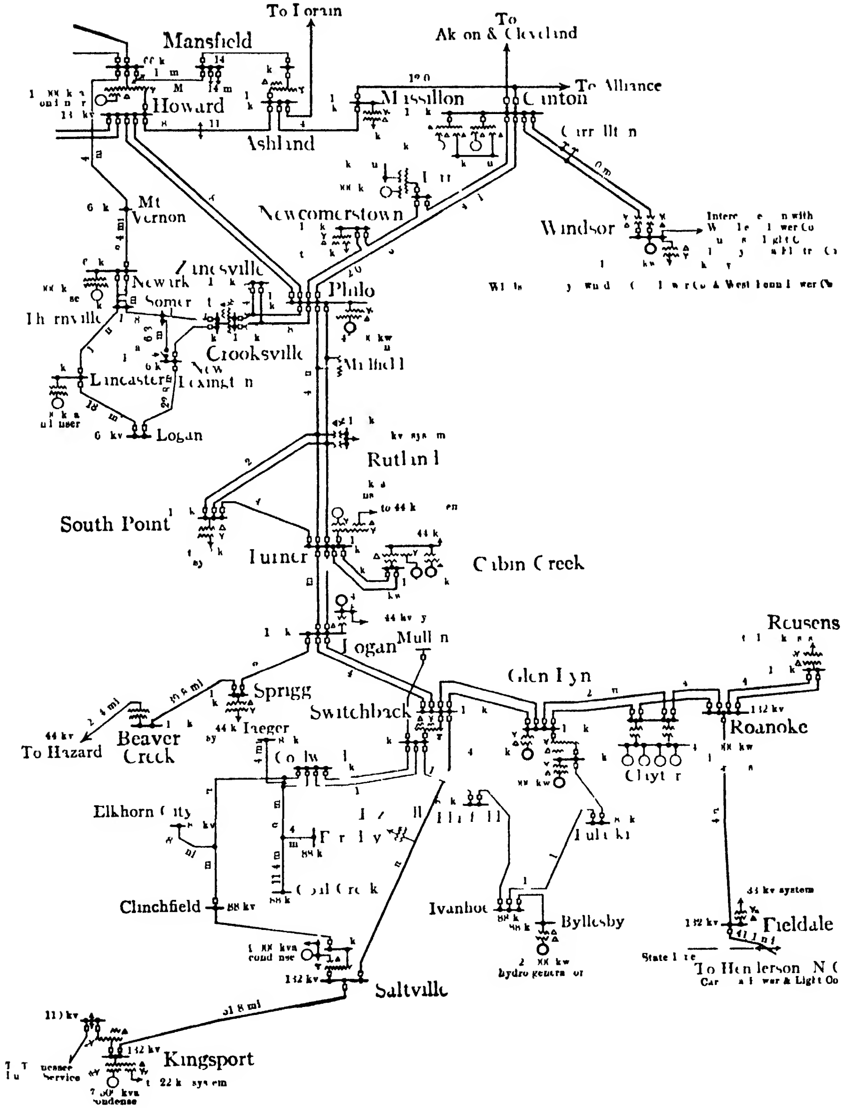
case. Also, in many practical cases the complete solution of the network is not desired; currents and voltages at terminal or other points of special interest suffice.

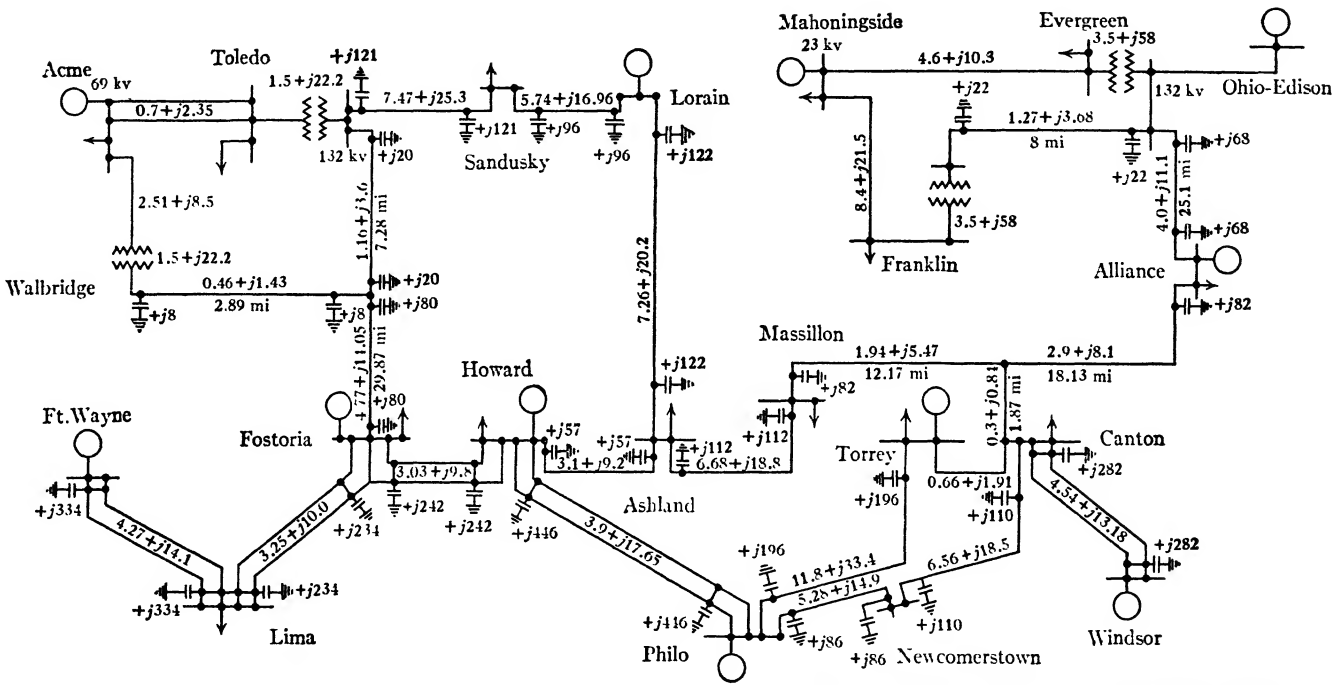
In making solutions of complicated linear networks, the use of ratios frequently is advantageous. For example, if the current in some branch j of a network is desired, a convenient value of current I'_j may be assumed to exist in that branch. By using that hypothetical current as a starting point, computations can be carried back through the network to some branch k for which the current or voltage is known. The current I'_k or voltage V'_k computed for this branch of course does not agree with the actual current I_k or voltage V_k known to exist there unless the assumption made for the current I'_j is correct, which is extremely unlikely. However, the *complex ratios* I'_j/I'_k or I'_j/V'_k of the assumed current in branch j to the calculated current or voltage in branch k are the same respectively as the complex ratios I_j/I_k or I_j/V_k of the actual quantities. Hence with one of these ratios and a corresponding actual current or voltage known, the unknown current I_j can be computed. For obtaining a numerical answer this procedure is sometimes simpler algebraically than the more conventional procedures of working directly from known to unknown quantities by one of the formal methods previously presented. It is assumed, of course, that all impedances involved are known.

Network Analyzer at Massachusetts Institute of Technology (footnote 2, p. 442).



Single line diagram of the 132 kv system and intermediate voltage networks connected to the 132 kv system of the American Gas and Electric Co.


Mansfield
Howard
To Alliance
Mt Vernon
Crooksville
Lancaster
Logan
Philo
South Point
Sprigg
Switchback
Beaver Creek
To Hazard
44 kv
Elkhorn City
Clinchfield
88 kv
Reusens
Roanoke
Fieldale
132 kv
Ivanhoe
Byllesby
Saltville
Kingsport
132 kv
22 kv system



Courtesy American Gas and Electric Co. and Ohio Public Service Co.

Reduction of a portion of the systems of the American Gas and Electric Co. and the Ohio Public Service Co. (partially shown on the preceding diagram) for representation on the Network Analyzer. The complex numbers beside the lines and transformers represent percentage impedance on a 100,000 kva, 132 kv base. The numbers beside the shunt capacitances represent actual system micromhos susceptance. The representation on the Network Analyzer is on a scale of 200 volts, 2 amperes, or 400 va base.

8. Illustrative example of the loop method

In the analysis of the impedance bridge an expression for the detector current as a function of the impedances involved is of interest. The general diagram for such a bridge is shown in Fig. 7. The problem is to

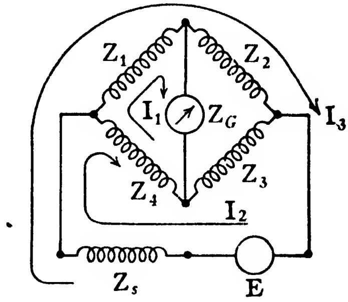


FIG. 7. Impedance bridge

find the current in the detector element Z_G. Impedance Z_S is the impedance of the source E, and impedances Z_1, Z_2, Z_3, and Z_4 are the bridge arms.

Solution: The loop current I_3 is not chosen in a mesh in order that the loop current I_1 may represent directly the result sought. The loop equations are

$$
\begin{aligned}
(Z_4 + Z_1 + Z_G)I_1 \quad & -Z_4 I_2 & +Z_1 I_3 &= 0, && [54a]\\
-Z_4 I_1 + & (Z_4 + Z_3 + Z_S)I_2 & +Z_S I_3 &= E, && [55a]\\
Z_1 I_1 \quad & +Z_S I_2 + & (Z_1 + Z_2 + Z_S)I_3 &= E. && [57a]
\end{aligned}
$$

The solution for I_1 is

$$
I_1 = \frac{\begin{vmatrix} 0 & -Z_4 & Z_1 \\ E & Z_4 + Z_3 + Z_S & Z_S \\ E & Z_S & Z_1 + Z_2 + Z_S \end{vmatrix}}{\begin{vmatrix} Z_4 + Z_1 + Z_G & -Z_4 & Z_1 \\ -Z_4 & Z_4 + Z_3 + Z_S & Z_S \\ Z_1 & Z_S & Z_1 + Z_2 + Z_S \end{vmatrix}} \qquad [58a]
$$

$$
= \frac{E[Z_1Z_S - Z_4Z_S - Z_1(Z_4 + Z_3 + Z_S) + Z_1(Z_1 + Z_2 + Z_S)]}{\{(Z_4 + Z_1 + Z_G)(Z_4 + Z_3 + Z_S)(Z_1 + Z_2 + Z_S) - 2Z_4Z_SZ_1 - Z_1^2(Z_4 + Z_3 + Z_S) - Z_S^2(Z_4 + Z_1 + Z_G) - Z_4^2(Z_1 + Z_2 + Z_S)\}}
$$

$$
= E\frac{Z_1Z_3 - Z_2Z_4}{\{Z_1Z_2Z_3 + Z_1Z_2Z_4 + Z_1Z_3Z_4 + Z_2Z_3Z_4 + Z_1Z_2Z_S + Z_1Z_3Z_S + Z_1Z_3Z_G + Z_1Z_4Z_G + Z_2Z_3Z_G + Z_2Z_4Z_S + Z_2Z_4Z_G + Z_3Z_4Z_S + Z_1Z_GZ_S + Z_2Z_GZ_S + Z_3Z_GZ_S + Z_4Z_GZ_S\}}
$$

Since the denominator of Eq. 58a cannot be infinite when all the Z's are finite, I_1 is zero when

$$Z_1Z_3 = Z_2Z_4, \tag{68}$$

which checks the general equation of balance for the impedance bridge obtainable by much simpler procedure.

9. The coupling network

A somewhat less general case of the ℓ-loop network than that just discussed has widespread application in both the power and communications fields. This case is the ℓ-loop network to which sources or their equivalent are connected in only two of the ℓ loops. For example, the entire transmission system used to connect an electrical generating station with a distant load center, including a long transmission line and transformers at both ends, is usually analyzed as an ℓ-loop network with the generating station as one source and the load as an equivalent of a negative source. Most of the circuits used in communications work are also analyzed in this same way. Such an ℓ-loop network with two pairs of terminals, one pair in each of two loops, to which sources (or loads which, from the point of view of the ℓ-loop network, are treated as sources) can be connected is often called a *coupling network*. It is also a *two-terminal-pair network*, so called from the fact that it has two pairs of terminals. The latter term is often shortened to *two-terminal pair*.

The coupling network is often rather complex and, in a broad use of the term, may, as mentioned above, contain long transmission lines, or even energy-conversion devices, so that between the two pairs of terminals a mechanical, acoustical, or electric-wave link may exist. In the latter cases the equivalent electrical behavior of the mechanical or other link must be known before the combination of such a link and electric circuits can be treated as a two-terminal-pair, or coupling, network.* The coupling network is treated in this chapter as applied to lumped linear passive networks only. However, this concept of a coupling network may also be applied to the study of networks containing energy sources, as, for example, the equipment used in completing a connection between two telephones through a common-battery exchange or through long-distance circuits having a radio link.

The two-terminal-pair network can be shown diagrammatically as in Fig. 8. A pair of leads 1–1′ represents a portion of one of the loops contained in the box, while the other pair 2–2′ similarly represents a portion of another loop. Although the E's and I's associated with these terminals are designated with subscripts 1 and 2, they are usually not adjacent loops They are thus designated merely for convenience in writing.

* Electromechanically coupled systems are treated in Ch. XII.

While the coupling network itself is concealed in a box and only two pairs of terminals are brought out, the complete internal structure is assumed to be known and is used when the equations are written for its performance. As pointed out previously, the coupling network contains no sources. Any currents or voltages appearing in it must therefore be a result of the application of one or more sources to its terminals. In order that the analysis of the coupling network may be independent of the circuits to which it is connected, the nature of the sources or loads connected to the terminal pairs is not specified. In other words, E_1, I_1, E_2, and I_2 are all treated as unknown quantities. It is seen in Art. 3 that if any two of these four quantities are known, an explicit solution for the coupling network can be obtained. If, however, all four of these quantities are retained as unknowns, the most explicit form in which the solution can be put is two equations relating them. Then two additional equations relating them or the values of two of them are requisite for a complete solution. This is precisely the result desired, for when an external circuit is connected to each end, the two additional values or equations are obtained, one from each external circuit. For example, if a pure voltage source is connected to terminals 1 1′, E_1 becomes known, and, if a simple impedance Z_L is connected to terminals 2 2′, the relation

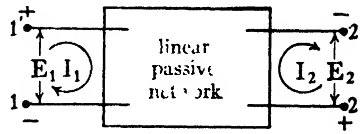


FIG. 8. Two terminal-pair, or coupling, network.

$$E_2 = -I_2Z_L = -V_2. \tag{69}$$

can be used. These two additional relations taken with the two coupling network equations uniquely fix all the currents and voltages in the system. The result desired here, however, is the pair of equations relating the four unknown quantities E_1, I_1, E_2, and I_2. In this presentation of the coupling network, E_1 or E_2 may be used to indicate voltage rises which are not necessarily caused by a source voltage. As indicated by Eq. 69, a passive element may be connected to one pair of terminals.

Since the coupling network is an ℓ-loop network, the general equations 58 to 61 apply, but with the restriction that source voltages E_3, $E_4 \cdots E_\ell$ are zero. From these equations – if it is assumed, as stated above, that the internal structure of the network and the values of its parameters are known — the ℓ currents I_1, $I_2 \cdots I_\ell$ can be expressed in terms of E_1 and E_2. But in the two terminal pair, only I_1 and I_2 are of interest; consequently the equations containing the remaining currents, while true, are irrelevant. For the two-terminal-pair ℓ-loop network these equations apply:

$$I_1 = y_{11}E_1 + y_{12}E_2, \qquad ▶[58b]$$

$$I_2 = y_{12}E_1 + y_{22}E_2. \qquad ▶[59a]$$

As indicated by Eqs. 62 and 63, the y's of Eqs. 58b and 59a are

$$y_{11} = \frac{M_{11}}{D_z}, \tag{63a}$$

$$y_{22} = \frac{M_{22}}{D_z}, \tag{63b}$$

$$y_{12} = y_{21} = \frac{M_{21}}{D_z} = \frac{M_{12}}{D_z}. \tag{62a}$$

It is well to recall at this point that these y's are functions of ω even though this fact is not expressly stated in the notation. This is evident from the fact that they are obtained above from the Z's of Eqs. 54 to 57, which are functions of ω.

It is worth while to emphasize that because of the theorem of reciprocity, the coefficient y_{12} appears twice and that therefore only *three* parameters are necessary to characterize a general coupling network uniquely. The term " parameter " is used here in a more general sense than when defining the elements R, L, and S in that it here designates a coefficient that is dependent upon frequency.

The following physical interpretation can be given to these y parameters: If the terminals 2–2′ of pair 2 are short-circuited, E_2 is zero, and Eq. 58b gives

$$y_{11} = \frac{I_1}{E_1} \tag{70}$$

from which it is seen that y_{11} is the admittance looking into end 1 with end 2 short-circuited. Similarly, if terminals 1-1′ of pair 1 are short-circuited, then E_1 is zero, and the second equation gives

$$y_{22} = \frac{I_2}{E_2}, \tag{71}$$

which shows that y_{22} is the admittance looking into end 2 with end 1 short-circuited. The admittances y_{11} and y_{22} are called the *short-circuit input admittances* or *short-circuit driving-point admittances* with respect to ends 1 and 2, respectively.

Now if end 2 is again short-circuited and E_1 is applied, Eq. 59a gives

$$y_{12} = \frac{I_2}{E_1}. \tag{72}$$

On the other hand, if end 1 is short-circuited and E_2 is applied,

$$y_{12} = \frac{I_1}{E_2}. \tag{73}$$

The admittance y_{12} is therefore the ratio of the vector current at the short-circuited end to the vector voltage applied at the other end, which ratio, by the reciprocity theorem, is independent of the particular end short-circuited. This admittance is called the *short-circuit transfer admittance*. In many practical cases the y's can be written by inspection of the network and making use of Eqs. 70 to 73.

Equations 58b and 59a are explicit expressions for the currents at the terminals of a two-terminal-pair network in terms of the terminal voltages. The voltages are often required when the currents are known, and can be determined if Eqs. 58b and 59a are solved for the voltages E_1 and E_2 in terms of the currents I_1 and I_2. This solution yields the equations

$$E_1 = z_{11}I_1 + z_{12}I_2, \tag{58c}$$

$$E_2 = z_{12}I_1 + z_{22}I_2, \tag{59b}$$

in which

$$z_{11} = \frac{y_{22}}{D_y}, \tag{74}$$

$$z_{22} = \frac{y_{11}}{D_y}, \tag{75}$$

$$z_{12} = -\frac{y_{12}}{D_y}, \tag{76}$$

$$D_y = y_{11}y_{22} - y_{12}^2. \tag{77}$$

Here D_y is the determinant of the y's of Eqs. 58b and 59a, and is different from D_Y.

The three parameters z_{11}, z_{22}, and z_{12} are an alternative set to the three y's for uniquely characterizing a coupling network. These z's, like the y's, are functions of the angular frequency ω of the sources.

Physically the z's have very simple interpretations. If end 2 is open, I_2 is zero, and Eq. 58c gives

$$z_{11} = \frac{E_1}{I_1}, \tag{78}$$

which shows that z_{11} is the impedance looking into end 1 with end 2 open. Similarly, z_{22} is seen to be the impedance looking into end 2 with end 1 open. The impedances z_{11} and z_{22} are called the *open-circuit input impedances* or *open-circuit driving-point impedances* with respect to ends 1 and 2 respectively.

If I_1 is applied to end 1, and end 2 is open, Eq. 59b gives

$$z_{12} = \frac{E_2}{I_1}, \tag{79}$$

while the application of the theorem of reciprocity gives

$$z_{12} = \frac{E_1}{I_2} \tag{79a}$$

for end 1 open-circuited and I_2 applied to end 2. The impedance z_{12} is called the *open-circuit transfer impedance* with respect to ends 1 and 2.

It is important to note the difference between the Z's of Eqs. 54 to 57 and the z's of Eqs. 58c and 59b, since both are impedance parameters of the same network. If the loops are numbered the same for these two sets of equations, the difference between Z_{11} and z_{11} is that Z_{11} equals E_1/I_1 when all loops but loop 1 are open-circuited, whereas z_{11} equals E_1/I_1 with loop 2 open-circuited but with electromotive forces of all other loops but loop 1 short-circuited. Similarly, Z_{12} equals E_2/I_1 with loops 2, 3, $4 \cdots \ell$ open-circuited, while z_{12} equals E_2/I_1 with loop 2 open-circuited but with electromotive forces of loops $3, 4 \cdots \ell$ short-circuited.

As with the y's, the z's can in many practical cases be set down by inspection of the circuit diagram. If the z's are obtained by inspection and the y's are wanted, the latter can readily be obtained by reinverting Eqs. 58c and 59b to obtain Eqs. 58b and 59a again, a fact which shows that the y's are related to the z's by

$$y_{11} = \frac{z_{22}}{D_z}, \tag{80}$$

$$y_{22} = \frac{z_{11}}{D_z}, \tag{81}$$

$$y_{12} = -\frac{z_{12}}{D_z}, \tag{82}$$

$$D_z = z_{11}z_{22} - z_{12}^2. \tag{83}$$

Here D_z is not the same as D_Z, since the z's are not loop and mutual impedances.

As a recapitulation of the coupling-network parameters, Fig. 9 is given, showing the physical interpretation of the y's and z's as they have been given in the foregoing paragraphs, together with the general relations among the parameters and the terminal currents and voltages.

10. General circuit constants A, B, C, D

It is shown in the foregoing section that three parameters (which are, in general, functions of frequency) suffice to fix uniquely the steady-state response of a two-terminal-pair network to sinusoidal impressed terminal forces. The three z's and the three y's are two such sets of parameters.

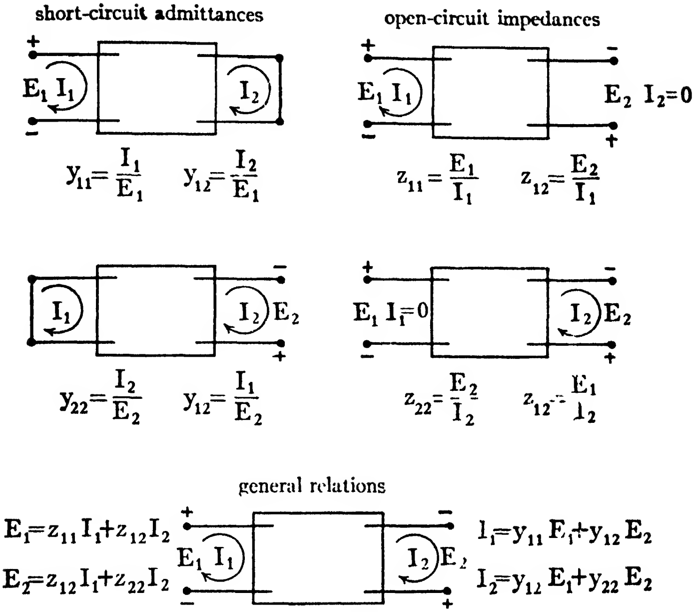
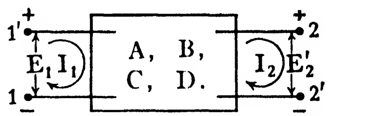


FIG. 9. Current and voltage relations at the terminals of a coupling network.

They are not, however, the only sets of three parameters that can be used. For power engineering it is usually convenient to express the voltage and current at one end of a two-terminal-pair network in terms of the voltage and current at the other end. Such equations are much used by the power engineer, in preference to the equations giving the two currents in terms of the two voltages, and vice versa.



$E_1 = AE_2' + BI_2$ $\quad E_2' = DE_1 - BI_1$

$I_1 = CE_2' + DI_2$ $\quad I_2 = -CE_1 + AI_1$

FIG. 10. The two-terminal-pair network in terms of general circuit constants.

The sign conventions customarily used by the power engineer in his treatment of the coupling network, as shown in Fig. 10, differ in one particular from those used thus far. Here the conventions for the directions of E_1, I_1, and I_2 are the same as those used in Figs. 8 and 9. The voltage E_2' at the terminals of loop 2 is, however, the amount by which 2 is positive in a vector sense with respect to $2'$, while E_2 of Figs. 8 and 9 is the amount by which $2'$ is positive with respect to 2. The networks of Fig. 8 and Fig. 10 are the *same* network, and E_2' and E_2 describe the voltage across the same two terminals, 2 and $2'$. Because of the difference in the definitions of E_2' and E_2, however,

$$E_2' = -E_2. \qquad [84]$$

The power engineer usually thinks of the terminals 1–1′ as the *sending-end* terminals, the terminals 2′–2 as the *receiving-end*, or *load-end*, terminals.

The equations of the coupling network as used in power circuits are customarily written[3]

$$E_1 = AE_2' + BI_2, \quad ▶[85]$$

$$I_1 = CE_2' + DI_2, \quad ▶[86]$$

in which the parameters A, B, C, and D are called *general circuit constants*, of which only three are independent.

These general circuit constants are in reality no more general than the z's and y's of the previous article; in fact, the parameters of any one of these three sets are readily expressible in terms of those of either of the two remaining sets. Thus, substituting I_1 as obtained from Eq. 59b into Eq. 58c, gives

$$E_1 = \frac{z_{11}}{z_{12}} E_2 - \frac{D_z}{z_{12}} I_2. \quad [87]$$

Solving Eq. 59b for I_1 gives

$$I_1 = \frac{1}{z_{12}} E_2 - \frac{z_{22}}{z_{12}} I_2. \quad [88]$$

Equations 87 and 88, when rewritten in terms of E_2', become

$$E_1 = -\frac{z_{11}}{z_{12}} E_2' - \frac{D_z}{z_{12}} I_2, \quad [87a]$$

$$I_1 = -\frac{1}{z_{12}} E_2' - \frac{z_{22}}{z_{12}} I_2, \quad [88a]$$

which, when compared with Eqs. 85 and 86, show that

$$A = -\frac{z_{11}}{z_{12}}, \quad [89]$$

$$B = -\frac{D_z}{z_{12}}, \quad [90]$$

$$C = -\frac{1}{z_{12}}, \quad [91]$$

$$D = -\frac{z_{22}}{z_{12}}. \quad [92]$$

[3] R. D. Evans and H. K. Sels, "Transmission-line Constants and Resonance," *Elec. J.*, XVIII (1921), 306–309; R. D. Evans and H. K. Sels, "Transmission Lines and Transformers," *id.*, 356–359; William Nesbit, *Electrical Characteristics of Transmission Circuits* (East Pittsburgh: Westinghouse Technical Night School Press, 1926); Cecil Dannatt and J. W. Dalgleisch, *Electric Power Transmission and Interconnection* (London: Sir Isaac Pitman & Sons, Ltd., 1930); O. G. C. Dahl, *Electric Circuits*, Vol. I: *Theory and Application* (New York: McGraw-Hill Book Company, Inc., 1928), Ch. ix.

Solution of Eqs. 85 and 86 for E_1 and E_2' in terms of I_1 and I_2 and application of the reciprocity theorem show readily that

$$AD - BC = 1. \qquad [93]$$

This result can also be verified from Eqs. 89 to 92. Equation 93 shows that only three of the four general circuit constants are independent, since, when any three are known, the fourth can be found.

The general circuit constants are also easily expressed in terms of the coupling network y's by using Eqs. 74 to 77 in Eqs. 89 to 92, giving as results

$$A = \frac{y_{22}}{y_{12}}, \qquad [94]$$

$$B = \frac{1}{y_{12}}, \qquad [95]$$

$$C = \frac{D_y}{y_{12}}, \qquad [96]$$

$$D = \frac{y_{11}}{y_{12}}. \qquad [97]$$

If the z's or y's are desired in terms of the general circuit constants, they are found directly from Eqs. 89 to 92 or Eqs. 94 to 97:

$$z_{12} = -\frac{1}{C}, \qquad [98]$$

$$z_{11} = \frac{A}{C}, \qquad [99]$$

$$z_{22} = \frac{D}{C}, \qquad [100]$$

$$y_{12} = \frac{1}{B}, \qquad [101]$$

$$y_{11} = \frac{D}{B}, \qquad [102]$$

$$y_{22} = \frac{A}{B}. \qquad [103]$$

Equations 98 to 103 show that if A and D are equal, the network is entirely symmetrical as viewed from its terminals. This condition is true, for example, of the equivalent lumped circuit of any two-wire transmission line having uniform conductors and spacing.

A useful physical interpretation can be given to the general circuit constants, useful both because it aids in visualizing their meaning and because it shows how they can be obtained by measurements on an actual network. From Eqs. 85 and 86 it is seen that A is the ratio E_1/E_2' when I_2 is zero, or, differently stated, A is an open-circuit voltage ratio; B is the ratio E_1/I_2 when E_2' is zero, or B is a short-circuit transfer impedance; C is an open-circuit transfer admittance, the ratio I_1/E_2' when I_2 is zero; D is the short-circuit current ratio I_1/I_2, that is, the ratio with E_2' equal to zero. It is seen by inspection that A and D are dimensionless complex numbers, that B has the dimensions of an impedance, and that C has the dimensions of an admittance.

In concluding this brief discussion of general circuit constants, it is emphasized again that they provide only one of several ways of handling the two-terminal-pair network. Their importance lies in the fact that such extensive analyses and tabulations are expressed in terms of them that neither the power nor the communications engineer should be unfamiliar with them.

11. *T* AND π, OR *Y* AND Δ CIRCUITS

Two very simple but important networks used extensively in both the power and the communications fields are now considered as illustrations of the application of the relations developed for the two-terminal-pair, or

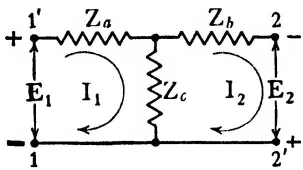


FIG. 11. *T* or *Y* network.

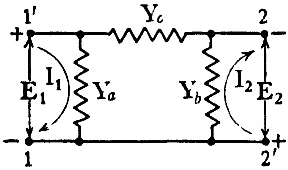


FIG. 12. π or Δ network.

coupling, network. These are the *T* or *Y* network shown in Fig. 11 and the π or Δ network shown in Fig. 12. The resemblance between the network structures and the forms of the descriptive symbols have led to these designations. The *T* and π networks are rather special cases of the coupling network both because they are simple and because they have only three independent terminals, terminals 1 and $2'$ being connected.

11a. *T* OR *Y* NETWORK

From the consideration of network geometry, it is seen that the *T* network, involving three nodes and two loops, is probably more easily treated on the loop basis and therefore in terms of impedances rather

than admittances. Because of the simplicity of the network the loop equations need not even be written in order to find the coupling parameters of the network, as values of these parameters can be written by inspection, their physical definitions being used. Thus the open-circuit driving-point and transfer impedances in terms of the branch impedances are

$$z_{11} = Z_a + Z_c, \qquad [104]$$

$$z_{22} = Z_b + Z_c, \qquad [105]$$

$$z_{12} = -Z_c. \qquad [106]$$

The corresponding y's are not so easily written down by inspection, but are readily obtainable from the z's by use of Eqs. 80 to 83. They are

$$y_{11} = \frac{Z_b + Z_c}{D_z}, \qquad [107]$$

$$y_{22} = \frac{Z_a + Z_c}{D_z}, \qquad [108]$$

$$y_{12} = \frac{Z_c}{D_z}, \qquad [109]$$

in which

$$D_z = z_{11}z_{22} - z_{12}^2 = Z_aZ_b + Z_bZ_c + Z_cZ_a. \qquad [110]$$

The general circuit constants of the T network are obtained by substituting Eqs. 107 to 109 in Eqs. 94 to 97, or Eqs. 104 to 106 in Eqs. 89 to 92, and are as follows:

$$A = \frac{Z_a + Z_c}{Z_c} = \frac{Z_a}{Z_c} + 1, \qquad [111]$$

$$B = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_c} = \frac{Z_aZ_b}{Z_c} + Z_b + Z_a, \qquad [112]$$

$$C = \frac{1}{Z_c}, \qquad [113]$$

$$D = \frac{Z_b + Z_c}{Z_c} = \frac{Z_b}{Z_c} + 1. \qquad [114]$$

To ascertain the branch impedances of a T network that is equivalent to a network whose general circuit constants are known, the functions Z_a, Z_b, and Z_c are found by solving Eqs. 111 to 114 for them, beginning

with Z_c. This solution gives

$$Z_a = \frac{A - 1}{C}, \qquad [115]$$

$$Z_b = \frac{D - 1}{C}, \qquad [116]$$

$$Z_c = \frac{1}{C}. \qquad [117]$$

In general a T network that is equivalent at one frequency to a given network is equivalent to it only at that frequency.

11b. π OR Δ NETWORK

The relations for the π network corresponding to those which have just been developed for the T network are now treated. By examination of Fig. 12 from the point of view of network geometry it is seen that the π network probably lends itself to simpler formulation on the node basis than on the loop basis. Therefore the parameters are more convenient if they are expressed in terms of admittances rather than impedances. This greater convenience is independent of the way in which the relations among the parameters are obtained.

The short-circuit input and transfer admittances can be formulated by inspection from their physical definitions as follows:

$$y_{11} = Y_a + Y_c, \qquad [118]$$

$$y_{22} = Y_b + Y_c, \qquad [119]$$

$$y_{12} = Y_c. \qquad [120]$$

By Eqs. 74 to 76 the corresponding set of open-circuit impedances is found to be

$$z_{11} = \frac{Y_b + Y_c}{D_y}, \qquad [121]$$

$$z_{22} = \frac{Y_a + Y_c}{D_y}, \qquad [122]$$

$$z_{12} = -\frac{Y_c}{D_y}, \qquad [123]$$

in which, from Eq. 77,

$$D_y = Y_aY_b + Y_bY_c + Y_cY_a. \qquad [124]$$

The general circuit constants for the π network are obtained by sub-

stituting Eqs. 121 to 123 in Eqs. 89 to 92, or Eqs. 118 to 120 in Eqs. 94 to 97, and are as follows:

$$A = \frac{Y_b + Y_c}{Y_c}, \tag{125}$$

$$B = \frac{1}{Y_c}, \tag{126}$$

$$C = \frac{Y_aY_b + Y_bY_c + Y_cY_a}{Y_c}, \tag{127}$$

$$D = \frac{Y_a + Y_c}{Y_c}. \tag{128}$$

When either the short-circuit admittances y_{11}, y_{22}, and y_{12} or the general circuit constants A, B, C, and D are known and the branch admittances Y_a, Y_b, and Y_c of the π circuit are desired, they can be obtained from Eqs. 118 to 120 or Eqs. 125 to 128, respectively, as

$$Y_a = y_{11} - y_{12}, \tag{129}$$

$$Y_b = y_{22} - y_{12}, \tag{130}$$

$$Y_c = y_{12}, \tag{131}$$

or

$$Y_a = \frac{D - 1}{B}, \tag{132}$$

$$Y_b = \frac{A - 1}{B}, \tag{133}$$

$$Y_c = \frac{1}{B}. \tag{134}$$

11c. Equivalent T and π networks

One more aspect of T and π networks is of interest at this point. Provided certain relations given below obtain, a π network can be equivalent to a given T network, and vice versa, as far as the behavior of the terminal functions E_1, I_1, E_2, and I_2 is concerned. Two networks are said to be equivalent with respect to stated conditions when under these stated conditions one is indistinguishable from the other. In this case the T and π networks are equivalent when the relations among the terminal E's and I's are the same for both. This condition is true if the z's for the T network as given by Eqs. 104 to 106 and for the π network as given by Eqs. 121 to 123 are identical. Or, alternatively, the networks are equiva-

lent if the y's of Eqs. 107 to 109 for the T network are identical with the y's of Eqs. 118 to 120 for the π network. Thus the two networks are equivalent if

$$Z_c = \frac{Y_c}{D_y}, \tag{135}$$

$$Z_b = \frac{Y_a}{D_y}, \tag{136}$$

$$Z_a = \frac{Y_b}{D_y}, \tag{137}$$

or

$$Y_c = \frac{Z_c}{D_z}, \tag{138}$$

$$Y_b = \frac{Z_a}{D_z}, \tag{139}$$

$$Y_a = \frac{Z_b}{D_z}. \tag{140}$$

Multiplying Eq. 138 by Eq. 135 shows that

$$D_y D_z = 1. \tag{141}$$

Since, in general, three parameters can uniquely characterize a coupling network, two coupling networks are evidently equivalent if any three corresponding independent parameters are the same for both networks. In other words, they are equivalent if either the three corresponding y's the three corresponding z's, or three of the corresponding general circuit constants are the same, respectively, for both networks. Thus for equivalence between a T and a π network the condition that the A's, B's, and C's from Eqs. 111 to 113 and Eqs. 125 to 127 are the same for both leads to the same relations as Eqs. 138 to 140. These equations are frequently very useful in simplifying complicated networks. By their use any group of three impedances connected in Y can be replaced by an equivalent Δ-connected group, and vice versa.*[4] That is, either the T or π network is really a three-terminal network since the terminals 1 and $2'$ are the same, and they can be used interchangeably to interconnect three terminals provided the relations of Eqs. 138 to 140 are true.

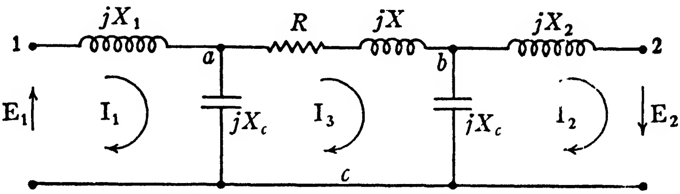
* This is proved for resistances in Art. 10, Ch. II.

[4] There is also a theorem showing a general equivalence between star and mesh networks. Any star can be replaced by an equivalent mesh, but the transformation from mesh to star is indeterminate when more than three elements are involved. A. Rosen, "A New Network Theorem," *I.E.E.J.*, LXII (1924), 916–918; G. A. Campbell, "Direct Capacity Measurements," *B.S.T.J.*, I (1922), 18–38.

12. Illustrative example of a coupling network

The circuit shown in Fig. 13 frequently occurs in power-system stability problems. The following quantities are to be calculated:

(a) the short-circuit transfer admittance, I_2/E_1 when E_2 is equal to zero;
(b) the constants for the equivalent T;
(c) the constants for the equivalent π;
(d) the general circuit constants.



$R = 5.66$ ohms $\qquad X_1 = X_2 = 240$ ohms
$X = 81.8$ ohms $\qquad X_c = -3{,}790$ ohms

Fig. 13. Circuit for example of Art. 12.

Solution: The mesh equations are:

$$
\begin{aligned}
&j(X_1 + X_c)I_1 && 0 && -jX_cI_3 = E_1, && [54b]\\
&-jX_cI_1 && -jX_cI_2 && +[R + j(X + 2X_c)]I_3 = 0, && [55b]\\
&0 && j(X_2 + X_c)I_2 && -jX_cI_3 = E_2, && [57b]
\end{aligned}
$$

$$
I_2 = \frac{\begin{vmatrix} j(X_1 + X_c) & E_1 & -jX_c \\ -jX_c & 0 & R + j(X + 2X_c) \\ 0 & E_2 & -jX_c \end{vmatrix}}{\begin{vmatrix} j(X_1 + X_c) & 0 & -jX_c \\ -jX_c & -jX_c & R + j(X + 2X_c) \\ 0 & j(X_2 + X_c) & -jX_c \end{vmatrix}}. \qquad [59c]
$$

Since the short-circuit transfer admittance is wanted, E_2 is zero in the above solution for I_2 because of the short circuit at terminals 2.

$$
I_2 = \frac{-E_1\begin{vmatrix} -jX_c & R + j(X + 2X_c) \\ 0 & -jX_c \end{vmatrix}}{D_Z} = \frac{E_1X_c^2}{D_Z}, \qquad [59d]
$$

$$
\left.\begin{aligned}
D_Z = {} & -jX_c^2(X_1 + X_c) - jX_c^2(X_2 + X_c)\\
& - [-(X_1 + X_c)(X_2 + X_c)\{R + j(X + 2X_c)\}].
\end{aligned}\right\} \qquad [142]
$$

Since $X_1 = X_2$,

$$
\left.\begin{aligned}
D_Z &= -j2(X_c^3 + X_1X_c^2) + (X_1^2 + 2X_1X_c + X_c^2)\{R + j(X + 2X_c)\}\\
&= R(X_1^2 + 2X_1X_c + X_c^2) + j[X(X_1^2 + 2X_1X_c + X_c^2) + 2X_1^2X_c\\
&\qquad + 4X_1X_c^2 - 2X_1X_c^2]\\
&= R(X_1^2 + 2X_1X_c + X_c^2) + j[X(X_1^2 + 2X_1X_c + X_c^2)\\
&\qquad + 2(X_1^2X_c + X_1X_c^2)].
\end{aligned}\right\} \qquad [142a]
$$

Putting in numerical values:

$$X_1^2 = 240^2 \qquad = \quad 0.006 \times 10^7, \tag{143}$$

$$2X_1X_c = -2 \times 240 \times 3{,}790 = -0.182 \times 10^7, \tag{144}$$

$$X_c^2 = 3{,}790^2 \qquad = \quad 1.436 \times 10^7, \tag{145}$$

$$(X_1^2 + 2X_1X_c + X_c^2) \qquad = \quad 1.260 \times 10^7, \tag{146}$$

$$X(X_1^2 + 2X_1X_c + X_c^2) = 81.8 \times 1.260 \times 10^7 = 1.030 \times 10^9, \tag{147}$$

$$2X_1^2X_c = -2 \times 240^2 \times 3{,}790 \qquad = -0.436 \times 10^9, \tag{148}$$

$$2X_1X_c^2 = 2 \times 240 \times 3{,}790^2 \qquad = 6.90 \times 10^9, \tag{149}$$

$$[X(X_1^2 + 2X_1X_c + X_c^2) + 2(X_1^2X_c + X_1X_c^2)] = 7.49 \times 10^9, \tag{150}$$

$$R(X_1^2 + 2X_1X_c + X_c^2) = 5.66 \times 1.260 \times 10^7 = 7.14 \times 10^7, \tag{151}$$

$$D_Z = (0.0714 + j7.49) \times 10^9, \tag{152}$$

$$\frac{\mathbf{I}_2}{\mathbf{E}_1} = \frac{3{,}790^2 \times 10^{-9}}{0.0714 + j7.49} = \frac{0.01436}{0.0714 + j7.49} = (0.0183 - j1.92)10^{-3} \text{ mho.} \tag{153}$$

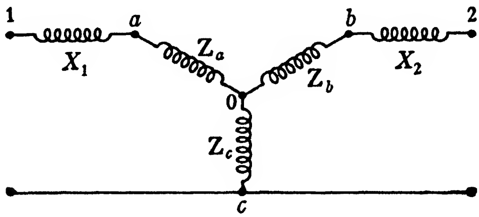


FIG. 13a. Equivalent T for circuit of Fig. 13.

The portion of the circuit between a-b-c can be replaced by an equivalent T. The equivalent T resulting for the whole circuit then can be replaced by an equivalent π. The result of the first step is illustrated by Fig. 13a, in which

$$\left.\begin{aligned} Z_a &= \frac{(-j3{,}790)(5.66 + j81.8)}{(-j3{,}790) + (5.66 + j81.8) + (-j3{,}790)} = \frac{310{,}000 - j21{,}440}{5.66 - j7{,}498} \\ &= 2.89 + j41.3 \text{ ohms,} \end{aligned}\right\} \tag{137a}$$

$$\left.\begin{aligned} Z_b &= \frac{(5.66 + j81.8)(-j3{,}790)}{(5.66 + j81.8) + (-j3{,}790) + (-j3{,}790)} = \frac{310{,}000 - j21{,}440}{5.66 - j7{,}498} \\ &= 2.89 + j41.3 \text{ ohms,} \end{aligned}\right\} \tag{136a}$$

$$\left.\begin{aligned} Z_c &= \frac{(-j3{,}790)(-j3{,}790)}{(-j3{,}790) + (-j3{,}790) + (5.66 + j81.8)} = \frac{-14{,}360{,}000}{5.66 - j7{,}498} \\ &= -1.45 - j1{,}915 \text{ ohms.} \end{aligned}\right\} \tag{135a}$$

The constants for the whole circuit are therefore

$$Z_{01} = Z_a + Z_{a1} = (2.89 + j41.3) + j240 = 2.89 + j281 \text{ ohms,} \tag{154}$$

$$Z_{02} = Z_b + Z_{b2} = 2.89 + j41.3 + j240 = 2.89 + j281 \text{ ohms,} \tag{155}$$

$$Z_{0c} = Z_c \qquad = -1.45 - j1{,}915 \text{ ohms.} \tag{156}$$

Replacing the above T-connected circuit by the equivalent π-connected circuit gives

$$\mathbf{Z}_{12} = \frac{\{(-1.45 - j1915)(2.89 + j281) + (2.89 + j281)(-1.45 - j1{,}915) + (2.89 + j281)(2.89 + j281)\}}{-1.45 - j1{,}915}$$

$$= \frac{(997 - j10.26)10^3}{-1.45 - j1{,}915} = 4.95 + j521 \text{ ohms}, \qquad [138a]$$

$$\mathbf{Z}_{2c} = \frac{(997 - j10.26)10^3}{2.89 + j281} = 0 - j3{,}540 \text{ ohms}, \qquad [139a]$$

$$\mathbf{Z}_{1c} = \frac{(997 - j10.26)10^3}{2.89 + j281} = 0 - j3{,}540 \text{ ohms} \qquad [140a]$$

The general circuit constants for the T-connected circuit are

$$\mathbf{A} = 1 + \frac{\mathbf{Z}_{01}}{\mathbf{Z}_{0c}} \qquad [111a]$$

$$\mathbf{B} = \mathbf{Z}_{01} + \mathbf{Z}_{02} + \frac{\mathbf{Z}_{01}\mathbf{Z}_{02}}{\mathbf{Z}_{0c}} \qquad [112a]$$

$$\mathbf{C} = \frac{1}{\mathbf{Z}_c} \qquad [113a]$$

$$\mathbf{D} = 1 + \frac{\mathbf{Z}_{02}}{\mathbf{Z}_{0c}} \qquad [114a]$$

$$\mathbf{A} = 1 + \frac{2.89 + j281}{-1.45 - j1{,}915} = 0.853 + j0.00140, \qquad [157]$$

$$\left.\begin{aligned}\mathbf{B} &= (2.89 + j281) + (2.89 + j281) + \frac{(2.89 + j281)(2.89 + j281)}{(-1.45 - j1{,}915)}\\ &= 4.97 + j520,\end{aligned}\right\} \qquad [158]$$

$$\mathbf{C} = \frac{1}{-1.45 - j1{,}915} = (-0.396 + j521)10^{-6}, \qquad [159]$$

$$\mathbf{D} = 1 + \frac{2.89 + j281}{-1.45 - j1{,}915} = 0.853 + j0.00140. \qquad [160]$$

13. Steady-state node equations for the general multi-branch network

In this article the general formulation of the equations for a complicated network is carried out on the node basis. In the same way that the results obtained in Art. 4a for a four-loop network were generalized to the l-loop case in Art. 5, this article generalizes to n-node form the results obtained for a five-node network in Art. 4b. Thus the development in this article is a parallel and alternative method to that of Art. 5. Either method leads to the same results, but one is sometimes much less laborious than the other.

By applying to a network having n independent nodes the processes

which are employed in Art. 4b, n equations relating the n unknown node potentials to the known source currents can be written. Consideration is given here to the case of steady-state conditions in which all the sources have the same frequency; so the equations are written in terms of complex voltages, currents, and admittances. These equations have the form

$$Y_{aa}V_a + Y_{ab}V_b + \cdots + Y_{aj}V_j + \cdots + Y_{an}V_n = I_a, \qquad [161]$$
$$Y_{ba}V_a + Y_{bb}V_b + \cdots + Y_{bj}V_j + \cdots + Y_{bn}V_n = I_b, \qquad [162]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Y_{ka}V_a + Y_{kb}V_b + \cdots + Y_{kj}V_j + \cdots + Y_{kn}V_n = I_k, \qquad [163]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Y_{na}V_a + Y_{nb}V_b + \cdots + Y_{nj}V_j + \cdots + Y_{nn}V_n = I_n. \qquad [164]$$

In these equations the one containing I_k, which is written for node k, may be thought of as typical of all of them, and the quantities may be defined for this equation. The quantity I_k is the total source current directed toward node k, expressed as a vector; V_k is the vector potential of node k above the reference node; Y_{kk} is the total self-admittance of node k; j designates any other node having a potential V_j; Y_{kj} or Y_{jk} is the mutual admittance of nodes k and j.

In general, the node potentials are to be determined in terms of the currents. These can be obtained, of course, by solving Eqs. 161 to 164 through use of Cramer's rule or any other method desired. Cramer's rule leads to these relations:

$$V_a = \frac{M_{aa}}{D_Y} I_a + \frac{M_{ba}}{D_Y} I_b + \cdots + \frac{M_{ka}}{D_Y} I_k + \cdots + \frac{M_{na}}{D_Y} I_n, \qquad ▶[165]$$

$$V_b = \frac{M_{ab}}{D_Y} I_a + \frac{M_{bb}}{D_Y} I_b + \cdots + \frac{M_{kb}}{D_Y} I_k + \cdots + \frac{M_{nb}}{D_Y} I_n, \qquad ▶[166]$$

$$V_k = \frac{M_{ak}}{D_Y} I_a + \frac{M_{bk}}{D_Y} I_b + \cdots + \frac{M_{kk}}{D_Y} I_k + \cdots + \frac{M_{nk}}{D_Y} I_n, \qquad ▶[167]$$

$$V_n = \frac{M_{an}}{D_Y} I_a + \frac{M_{bn}}{D_Y} I_b + \cdots + \frac{M_{kn}}{D_Y} I_k + \cdots + \frac{M_{nn}}{D_Y} I_n. \qquad ▶[168]$$

The coefficients of the I's of Eqs. 165 to 168 have a physical interpretation similar to that given for the coefficients of the E's of Eqs. 58 to 61; they are open-circuit transfer and driving-point impedances:

$$z_{kj} = \frac{M_{jk}}{D_Y}, \qquad [169]$$

$$z_{kk} = \frac{M_{kk}}{D_Y}. \qquad [170]$$

In general, of course, they are functions of ω.

The theorem of reciprocity applies to this case as well as to the case in which the voltages are the impressed forces and the currents are the responses. Here the statement of the theorem is that the voltage of any node k resulting from the application of a source current $\mathbf{I}_j$ to any node j, all other source currents being zero, is identical with the voltage of node j resulting from the application of an equal source current $\mathbf{I}_k$ to node k, all other source currents being zero. In other words, the points of application of a pure current source and an ideal voltage-measuring device can be interchanged without affecting the indications of the voltage-measuring device. This is merely another way of saying that $\mathbf{M}_{jk}$ and $\mathbf{M}_{kj}$ are equal.

Equations 165 to 168 are the general steady-state solution of the n-node network for n impressed vector currents $\mathbf{I}_a$, $\mathbf{I}_b \cdots \mathbf{I}_n$, all of the same angular frequency ω. If other angular frequencies are present in the impressed currents, separate solutions must be made for each frequency. If desired, these solutions can be superposed, because of the linearity of the system, to obtain the resultant node voltages.

Occasionally the engineer applies the foregoing analysis in its complete form to networks in which n is four or five or perhaps even more. More frequently, however, the results are useful as applied to the n-node two-terminal-pair network which is considered in the next article.

14. Coupling-network parameters from node equations

One of the useful applications of Eqs. 165 to 168 is to the two-terminal-pair network. The first such application, which can be used in the majority of practical cases, is that in which terminals 1 and 2′ of Fig. 8 are the same node and can be used as the reference node in writing Eqs. 161 to 164. Simple examples of networks in which this condition obtains are the

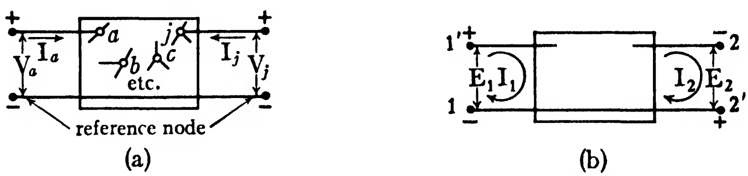


Fig. 14. Coupling network analyzed by node method.

T and π networks of Figs. 11 and 12. The second application, which is included here in the interests of generality, is that in which no two terminals of the coupling network are the same and in which, therefore, the terminal voltage at one end of the coupling network is obtained as the difference in two node potentials. While this situation occurs less frequently in practice than the first, it is important that the methods presented here be general enough to include it.

In Fig. 14 are shown two schematic diagrams of the same network. Diagram (a) has the notation and sign conventions used in the node formulation, and diagram (b) has those used for the coupling network of Fig. 8. Nodes a and j become terminals 1′ and 2, respectively. The relations between the coupling network and node quantities are evidently as follows:

$$\mathrm{E}_1 = \mathrm{V}_a, \tag{171}$$

$$\mathrm{E}_2 = -\mathrm{V}_j, \tag{172}$$

$$\mathrm{I}_2 = -\mathrm{I}_j, \tag{173}$$

$$\mathrm{I}_1 = \mathrm{I}_a. \tag{174}$$

The only sources present are I_a and I_j; all other source currents are zero by the definition of a two-terminal-pair network. Using Eqs. 171 to 174 in the two general equations which give V_a and V_j, and remembering that M_{aj} equals M_{ja}, one finds

$$\mathrm{E}_1 = \frac{\mathrm{M}_{aa}}{\mathrm{D}_Y}\mathrm{I}_1 - \frac{\mathrm{M}_{aj}}{\mathrm{D}_Y}\mathrm{I}_2, \tag{165a}$$

$$\mathrm{E}_2 = -\frac{\mathrm{M}_{aj}}{\mathrm{D}_Y}\mathrm{I}_1 + \frac{\mathrm{M}_{jj}}{\mathrm{D}_Y}\mathrm{I}_2. \tag{166a}$$

A comparison of Eqs. 165a and 166a with Eqs. 58c and 59b for the two terminal pair shows that the open-circuit driving-point and transfer impedances characterizing the two terminal pair are given in terms of the impedances obtained by the node formulation by

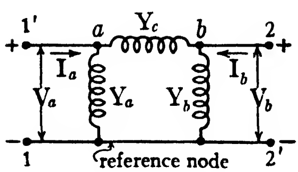


Fig. 15. π network analyzed by node method.

$$\mathrm{z}_{11} = \frac{\mathrm{M}_{aa}}{\mathrm{D}_Y}, \tag{175}$$

$$\mathrm{z}_{22} = \frac{\mathrm{M}_{jj}}{\mathrm{D}_Y}, \tag{176}$$

$$\mathrm{z}_{12} = -\frac{\mathrm{M}_{aj}}{\mathrm{D}_Y}. \tag{177}$$

As a simple illustration, the z's for the π network of Fig. 15 are obtained by use of the node formulation just developed. The node equations are

$$\mathrm{I}_a = (\mathrm{Y}_a + \mathrm{Y}_c)\mathrm{V}_a - \mathrm{Y}_c\mathrm{V}_b, \tag{178}$$

$$\mathrm{I}_b = -\mathrm{Y}_c\mathrm{V}_a + (\mathrm{Y}_b + \mathrm{Y}_c)\mathrm{V}_b. \tag{179}$$

Here node j of Eqs. 172 and 173 becomes b. Solving Eqs. 178 and 179 for

the V's gives for the cofactors M:

$$M_{aa} = Y_b + Y_c, \tag{180}$$

$$M_{bb} = Y_a + Y_c, \tag{181}$$

$$M_{ab} = Y_c, \tag{182}$$

and by Eqs. 175 to 177 the z's become

$$z_{11} = \frac{Y_b + Y_c}{D_Y}, \tag{121}$$

$$z_{22} = \frac{Y_a + Y_c}{D_Y}, \tag{122}$$

$$z_{12} = -\frac{Y_c}{D_Y}. \tag{123}$$

in which

$$D_Y = Y_aY_b + Y_bY_c + Y_cY_a. \tag{124}$$

For the special case of the π network, D_Y and D_y are equal.

Of course the π network is so easily solved by any method that the procedure just used seems unduly complicated and indirect. In more complicated networks, however, the node method may easily be a less laborious way than the loop method of Art. 5 to obtain the desired coupling-network parameters. Once the z's have been obtained, all the results of Arts. 9 and 10 expressed in terms of these z's can be used. In other words, the z's given by Eqs. 175 to 177 are identical with those of Arts. 9 and 10.

The foregoing results apply to the first case, that in which the terminals 1 and 2′ of the coupling network are common and are used as the reference node in the formulation of the node equations. The second case, in which all four terminals 1, 1′, 2, and 2′ are separate nodes of the network, is more general and also slightly more complicated. Such a case is shown in Fig. 16. This network contains n independent nodes $a, b \cdots n$. Of these, nodes a, j, and k are brought out as terminals 1′, 2, and 2′, respectively, of the coupling network. Terminal 1 is taken as the reference node. In the general node formulation, a source

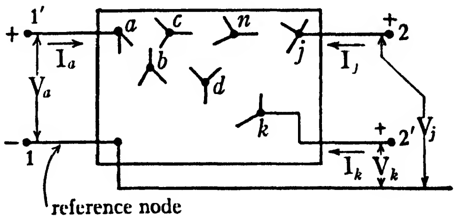


FIG. 16. n-node network analyzed as a two terminal pair.

current is expressed as a current directed toward the node. These are shown as I_a, I_j, and I_k, subsequently expressed in terms of the currents I_1 and I_2 of the coupling network. Voltages V_a, V_j, and V_k are the vector amounts by which the potentials of nodes a, j, and k, respectively, exceed the potential of the reference node.

Since the parameters of the network of Fig. 16 are assumed to be known, equations like Eqs. 161 to 164 can be written by following the procedure carried out in detail in Art. 9. When solved for the V's, the resulting equations have the form of Eqs. 165 to 168. Only the equations involving the potentials of nodes a, j, and k need be considered. Furthermore, since sources are connected only to these same nodes, no terms other than those in I_a, I_j, and I_k are present. Equations 165 to 168 reduce therefore to

$$V_a = \frac{M_{aa}}{D_Y} I_a + \frac{M_{ja}}{D_Y} I_j + \frac{M_{ka}}{D_Y} I_k, \qquad [165b]$$

$$V_j = \frac{M_{aj}}{D_Y} I_a + \frac{M_{jj}}{D_Y} I_j + \frac{M_{kj}}{D_Y} I_k, \qquad [166b]$$

$$V_k = \frac{M_{ak}}{D_Y} I_a + \frac{M_{jk}}{D_Y} I_j + \frac{M_{kk}}{D_Y} I_k. \qquad [167a]$$

Equations 165b to 167a can be rewritten in terms of the coupling-network parameters if the V's and I's of Fig. 16 are expressed in terms of the E's and I's of the coupling network. Thus a comparison of Figs. 16 and 8 shows them to be identical if

$$E_1 = V_a, \qquad [183]$$

$$E_2 = V_k - V_j, \qquad [184]$$

$$I_1 = I_a, \qquad [185]$$

$$I_2 = I_k = -I_j. \qquad [186]$$

Substituting the values for I_k and I_j from Eq. 186 in Eqs. 165b to 167a, subtracting the second equation from the third to obtain E_2, and combining terms in like currents give the two equations

$$E_1 = \frac{M_{aa}}{D_Y} I_1 + \left(\frac{M_{ak} - M_{aj}}{D_Y}\right) I_2, \qquad [165c]$$

$$E_2 = \left(\frac{M_{ak} - M_{aj}}{D_Y}\right) I_1 + \left(\frac{M_{jj} + M_{kk} - 2M_{jk}}{D_Y}\right) I_2. \qquad [166c]$$

These two equations take the standard form for the coupling-network equations in terms of the z's if

$$z_{11} = \frac{M_{aa}}{D_Y}, \qquad [187]$$

$$z_{22} = \frac{M_{jj} + M_{kk} - 2M_{jk}}{D_Y}, \qquad [188]$$

$$z_{12} = \frac{M_{ak} - M_{aj}}{D_Y}. \qquad [189]$$

These z's are the same as the z's of Arts. 9 and 10, and as soon as its z's are found by Eqs. 187 to 189, the network of Fig. 16 can be treated as a coupling network by the methods of these two articles.

15. Thévenin's theorem

In Art 9, Ch. II, a useful relation known as Thévenin's theorem is demonstrated for a direct-current network containing resistances only. This theorem suitably extended is also true for alternating-current steady-state conditions for the general linear network containing any number of constant vector-voltage or current sources. In fact, the theorem can also be applied to transient conditions, but this subject lies beyond the present treatment.

Though usually not phrased in this way, Thévenin's theorem for the case considered here may be stated as follows:

▶In the alternating-current steady state and for any given single frequency, any network containing only linear passive elements and constant vector-voltage or current sources is, when viewed from any given pair of terminals, indistinguishable from a simple vector-voltage source consisting of a constant vector electromotive force in series with a constant impedance. The vector electromotive force is equal to the open-circuit voltage across the given pair of terminals, and the series impedance is equal to the impedance of the given network as viewed from the same terminals with all sources replaced by connections of zero impedance. The vector current source, which is the equivalent of this vector-voltage source, as shown in Fig. 11, p. 390, may likewise be substituted for the given network.◀

This theorem is useful when the characteristics of one branch of a network are to be changed by varying its parameters or where two networks are to be connected through two pairs of terminals. In the latter case, if each network is converted to a simple series circuit, both the current between the two networks and the terminal voltages of the networks may be calculated readily.

The proof of Thévenin's theorem for the alternating-current steady state is identical with the proof in Art. 9, Ch. II, if resistance symbols are replaced by complex impedance symbols, and voltages and currents are regarded as complex.

16. Illustrative example of Thévenin's theorem

As an example of an application of Thévenin's theorem, the three-loop circuit of Fig. 17a is replaced by the circuit of Fig. 17b, which has identical characteristics when viewed from the terminals a and b.

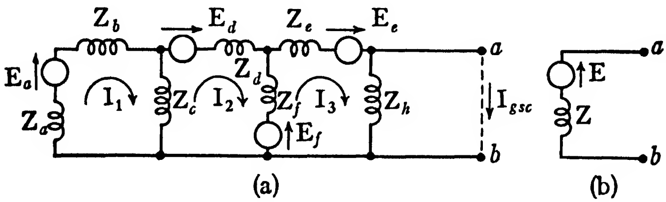


FIG. 17. Application of Thévenin's theorem to a three-loop network.

Solution: In this network, the loop-voltage equations are:

$$Z_{11}I_1 + Z_{12}I_2 + Z_{13}I_3 = E_1 = E_a, \qquad [54c]$$

$$Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 = E_2 = E_d - E_f, \qquad [55c]$$

$$Z_{31}I_1 + Z_{32}I_2 + Z_{33}I_3 = E_3 = E_f + E_e, \qquad [57c]$$

in which

$$Z_{11} = Z_a + Z_b + Z_c, \qquad [190]$$

$$Z_{12} = -Z_c, \qquad [191]$$

$$Z_{13} = 0, \qquad [192]$$

$$Z_{22} = Z_c + Z_d + Z_f, \qquad [193]$$

$$Z_{23} = -Z_f, \qquad [194]$$

$$Z_{33} = Z_f + Z_e + Z_h. \qquad [195]$$

In the equivalent circuit,

$$E = V_{ab} = I_3Z_h. \qquad [196]$$

Applying Cramer's rule to obtain I_3 gives for E,

$$E = \frac{Z_h}{D_Z}(M_{13}E_1 + M_{23}E_2 + M_{33}E_3). \qquad [197]$$

When a connection of zero impedance is made from a to b,

$$I_{gsc} = I_{3sc} = \frac{1}{D_{Zsc}}(M_{13}E_1 + M_{23}E_2 + M_{33}E_3), \qquad [61a]$$

in which the determinant D_{Zsc} is the determinant of the Z's with Z_{33} equal to $Z_f + Z_e$ instead of $Z_f + Z_e + Z_h$ as before. Then

$$Z = \frac{V_{ab}}{I_{gsc}} = \frac{V_{ab}}{I_{3sc}} = \frac{Z_h D_{Zsc}}{D_Z}. \qquad [198]$$

Equations 197 and 198 give the necessary values of E and Z for equivalence.

It should be noted that the values of the parenthetical terms in Eqs. 197 and 61a are identical, because in both the three cofactors are taken with respect to column 3 of the determinant of Z's, so that no column-3 terms are involved; hence the change in the Z_{33} term caused by short-circuiting a and b does not affect the expansion of the minors, although it does affect the value of the determinant D_Z. The value of Z found in Eq. 198 is the same as the impedance of the network as viewed from terminals a and b with all sources replaced by short circuits. Other useful applications of Thévenin's theorem can be worked out by the student. In general, labor is saved if the circuit diagram is so arranged for analysis that only one loop current (I_3 in this case) passes between the terminals a and b to which the theorem is applied.

Once the equivalent elements of Fig. 17b are obtained, the current I in any impedance Z_L connected to the terminals a–b evidently can be found from

$$I = \frac{E}{Z + Z_L}. \qquad [199]$$

This is the usual statement of Thévenin's theorem in which E is understood to be the open-circuit voltage and Z is interpreted as the short-circuit self-impedance of the circuit viewed from the given pair of terminals.

PROBLEMS

1. What are the Kirchhoff equations in complex form for the network of Fig. 18
(a) on the node basis, using the ground as the reference node?
(b) on the loop basis?

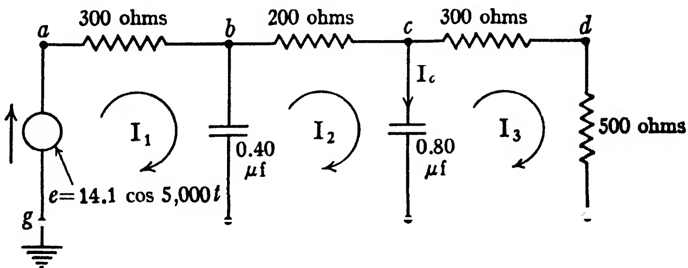


FIG. 18. Circuit for Prob. 1.

What is the general expression for the complex ratio of I_c to I_1 in terms of symbols representing network parameters? What value has the expression for the parameters of Fig. 18?

2. The parameter values indicated for the circuit of Fig. 19 are in ohms, henrys, and megadarafs. The values of the vector-voltage and vector-current sources are in

volts and amperes at 0 phase angle. For the given loop reference arrows, what are the loop and mutual parameters? What are the steady-state voltage equilibrium equations for ω equal to 1,000 radians/sec?

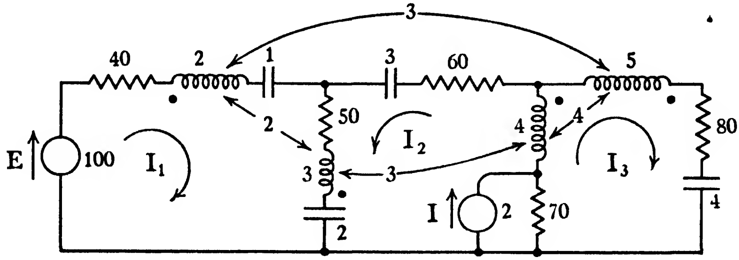
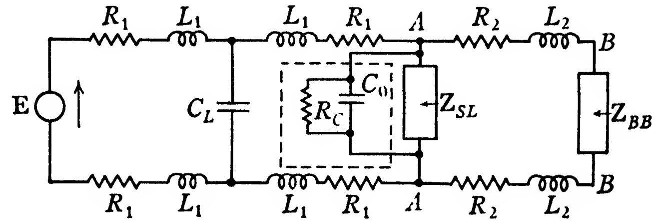


FIG. 19. Circuit for Prob. 2.

3. The diagram of Fig. 20 is a simplified approximation of one phase of a high-tension transmission line connecting a generating station at E to a substation Z_{SL} at A–A, supplying a general community load, and to a large industrial plant at B–B. The power factor of the general load is improved at the substation by means of a synchronous condenser, represented by R_C and C_0 inclosed by the dotted rectangle.



$E = 75{,}000$ v	$R_2 = 23.4$ ohms
$f = 60 \sim$	$L_2 = 0.068$ h
$R_1 = 8.2$ ohms	$C_0 = 10\ \mu$f
$L_1 = 0.065$ h	$R_C = 12{,}500$ ohms
$C_L = 0.5\ \mu$f	$Z_{SL} = 165 + j103$ ohms
$Z_{BB} = 705 + j232$ ohms	

FIG. 20. Simplified equivalent circuit of power system, Prob. 3.

What are the power delivered by the generating station and the corresponding power factor? The method to be used in this problem is to determine the vector value of generator current

(a) by reducing the network by combining impedances;
(b) by the loop method, using the three major loops, and using the process of elimination for solving for the generator current;
(c) by the loop method, using determinants.

4. What are the driving-point impedances of the circuits of Fig. 21 for which the parameters satisfy the relation

$$\frac{L_1}{C_2} = \frac{L_2}{C_1} = \frac{L}{C} = R^2?$$

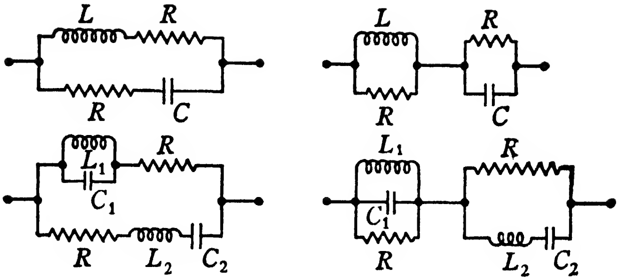


FIG. 21. Circuits for calculating driving-point impedances, Prob. 4.

5. The bridge circuit of Fig. 22 is used to determine the inductance of a coil. The value of L can be calculated from the values of the parameters in the bridge arms when the bridge is balanced, that is, when the current through the detector is 0.

(a) What are the loop equations and the general solution for the loop currents I_1, I_2, and I_3?

(b) What is the relation between the unknown L and the parameters C, R_C, R_1, R_2, and R_L for the condition when the bridge is balanced?

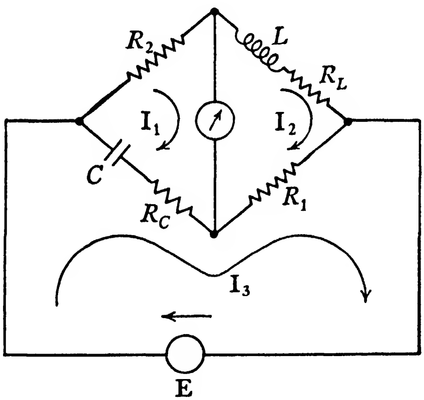


FIG. 22. Hay bridge, Prob. 5.

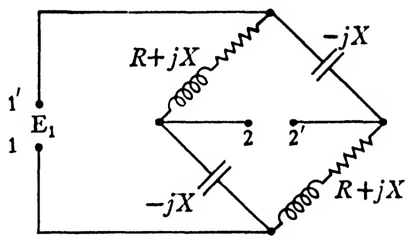


FIG. 23. Monocyclic square, Prob. 6.

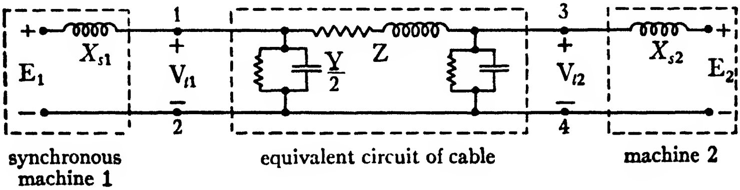
6. Figure 23 shows the circuit of a coupling network known as a *monocyclic square*, used in connection with a proposed method of high-voltage power transmission to transform a constant voltage, impressed on one diagonal, into a constant current across the other diagonal, and inversely. The resistance R of the reactor coils is made as small as practicable and economical. If a load of impedance Z_L is connected to terminals 2 and 2′,

(a) What is the expression for the transfer admittance Y_{12} between the source E_1 and the load Z_L? In the idealized case of zero resistance, is the current through the load Z_L independent of the load impedance and hence does the network transform a constant voltage into a constant current?

(b) If in the idealized case of zero resistance a constant current is impressed across 1–1′, does a constant voltage appear across 2–2′ regardless of the load impedance Z_L?

(c) *What is the expression for the input admittance at terminals 1–1′ with terminals 2–2′ short-circuited? What is the expression for the transfer admittance giving the current at short-circuited terminals 2–2′ per volt impressed at terminals 1–1′?*

(d) For the idealized case of zero resistance, what are the network equations on the node-potential basis? By determinants, what is the solution for the voltage across the load impedance Z_L? What is the explanation of the result?



$$E_1 = E_2 = 10{,}000 \text{ v}$$
$$X_{s1} = X_{s2} = 20 \text{ ohms}$$
$$Z = 4 + j4 \text{ ohms}$$
$$Y = (20 + j2{,}200) \times 10^{-6} \text{ mho}$$

FIG. 24. Synchronous machines connected by cable, Prob. 7.

7. The circuit of Fig. 24 represents in the first approximation two synchronous rotating machines connected electrically through a cable. As shown, a synchronous machine can be represented electrically in the steady state by a constant vector voltage E in series with a reactance X_s.

A synchronous machine operates as a motor if V_t leads E in phase, and as a generator if E leads V_t in phase. In the circuit of Fig. 24 machine 1 acts as a generator delivering power to machine 2 if E_1 is leading E_2.

(a) What are the Kirchhoff's equations on the loop basis? Numerical values are to be given for all self- and mutual impedances.

(b) What is the vector power at the terminals + and − of E_1 and at the terminals + and − of E_2 as a function of the phase angle $(\psi_1 - \psi_2)$, by which the vector voltage E_1 leads the voltage E_2?

(c) When ψ_1 is appreciably larger than ψ_2, that is, when E_1 leads E_2 by more than some minimum angle, does machine 1 operate as a generator or as a motor? What is this minimum angle?

(d) As an additional interesting exercise this problem may be solved on the node basis, representing the two machines by their equivalent current sources. This results in a circuit of the form shown on Fig. 24a.

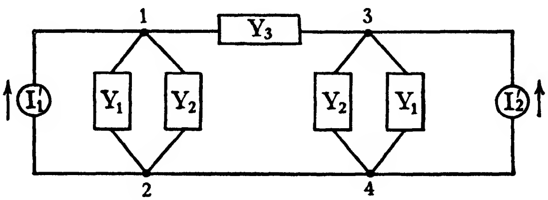
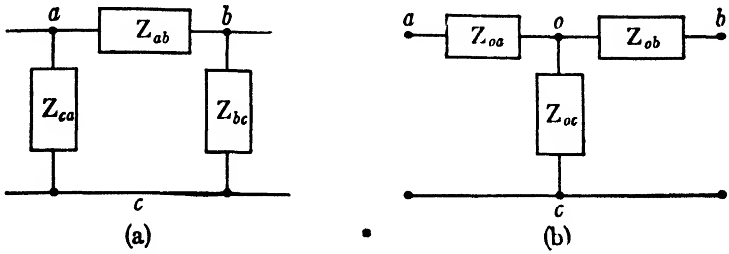


FIG. 24a. Modification of Fig. 24 for solution by the node method, Prob. 7.

8. At a given frequency what impedances Z_{oa}, Z_{oc}, and Z_{ob} make the two circuits of Fig. 25 exactly equivalent between their terminals *ab*, *bc*, and *ca*?



$$Z_{ab} = 10\,\epsilon^{j(\pi/4)} \text{ ohms}$$
$$Z_{bc} = 15\,\epsilon^{j(\pi/3)} \text{ ohms}$$
$$Z_{ca} = 5.0\,\epsilon^{j(\pi/6)} \text{ ohms}$$

FIG. 25. Equivalent π and T circuits, Prob. 8.

9. For the circuit of Fig. 26

(a) What are the equations which give E_1 and I_1 in terms of the circuit parameters and E_2' and I_2?
(b) What are the general circuit constants of the circuit?
(c) What is the open-circuit input impedance from end 1?
(d) What is the open-circuit transfer impedance?

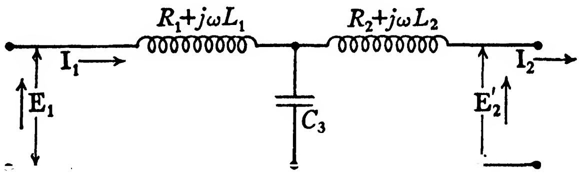


FIG. 26. Coupling network for Prob. 9.

10. The branches of the network at the left end of Fig. 27 contain only resistance with ohmic values as indicated. The network at the right is characterized by the follow-

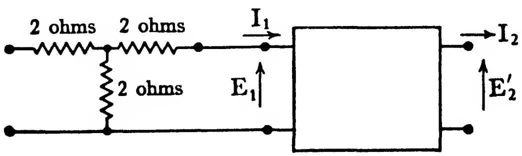


FIG. 27. Coupling networks for Prob. 10.

ing data: With the right-hand terminals open-circuited

$$\frac{E_1}{E_2'} = \frac{E_2'}{I_1} = 2 \textit{ numerically}; \qquad [201]$$

with the right-hand terminals short-circuited

$$\frac{I_1}{I_2} = 2. \qquad [202]$$

(a) What are the general circuit constants for the right-hand network?
(b) What is the T-circuit representation for the right-hand network?
(c) What are the equivalent T and the equivalent π circuits for the combination of both networks in cascade?
(d) What are the general circuit constants for the resultant combination?

11. A transformer has the following parameters:

R_1	L_1	R_2	L_2	M
15 ohms	1.0 h	0.60 ohm	0.04 h	0.18 h

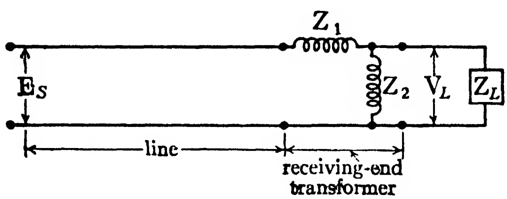
What are the values of the general circuit constants at an angular frequency of 5,000 radians/sec? The No. 1 side is the input side of the transformer.

12. A transmission line 225 miles in length supplies a load of 25,000 kw at 0.85 power factor lagging. The voltage at the load is 127 kv. The line has the following circuit constants:

$$A = D = 0.9/0.02 \text{ radian}, \qquad [203]$$

$$C = j10^{-3} \text{ mho}. \qquad [204]$$

(a) What is the voltage at the sending end referred to the load voltage?
(b) What are the power supplied at the sending end and the efficiency of the line?
(c) What is the magnitude of the voltage at the receiving end if the load is disconnected and the sending-end voltage held at the value found in part (a)?



$Z_1 = 3 + j40$ ohms
$Z_2 = 1{,}200 + j9{,}000$ ohms
Z_L = load of Prob. 12

FIG. 28. Diagram of transmission line, transformers, and load, Prob. 13.

13. The transmission line of Prob. 12 has a receiving-end transformer which may be represented by the equivalent circuit in Fig. 28, in which all values are referred to the high-tension side of the transformers.

(a) What is the sending-end voltage E_S, corresponding to the load of Prob. 12, referred to V_L?
(b) What are the sending-end power and power factor for the load of Prob. 12?
(c) If V_L is held constant at 127 kv, what is the sending-end voltage E_S for the condition of no load at the receiving end ($Z_L = \infty$) but with the transformer connected?

(d) What are the power input and power factor at the sending end under no-load conditions as described in (c)?

(e) What are the general circuit constants of the combined line and transformer?

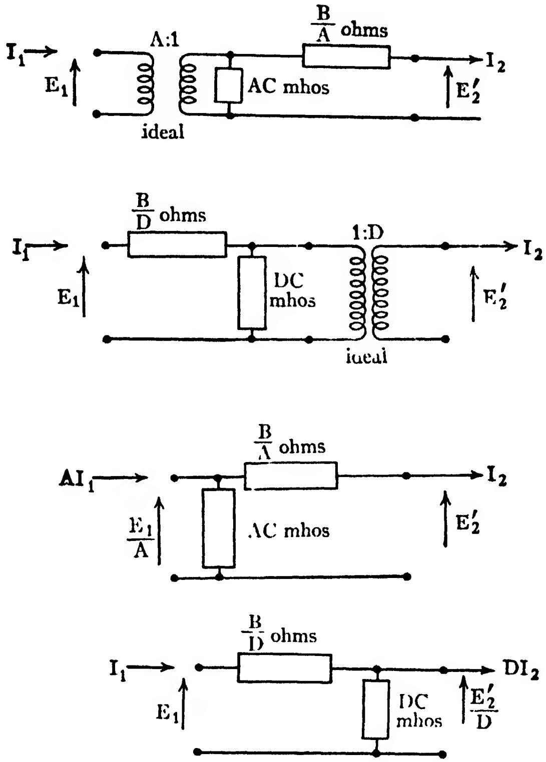


FIG. 29. Equivalent circuits of transmission system, Prob. 14.

14. A transmission system has the general circuit constants A, B, C, D. Can the circuits of Fig. 29 be used as equivalents?

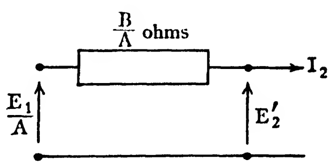


FIG. 30. Equivalent circuit of transmission system, Prob. 15.

15. By applying Thévenin's theorem to the network represented by the general-circuit-parameter equations, can it be shown that so far as receiving-end conditions are concerned, the transmission system may be represented by the circuit of Fig. 30, so that for any load impedance Z_L, the ratio of voltages becomes

$$\frac{E_1}{E_2'} = A + \frac{B}{Z_L} ? \qquad [205]$$

CHAPTER IX

Loci of Complex Functions

1. Impedance and admittance loci; circle diagrams

In Ch. IV the steady-state impedances, admittances, voltages, and currents in simple series and parallel circuits are studied from the analytical point of view. Graphical representations which aid in a visualization of the analytical expressions are also given, in the form of vector diagrams drawn in the complex plane. These vector diagrams are such convenient and powerful aids to thought in alternating-current circuit theory that facility in their construction and interpretation is essential to anyone working in electrical engineering. In Chs. VI and VIII the analytical study of alternating-current circuits in the steady state is extended to include methods of solving more complicated networks, and means are developed for representing the behavior of these networks in terms of circuit characteristics as viewed from one pair or two pairs of terminals. Thus the general circuit constants and Thévenin's theorem provide convenient ways of lumping together all but one element of a network so that the influence of this one branch may be readily studied. In this chapter these methods of analysis and demonstration are shown to be useful in the investigation of the characteristics of circuits having one parameter which varies according to known restrictions.

In the plots of steady-state time vectors in previous chapters, the locus of the terminal of any individual rotating vector is a circle concentric with the origin, and the angular displacement of the vector at any instant is proportional to the independent variable, time. In this chapter the use of this vector diagram is for the steady state only, but is extended to include a range of steady-state conditions by letting a vector sweep out a locus in the complex plane as either a parameter or the frequency is varied. This application differs from the concepts of previous chapters in that the locus is not swept out by a *time* vector. By means of this extended vector diagram, the circuit behavior can be visualized and quantitatively analyzed not only for a single steady-state condition but also for a range of conditions. This method is especially convenient and practically useful because these loci prove to be circles (or straight lines) in many useful cases. They are easily drawn after a small amount of computation. For these reasons such loci — or circle diagrams, as they are commonly called — are widely used in both the power and the communications fields.

A very simple example serves to illustrate the fundamental principles.

If it is desired to determine the locus in the complex plane of the current in a series circuit having a resistance R and an inductance L, as the frequency of the constant-amplitude voltage source is varied from zero to infinity, the general procedure is

(a) to plot the locus of the Z vector,
(b) to determine from the Z-vector locus the corresponding Y-vector locus,
(c) to multiply the Y-vector locus by the impressed vector voltage E to obtain the locus of the I vector.

The first step is to plot the locus of

$$Z = R + j\omega L \qquad [1]$$

in the complex plane as a function of the angular frequency ω (or the frequency f if preferred), as shown in Fig. 1. This locus evidently consists of a straight line parallel to the j axis, displaced R units to the right of it. It extends only above the real axis since $j\omega L$ has only positive imaginary values. On this locus can be marked off a scale of ω which is indicated in terms of an arbitrary unit ω_1.

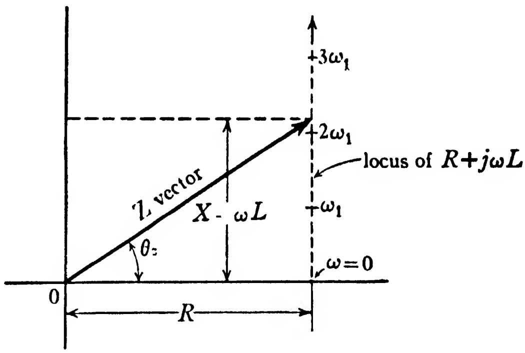


FIG. 1 Locus of vector impedance $Z = R + j\omega L$ as a function of angular frequency ω. (ω_1 is an arbitrary unit of ω.)

The second step involves the important new concept of obtaining from the locus of the complex impedance the locus of the complex admittance $Y(\omega)$, where

$$Y(\omega) \equiv \frac{1}{Z(\omega)}. \qquad [2]$$

Since this process of getting a reciprocal locus graphically is at the heart of the graphical circle-diagram methods, it requires detailed study.

The problem is now considered as one in analytical geometry in which the symbols represent merely algebraic functions. The complex function Z is given. Both R and X are, in the general case, real functions of a real variable. If this variable is called u, then Z is a function of u and

$$Z(u) = R(u) + jX(u), \qquad [1a]$$

in which functional notation is used to emphasize this dependence of Z,

R, and X upon u. If $Z(u)$ is plotted on a set of co-ordinates R and X, the plane thus determined is called the Z plane, and the locus of $Z(u)$ as u is varied over its range might look like the curve of Fig. 2a. It is desired to find the locus of the complex function $Y(u)$ where

$$Y(u) \equiv \frac{1}{Z(u)} = \frac{1}{R(u) + jX(u)}, \tag{2a}$$

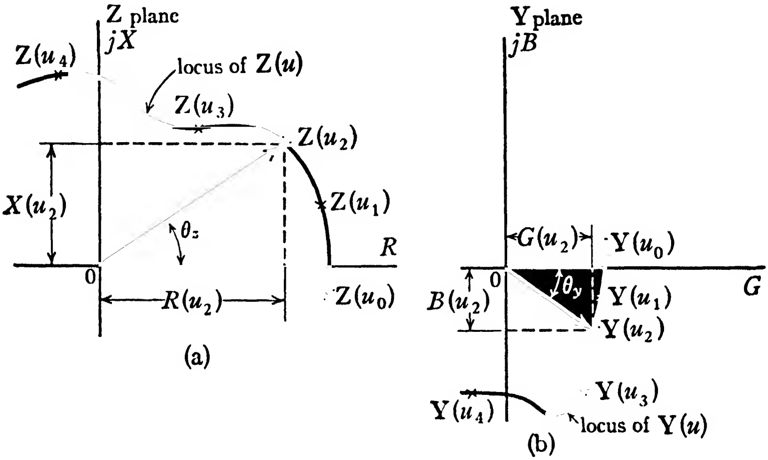


FIG. 2. Corresponding loci of Z vector in Z plane and Y vector in Y plane.

Dropping the functional notation for simplicity in writing and rationalizing the expression for Y gives these expressions:

$$Y \equiv \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = G + jB, \tag{2b}$$

$$G \equiv \frac{R}{R^2 + X^2}, \tag{3}$$

$$B \equiv \frac{-X}{R^2 + X^2}, \tag{4}$$

where G and B are conductance and susceptance functions, respectively, as defined in Ch. IV. Although the functional notation is omitted from these equations, it should be remembered that, in general, Z, Y, R, X, G, and B are all functions of the real variable u. The complex Y function, or locus of $Y(u)$ as u varies over its range, is next plotted on a new set of co-ordinate axes whose complex plane is called the Y plane. This locus, shown in Fig. 2b, is the *complex inversion* of the locus of Fig. 2a.

By taking the reciprocal of Y in Eq. 2b, R and X can be obtained in terms of G and B:

$$Z \equiv \frac{1}{Y} = \frac{1}{G + jB} = \frac{G - jB}{G^2 + B^2} = R + jX, \tag{2c}$$

$$R \equiv \frac{G}{G^2 + B^2}, \tag{5}$$

$$X \equiv \frac{-B}{G^2 + B^2}. \tag{6}$$

In Eqs. 2b and 2c the imaginary part of Y is opposite in sign to the imaginary part of Z. It is therefore convenient in the process of inversion to obtain Y from Z in two principal steps geometrically. In the first, the conjugate $\overline{Y}$ of Y is obtained, where

$$\overline{Y} = G - jB = \frac{R}{R^2 + X^2} + j\frac{X}{R^2 + X^2} = G + jB', \tag{7}$$

$$B' = -B = \frac{X}{R^2 + X^2}. \tag{8}$$

As the second step, Y is obtained from $\overline{Y}$ by mere substitution of $-B$ for the B' of $\overline{Y}$, that is, by taking the mirror image of $\overline{Y}$ with respect to the real or G axis.

In order to avoid the analytical geometry of the complex plane with which the student may not be familiar, several intermediate steps are taken to permit use of only the analytical geometry of the real plane for finding the $Y(u)$ locus. These steps are illustrated by inversion of a point $R + jX$ on the curve in the complex Z plane. From this point is determined a corresponding point (R, X) in a real plane whose abscissa is R and whose ordinate is X. This real plane is called the R-X plane. It is identical geometrically with the Z plane but its ordinates are real numbers and its analytic geometry is that of a real plane. Next, by Eqs. 3 and 8, the point (R, X) in the real R-X plane is transformed to the point (G, B') in a real G-B' plane defined by a G axis of abscissas and a B' axis of ordinates. This is *geometric inversion*. To obtain the point $G + jB'$ in the complex Y plane from this point (G, B') is merely a matter of renaming the axes. The final point $G + jB$, or $G - jB'$ in the complex Y plane, is then readily obtained by merely reflecting the point $G + jB'$ about the real or abscissa axis. These successive steps in locating a Y-plane point from a Z-plane point are indicated by diagrams (a) to (e) of Fig. 3.

Since a locus is a curve passing through a series of points, the correspondence of two loci depends upon the correspondence of points lying on

two different curves. In electric circuits the Z-plane locus is often either a circle or a straight line which can be treated as a portion of the circumference of an infinitely large circle. If a circle is located as

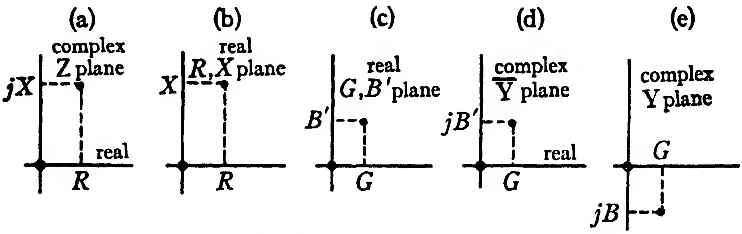


FIG. 3. Steps in inversion of point $R + jX$ in Z plane to point $G + jB$ in Y plane.

shown in Fig. 4 with respect to the co-ordinate axes, the equation for this circle, in terms of the co-ordinates R and X in the real R- X plane, is

$$(R - \alpha)^2 + (X - \beta)^2 = r^2, \quad [9]$$

which, for transformation by Eqs. 5 and 6, is better expanded to the form

$$R^2 + X^2 - 2\alpha R - 2\beta X + \alpha^2 + \beta^2 - r^2 = 0. \quad [9a]$$

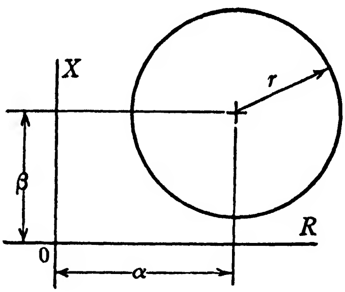


FIG. 4. Circular locus in R–X plane.

If the values for R and X from Eqs. 5 and 6, and the value for B' from Eq. 8 are substituted in Eq. 9a, this relation results:

$$\frac{1}{G^2 + (B')^2} - 2\alpha \frac{G}{G^2 + (B')^2} - 2\beta \frac{B'}{G^2 + (B')^2} + \alpha^2 + \beta^2 - r^2 = 0. \quad [9b]$$

Multiplying by $\dfrac{G^2 + (B')^2}{\alpha^2 + \beta^2 - r^2}$ gives

$$\frac{1}{\alpha^2 + \beta^2 - r^2} - \frac{2\alpha G}{\alpha^2 + \beta^2 - r^2} - \frac{2\beta B'}{\alpha^2 + \beta^2 - r^2} + G^2 + (B')^2 = 0, \quad [9c]$$

which, by completing the square, can be rewritten as

$$\left.\begin{aligned}\left(G - \frac{\alpha}{\alpha^2 + \beta^2 - r^2}\right)^2 + \left(B' - \frac{\beta}{\alpha^2 + \beta^2 - r^2}\right)^2 \\ = \frac{\alpha^2 + \beta^2}{(\alpha^2 + \beta^2 - r^2)^2} - \frac{\alpha^2 + \beta^2 - r^2}{(\alpha^2 + \beta^2 - r^2)^2} \\ = \frac{r^2}{(\alpha^2 + \beta^2 - r^2)^2}.\end{aligned}\right\} \qquad [9d]$$

This is recognized as the equation of a circle in terms of the $\overline{Y}$-plane co-ordinates G and B'. Thus a circle in the Z plane becomes a circle in the $\overline{Y}$ plane when Z and Y are related by Eq. 2 and $\overline{Y}$ is the conjugate of Y.

The $\overline{Y}$-plane circle can be determined by Eq. 9d or by such a graphical method as follows. From Eqs. 7 and 8 it is easily seen that

$$G^2 + (B')^2 = \frac{1}{R^2 + X^2}, \qquad [10]$$

and that

$$\frac{G}{B'} = \frac{R}{X}. \qquad [11]$$

The equivalents of these relations are expressed in words as follows:

(a) The distance from the origin of any point in the $\overline{Y}$ plane is the reciprocal of the distance from the origin of the corresponding point in the Z plane.

(b) The angles of the corresponding radius vectors in the $\overline{Y}$ plane and in the Z plane are equal.

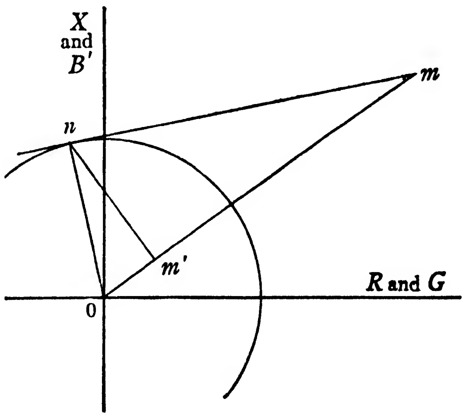


FIG. 5 Construction used for geometric inversion of point m to point m' with the aid of a circle of unit radius.

If it is desired to find the location in the $\overline{Y}$ plane of the point m', which is the geometric inverse of the point m in the Z plane of Fig. 5, the $\overline{Y}$ and Z planes are superposed with their R and G axes and their X and B' axes coincident. By rule (b) above, the geometrically inverse point must lie on the straight line $0m$ drawn from the origin through the point m. By rule (a), the distance $0m'$ from

the origin to the desired geometrically inverse point m' is equal to $1/(0m)$, which can easily be found by measurement of $0m$ and computation of its reciprocal.

A geometric construction may be employed as an alternative method of determining the distance $0m$. A circle of unit radius is drawn, having its center at the origin. From the point m a line mn is drawn tangent to the unit circle at n, and a perpendicular is constructed from n to $0m$, intersecting $0m$ at m'. Then, since triangles $0nm$ and $0m'n$ are similar,

$$\frac{0m'}{0n} = \frac{0n}{0m}, \tag{12}$$

$$(0m')(0m) = (0n)^2 = 1, \tag{12a}$$

$$0m' = \frac{1}{0m}. \tag{12b}$$

Since $0m'$ lies in the same direction as $0m$, the angles of these radius vectors with the R axis are equal, and m' is therefore the geometrically inverse point to m.

In Fig. 6 the Z-plane circle is transformed by applying the construc-

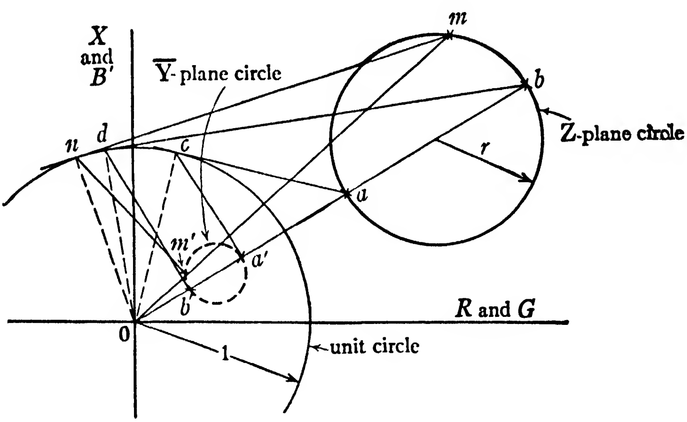


FIG. 6 Three points m, b, and a, transformed to their geometrically inverse points m', b', and a'.

tion outlined above to the three points m, b, and a, to obtain the geometrically inverse points m', b', and a' which determine the $\overline{\text{Y}}$-plane circle. This circle is the geometrically inverse locus of the Z-plane circle. For determining the inverse circle, the points a' and b', inverse to points a and b which lie in the same straight line on a diameter of the Z-plane circle, are adequate for determining the inverse circle, since $a'b'$ is a

diameter of that circle. It should be noted carefully that since the center of a circle is not a point on the circle, the center cannot be inverted as a means of finding the center of the inverse circle.

The Y circle, that is, the complex inversion of the Z circle, remains to be found. Since Y is merely the conjugate of $\overline{Y}$, the Y circle is the image of the $\overline{Y}$ circle about the axis of reals. Figure 7 shows the Z, $\overline{Y}$, and Y circles for a typical case in which the Z circle lies in the fourth quadrant. Unprimed, primed, and double-primed letters indicate corresponding points on the Z, $\overline{Y}$, and Y loci, respectively.

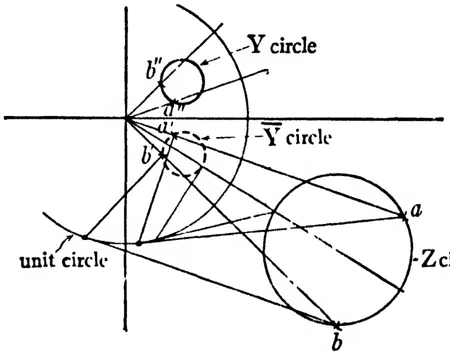


FIG. 7. Construction for complex inversion of circular Z locus.

In Fig. 8 is shown the construction for a special case in which the Z circle is infinitely large and is removed to infinity except for a portion of arc which is the straight line *abc*. The same construction as that previously described can be applied to all points on the straight line that lie outside the unit circle. For those lying inside, such as *b*, the construction of Fig. 5 can be reversed by erecting to *ob* at *b* a normal which intersects the unit circle at *n* and by extending a tangent to the unit circle at *n* to intersect *ob* at *b′*. The point *b′* is thus the geometrically inverse point to *b*.

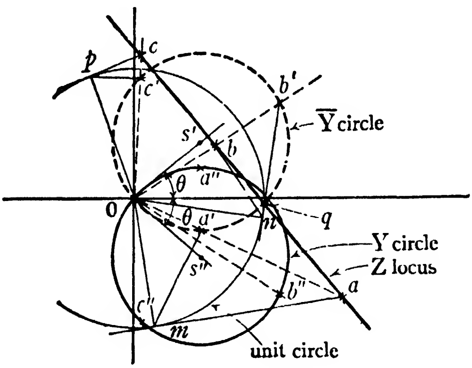


FIG. 8. Construction for complex inversion of straight-line Z locus.

The origin is a point on the inverse of any straight line, and a diameter through the origin is perpendicular to the straight line.

Figure 9 is another special case in which the Z circle incloses the origin and in which the reversed construction of the previous example must be used for the point *c*. In this case the Z and $\overline{Y}$ circles meet at the two points *q* which are on the unit circle.

In all the foregoing examples as well as in general, the Z and $\overline{Y}$ loci

are called geometrically inverse loci, as has been stated. The Z and Y loci are inverse in another sense, namely, in the sense that the Y locus is the conjugate of the geometrical inverse $\overline{Y}$ of the Z locus. In other words, Y and Z are ***inverse in the complex-function sense*** as described by Eqs. 2b and 2c. The simple term " inversion " as applied to loci in complex planes

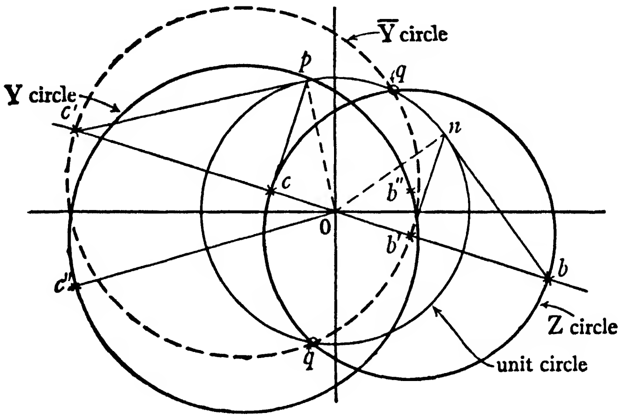


FIG. 9. Construction for complex inversion of circular Z locus inclosing the origin.

connotes this complex-function sense, which includes conjugation after geometrical inversion.

In developing the idea of inversion, the symbols Z and Y, used in electrical engineering to denote complex impedance and admittance functions, respectively, are employed here. The inversion process applies, however, to any two complex functions whose product is the real number unity. Here it is used primarily to go from impedance loci to admittance loci, and vice versa.

The second step in the circuit problem started on p. 479 is to obtain the locus of $Y(\omega)$ as ω varies from zero to infinity. Applying the theory for inversion just developed to the $Z(\omega)$ curve of Fig. 1 gives the $Y(\omega)$ locus as shown in Fig. 10. Since $I(\omega)$ equals $EY(\omega)$, if E is taken as real for convenience, the $I(\omega)$ locus can be found by drawing a semicircle E times as large as the $Y(\omega)$ semicircle, as shown in Fig. 10, or by adding new scales on the axes for the original circle.

From Fig. 10 the steady-state response of the series RL circuit to a constant-magnitude voltage can be fully visualized. When ω equals zero, the current has its maximum amplitude of E/R and is in phase with the voltage. As ω is increased, I decreases and lags the voltage by an increasing angle. As ω approaches infinity, I approaches zero, and the angle of

I approaches $-\pi/2$. For any particular ω, such as that corresponding to point a on the Z locus, the complex value of $Y(\omega)$ is $0a'$ and the complex current is $0a''$. Because of its circular nature, the locus of $I(\omega)$ is called a circle diagram.

While the example chosen for presenting the subject is very elementary, the essential features of impedance and admittance loci are illustrated.

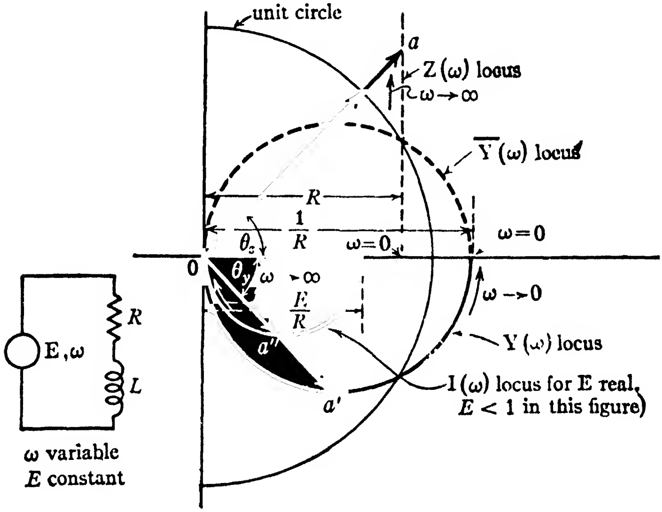


Fig. 10 Loci representing the behavior of series RL circuit as a function of frequency.

More complicated examples require merely a repeated application of the same basic ideas.[1]

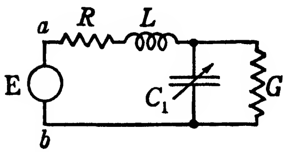
It is pointed out that a Z locus can be thought of as the locus of vector voltage per unit vector current and, correspondingly, that a Y locus is the vector current per unit vector voltage. Often circle diagrams are left in the admittance or impedance form to make them more generally useful. Since the response of any linear passive circuit is proportional to the impressed force, one has to multiply a Z by a current to get a voltage drop or multiply a Y by a voltage to get a current

2. Illustrative example of determination and use of loci

As an illustration of the ideas developed in the preceding article the behavior of the circuit shown in Fig. 11 is determined as the capacitance C_1 is varied over the range from zero to infinity, the frequency of the applied voltage being constant. It is desired to find:

[1] Jens L. La Cour and Ole S. Bragstad, *Theory and Calculation of Electric Currents* (New York: Longmans, Green and Company, 1913).

(a) the maximum and minimum values of effective current;
(b) what value or values of the capacitance, if any, make the source current in phase with the source voltage;
(c) if the values (b) do not exist, the minimum angle between this source current and source voltage.



$G = 1.25 \times 10^{-4}$ mho
$C_1 = 0 \longrightarrow \infty$
$R = 5{,}500$ ohms
$L = 1.85$ h
$\omega = 5{,}000$ radians per sec
$E = 7.50$ v

FIG. 11. Circuit having variable capacitance C_1, analyzed in Art. 2. The loci involved are presented in Fig. 12.

This circuit might represent an oscillator having a no-load vector voltage E, and an internal resistance R and inductance L, delivering power to a load conductance G which is shunted by the adjustable capacitance C_1. The circle-diagram method proves very effective in the solution of such a problem.

Solution: The general procedure is to obtain the admittance locus for the entire circuit as viewed from the generated voltage E, and from this locus to determine the desired results. In obtaining this admittance locus, the best starting point is the locus of the series or parallel combination of elements that contains the variable element, in this case the admittance locus of the GC_1 combination. This GC_1 admittance locus is then inverted to give the impedance locus of the combination. To this is added the impedance of R and L in series. The inversion of this result gives the desired admittance locus of the circuit as viewed from the terminals a–b, known as the driving-point admittance.

The admittance $\mathbf{Y}_{GC_1}$ of G and C_1 in parallel is

$$\mathbf{Y}_{GC_1} = 1.25 \times 10^{-4} + j5{,}000C_1, \qquad [13]$$

which, plotted in a complex plane, is a straight line parallel to the j axis and 1.25×10^{-4} unit to the right of it. The question of a suitable scale for plotting immediately arises, for the unit circle is several thousand times as large as G, and to plot the unit circle and the admittance to the same scale is therefore quite impracticable. Probably the simplest way out of this difficulty is to plot not $\mathbf{Y}_{GC_1}$ but some convenient multiple of $\mathbf{Y}_{GC_1}$ such that the graphical construction for inversion is practicable and accurate. An examination of the inversion process shows that the real part of this multiple of $\mathbf{Y}_{GC_1}$ should be of the order of unity and that $10^4\mathbf{Y}_{GC_1}$ is convenient. In Fig. 12 the locus of $10^4\mathbf{Y}_{GC_1}$ is shown. The geometrical inverse of $10^4\mathbf{Y}_{GC_1}$ is $10^{-4}\,\overline{\mathbf{Z}}_{GC_1}$; both this and the inverse $10^{-4}\,\mathbf{Z}_{GC_1}$ are shown in the figure.

Next the impedance $\mathbf{Z}_{RL}$ is calculated:

$$\mathbf{Z}_{RL} = R + j\omega L = 5{,}500 + j5{,}000 \times 1.85 = 5{,}500 + j9{,}250 \text{ ohms}. \qquad [14]$$

A scale of 10^{-4} times this is convenient for the diagram. The result of adding this complex constant $10^{-4}\mathbf{Z}_{RL}$ to $10^{-4}\mathbf{Z}_{GC_1}$ is to shift the $10^{-4}\mathbf{Z}_{GC_1}$ locus by $10^{-4}\mathbf{Z}_{RL}$ or $0.550 + j0.925$ as shown, giving the locus of the driving-point impedance multiplied by 10^{-4} or $10^{-4}\mathbf{Z}_{1a}$. As the final step, $10^{-4}\mathbf{Z}_{1a}$ is inverted to give $10^4\mathbf{Y}_{1a}$, $\mathbf{Y}_{1a}$ being the desired driving-point admittance.

The required results can now be obtained. The minimum current occurs when C_1

is zero (by inspection), for which Y_{1a} is $10^{-4} \times 0.60$ mho (scaled from diagram), and the minimum current is

$$I_{min} = 7.50 \times 0.60 \times 10^{-4} = 4.5 \times 10^{-4} \text{ amp.} \qquad [15]$$

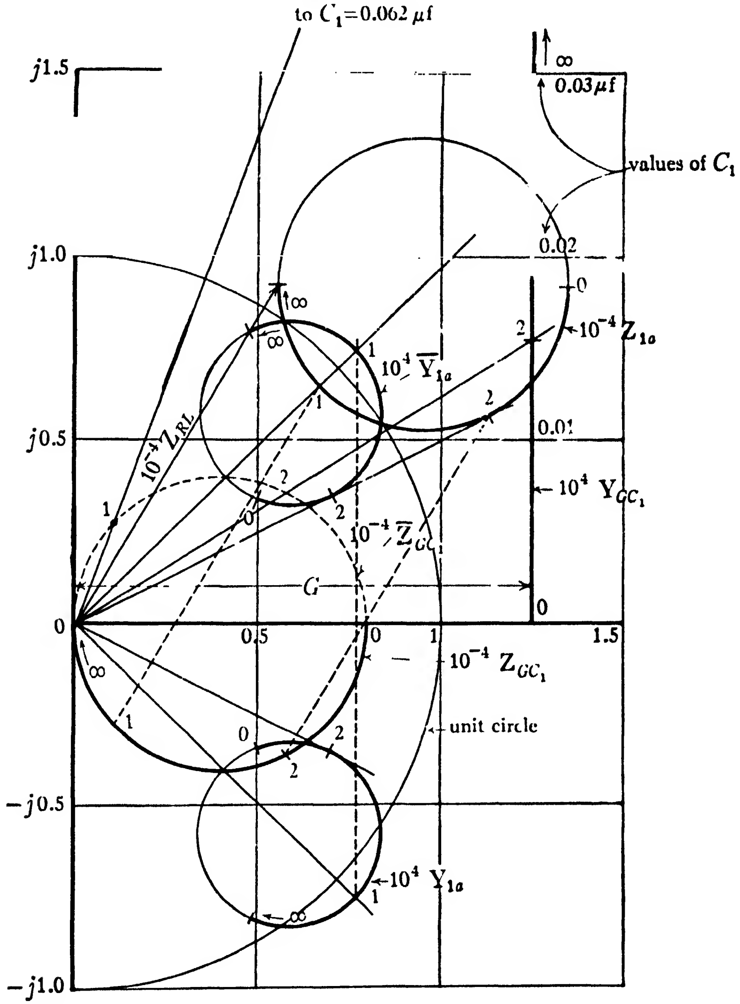


FIG. 12. Admittance and impedance loci used in analysis of circuit of Fig. 11.

The maximum admittance is 1.06×10^{-4} mho (scaled from diagram), and the maximum current is

$$I_{max} = 7.50 \times 1.06 \times 10^{-4} = 7.95 \times 10^{-4} \text{ amp.} \qquad [16]$$

To find the value of C_1 at which this occurs, the point of maximum Y_{1a} is traced back through the diagram and is found to correspond to $0.062\,\mu f$ approximately. The points involved are marked " 1."

For part (b) of the problem it is seen that for no value of C_1 is the current in phase with the voltage. For part (c) the points marked " 2 " apply, corresponding to a minimum angle of 26° and a capacitance of 0.0155 μf approximately. A large-scale, carefully constructed diagram is necessary here as in all graphical work to secure accurate results.

3. Circle diagrams using Thévenin's theorem

Since Thévenin's theorem, as presented in Art. 15, Ch. VIII, provides a ready method for studying the characteristics of a definite portion of a network, the theorem is often a useful tool in the rapid construction of loci by the circle-diagram method. A specific example serves to illustrate the general principle involved: All conditions are as described in the

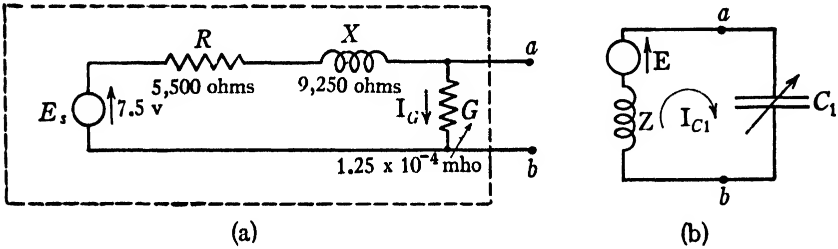


Fig. 13. Representation of circuit of Fig. 11 by Thévenin's theorem.

problem of Art. 2. Instead of determining the locus of the source current, however, determination of the locus of the current I_{C_1} in the condenser C_1 is desired. The circuit can be rearranged to apply Thévenin's theorem as shown in Fig. 13a.

When terminals a–b are open, as in Fig. 13a,

$$I_G = \frac{E_s}{R + jX + \dfrac{1}{G}} = \frac{7.5\underline{/0°}}{16{,}380\underline{/34.4°}}, \qquad [17]$$

$$E = V_{ab} = I_G \times \frac{1}{G} = \frac{7.5 \times 8{,}000}{16{,}380\underline{/34.4°}} = 3.66\underline{/-34.4°} \text{ v.} \qquad [18]$$

When terminals a–b are open-circuited, the impedance Z of the equivalent circuit of Fig. 13b is the impedance looking in at a–b with E_s replaced by a short circuit. This gives

$$Z = \frac{(R + jX)\dfrac{1}{G}}{R + jX + \dfrac{1}{G}} = \frac{(10{,}750\underline{/59.2°})(8{,}000)}{16{,}380\underline{/34.4°}} = 5{,}250\underline{/24.8°} \text{ ohms.} \qquad [19]$$

The variable capacitance C_1 is now added to the circuit by connecting it across the terminals a–b, as shown in Fig. 13b. The current I_{C_1} through the variable capacitance is

$$\mathrm{I}_{C_1} = \mathrm{E}\frac{1}{\mathrm{Z} + \dfrac{1}{j\omega C_1}}. \qquad [20]$$

The desired locus is readily determined by the following steps:

(a) The locus of $j\omega C_1$ is plotted.
(b) The locus (a) is inverted to obtain the locus $1/(j\omega C_1)$.
(c) The complex constant Z is added to the $1/(j\omega C_1)$ locus, to obtain the locus $\mathrm{Z} + 1/(j\omega C_1)$.
(d) The locus (c) is inverted to obtain the locus $1/[\mathrm{Z} + 1/(j\omega C_1)]$.
(e) The locus (d) is multiplied by the complex constant $\overline{E/\alpha}$ to obtain the desired locus of I_{C_1}. This step consists in rotating the locus (d) about the origin through the angle α and then multiplying its magnitudes by the factor E. The multiplication results in a circle of diameter E times as large with center E times as far from the origin as the locus (d).

4. Circle diagrams using the general circuit constants

In any network having a real variable ρ, such as frequency, capacitance or other parameters, the complex value of some circuit characteristic, such as input admittance or impedance, can be expressed as a function of ρ, here called $\mathrm{F}(\rho)$. If $\mathrm{F}(\rho)$ can be expressed in the form of Eq. 21, in which M, N, T, and U are complex constants, the locus of $\mathrm{F}(\rho)$ is a circle in the complex plane, a statement that is subsequently justified.[2]

$$\mathrm{F}(\rho) = \frac{\mathrm{M}\rho + \mathrm{N}}{\mathrm{T}\rho + \mathrm{U}}. \qquad \blacktriangleright[21]$$

Equation 21 is perfectly general and hence applicable to any problem which involves a circuit containing one variable parameter. However, by use of the method of analysis employing the general circuit constants A, B, C, and D, as presented in Art. 10, Ch. VIII, an expression of the form of Eq. 21 can be determined quickly. If it is desired to determine the circle diagram of the current, I, in the source branch of a network, as the resistance of any other branch is varied over a definite range, the branch containing the variable resistance R is considered the load, Z_L. This

[2] W. O. Schumann, "Zur Theorie der Kreisdiagramme," *Arch. f. Elek.*, XI (1922), 140–146; R. Richter, *Elektrische Maschinen* (Berlin: Julius Springer, 1930), Vol. II, Ch. i; A. C. Seletsky, "Cross Potential on a 4-Arm Network," *E.E.*, LII (1933), 861–867.

load may have constant series reactance X as shown in Fig. 14, it carries the current I_2, and it has the voltage V_2 across its terminals 2–2′.

The general circuit constants A, B, C, and D of the coupling network are determined by the methods outlined in Art. 10, Ch. VIII. The equations of equilibrium are

$$E_1 = AV_2 + BI_2, \qquad [22]$$

$$I_1 = CV_2 + DI_2. \qquad [23]$$

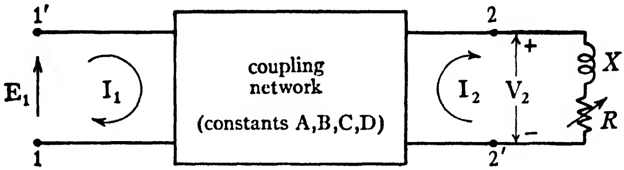


FIG. 14. Coupling network connected to a load having variable resistance.

The input impedance Z_{1a} is equal to E_1/I_1; so that its value may be determined by dividing Eq. 22 by Eq. 23, as follows:

$$Z_{1a} = \frac{E_1}{I_1} = \frac{AV_2 + BI_2}{CV_2 + DI_2}. \qquad [24]$$

Dividing numerator and denominator by I_2 and setting V_2/I_2 equal to Z_L give

$$Z_{1a} = \frac{AZ_L + B}{CZ_L + D}, \qquad [25]$$

and since

$$Z_L = R + jX, \qquad [26]$$

in which R is variable,

$$Z_{1a} = \frac{AR + (jAX + B)}{CR + (jCX + D)}. \qquad [25a]$$

Equation 25a is in the general form of Eq. 21. The input admittance

$$Y_{1a} = \frac{CZ_L + D}{AZ_L + B} = \frac{CR + (jCX + D)}{AR + (jAX + B)} \qquad [27]$$

is also in the general form of Eq. 21. Since the general circuit constants can be determined readily for any coupling network, the desired impedance or admittance function may evidently always be expressed by means of an equation of the general form of Eq. 21 or the special form of Eqs. 25a and 27.

It remains to be shown that the input impedance or admittance described by these expressions is represented by a circular locus in the complex plane. Carrying out the indicated division in Eq. 27 gives

$$Y_{1a} = \frac{C}{A} + \frac{D - \dfrac{BC}{A}}{AZ_L + B}, \qquad [27a]$$

and since

$$AD - BC = 1, \qquad [28]$$

then

$$Y_{1a} = \frac{C}{A} + \frac{1}{A^2} \times \frac{1}{Z_L + \dfrac{B}{A}}. \qquad [27b]$$

It is at once evident that Eq 27b is represented in the complex plane by a circular locus. The term Z_L is, in general, either a point or a straight line which, when added to the complex constant B/A, becomes another point or another straight line. This locus of Z_L + B/A is then inverted to obtain the $1/[Z_L + (B/A)]$ locus, which is known to be a circle, in accordance with the proof of Art. 1 where it is shown that straight lines are transformed into circles by the inversion process. Multiplying the resulting circular locus by the complex constant $1/A^2$ gives another circle of different size and different position. The resulting locus (representing $1/A^2 \times 1/[Z_L + (B/A)]$ is then added to the complex constant C/A, which process results in the determination of a circular locus for the final input admittance Y_{1a}.

The procedure involved in determining the Y_{1a} locus is now presented in detail, following which presentation it is possible to demonstrate a rapid method of constructing the locus and an appropriate scale without going through the step-by-step procedure. If in the coupling network of Fig. 14,

$$P = \frac{B}{A} + jX, \qquad [29]$$

$$\rho = R, \qquad [30]$$

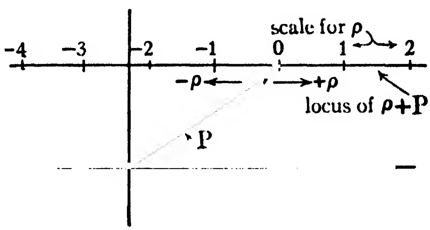


FIG. 15. Locus of ρ + P.

so that the factor Z_L + (B/A) of Eq. 27b, consisting of the real variable ρ and the complex constant P, equals ρ + P and may be shown as in Fig. 15,

Eq. 27b becomes

$$Y_{1a} = F(\rho) = \frac{C}{A} + \frac{1}{A^2} \times \frac{1}{\rho + P}. \qquad [27c]$$

By the method of Art. 1, $\rho + P$ can be inverted easily to obtain $1/(\rho + P)$. The two steps are as follows:

(a) the geometrical inversion, to give $[1/(\rho + P)]_k$ in which the subscript k denotes the conjugate of the expression;

(b) taking the conjugate of $[1/(\rho + P)]_k$ to give the circular locus of $1/(\rho + P)$.

These steps are shown in Fig. 16, in which one new step requires explanation. Any locus may be inverted by graphical means either by first

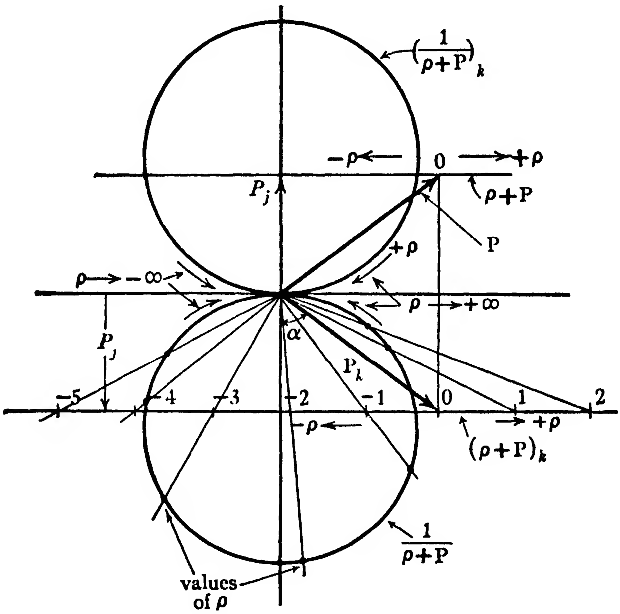


FIG. 16. Inversion of $\rho + P$ locus to $\dfrac{1}{\rho + P}$ locus, including a scale of ρ.

Note: $\dfrac{1}{\rho + P}$ and $(\rho + P)_k$ are plotted to different scales.

obtaining its geometrical inverse and from that its image, the desired complex inverse; or by constructing the image of the locus first and then obtaining the geometrical inverse of the image to give the required complex inverse. Thus in Fig. 16, $1/(\rho + P)$ is the geometrical inverse of $(\rho + P)_k$. This use of $(\rho + P)_k$ has the advantage of permitting the

immediate construction of a scale for ρ upon the $1/(\rho + P)$ circle as shown.

Once the locus of the factor $1/(\rho + P)$ in the second term for Y_{1a} in Eq. 27c has been obtained, together with an easily constructed scale for ρ, upon this locus, the next step is to multiply this locus by the complex constant

$$\frac{1}{A^2} = H\underline{/\eta}. \qquad [31]$$

In these steps the point corresponding to an infinite value of ρ remains at the origin. These two steps are shown in Fig. 17 for $1/A^2$ when equal

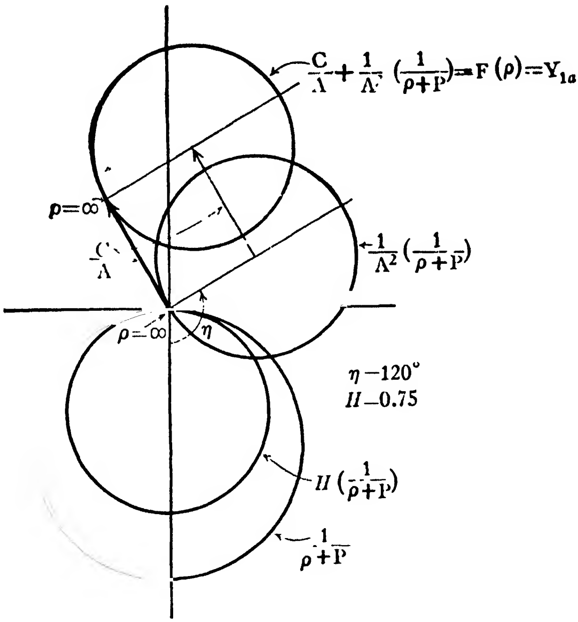


FIG. 17. Locus of $\frac{1}{\rho + P}$ multiplied by $\frac{1}{A^2}$ and added to $\frac{C}{A}$.

to 0.75 at an angle of 120 degrees, giving the locus of the term $\frac{1}{A^2}\left(\frac{1}{\rho + P}\right)$.

The following point should be carefully noted to avoid confusion in the rotation process: Before rotation, the diameter of the circle that goes out from the origin lies along the imaginary axis in the direction *opposite* to that along which lies the imaginary part P_j of P. The rotated position

of this diameter may be considered either as lying at an angle η from the imaginary axis opposite to P_j, or as lying at an angle $180 + \eta$ degrees from the imaginary axis on which P_j lies, both measured in the positive direction. The latter is perhaps the simpler working rule.

Finally, to the $\frac{1}{A^2}\left(\frac{1}{\rho + P}\right)$ locus must be added the complex constant C/A, or, in other words, the $\frac{1}{A^2}\left(\frac{1}{\rho + P}\right)$ locus is translated in the complex plane by C/A to obtain the desired locus of $F(\rho)$. This translation is illustrated in Fig. 17.

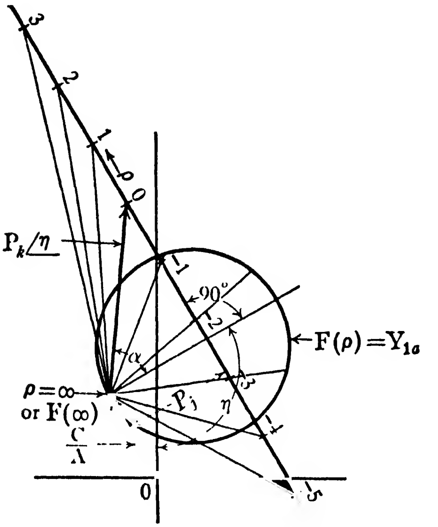


FIG. 18. Scale for ρ superimposed upon the final locus of Fig 17.

From Fig. 17 the rotated ρ scales have been omitted in order not to confuse the multiplication and translation operations. By these operations, however, the ρ scale is unaltered with respect to the circle and can be readily reconstructed directly upon the $F(\rho)$ circle as shown in Fig. 18. The vector $(P)_k$ is laid off exactly the same with respect to the diameter through the point corresponding to an infinite value of ρ for the $F(\rho)$ circle as it is with respect to the $1/(\rho + P)$ circle, and the same linear or uniform scale for ρ is laid off on the line through the end of this vector and perpendicular to this diameter.

The dimensions of the $F(\rho)$ circle can be readily obtained. From the definition of inversion the diameter D of the $1/(\rho + P)$ circle of Fig. 16 can be seen to be

$$D = \frac{1}{P_j}, \tag{32}$$

where

$$P = P_r + jP_j. \tag{33}$$

Consequently the diameter D_F of the $F(\rho)$ circle is

$$D_F = HD = \frac{H}{P_j} = \frac{1}{P_j} \times \frac{1}{A^2}, \tag{32a}$$

or the radius R_F is

$$R_F = \frac{1}{2P_j} \times \frac{1}{A^2}. \qquad [32b]$$

The foregoing discussion makes evident the fact that the final circle for $F(\rho)$ can be laid off directly as in Fig. 18 by the following instructions, without going through the steps of Figs. 15 to 17:

(a) The vector C/A is laid off from the origin which locates the $F(\infty)$ point.

(b) From $F(\infty)$, or the end of the C/A vector, the diameter D_F given by Eq. 32a is laid off, making an angle $180 + \eta$ degrees with the axis of imaginaries along which P_j lies. The $F(\rho)$ circle is drawn upon this diameter.

(c) From $F(\infty)$ the vector $P_k\underline{/\eta}$ is laid off. Its tip is the origin for a linear scale of ρ, η being the angle of $1/A^2$ defined by Eq. 31. This scale for ρ is perpendicular to the diameter laid off in (2) and is plotted to the same scale as P_k.

(d) The scale points for ρ on $F(\rho)$ are located on lines radiating from $F(\infty)$ and passing through the points for corresponding values on the linear scale for ρ.

5. Illustrative example of circle diagram obtained from general circuit constants

As an illustration, the method of Art. 4 is now used in the solution of the problem presented in Art. 2. For this purpose the variable capaci-

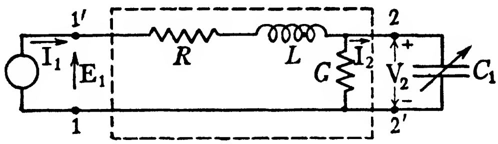


FIG. 19. Circuit having variable capacitance C_1, analyzed by use of general circuit constants

tance C_1 is considered as the load connected across terminals 2–2′ of the coupling network of Fig. 19.

Solution: In this circuit,

$$\left.\begin{aligned} z_{11} &= R + j\omega L + \frac{1}{G} = 5{,}500 + j9{,}250 + 8{,}000 \\ &= 13{,}500 + j9{,}250 = 16{,}380\underline{/34.4^\circ} \text{ ohms,} \end{aligned}\right\} \qquad [34]$$

$$z_{12} = -8{,}000 \text{ ohms,} \qquad [35]$$

$$z_{22} = 8{,}000 \text{ ohms.} \qquad [36]$$

By Eqs. 89 to 92, p. 454

$$\mathrm{A} = \frac{-z_{11}}{z_{12}} = \frac{-16{,}380\underline{/34.4^\circ}}{-8{,}000} = 2.04\underline{/34.4^\circ}, \tag{37}$$

$$\begin{aligned} \mathrm{B} &= \frac{-\mathrm{D}_z}{z_{12}} = \frac{-[(16{,}380\underline{/34.4^\circ})(8{,}000) - (-8{,}000)^2]}{-8{,}000} \\ &= 16{,}380\underline{/34.4^\circ} - 8{,}000 = 5{,}500 + j9{,}250 = 10{,}750\underline{/59.2^\circ}, \end{aligned} \tag{38}$$

$$\mathrm{C} = -\frac{1}{z_{12}} = 1.25 \times 10^{-4}\ \text{mho}, \tag{39}$$

$$\mathrm{D} = \frac{-z_{22}}{z_{12}} = \frac{-8{,}000}{-8{,}000} = 1. \tag{40}$$

As a check,

$$\mathrm{AD} = (2.04\underline{/34.4^\circ})1 = 1.684 + j1.152, \tag{41}$$

$$\begin{aligned} \mathrm{BC} &= 1.342\underline{/59.2^\circ} = 0.687 + j1.154 \\ \mathrm{AD} - \mathrm{BC} &= 0.997 - j0.002\text{'} \end{aligned} \tag{42}$$

which is 1 as required, within limits of slide-rule accuracy. The general circuit constant C should not be confused with the real variable C_1.

In this problem the variable is the capacitance C_1, rather than a resistance as in the development of Art. 2. If Eq. 27c is used when Z_L equals $1/(j\omega C_1)$, the real variable ρ appears as the reciprocal form $1/C_1$, which is undesirable to use in laying out scales. Instead, Eq. 27 is rewritten in the following form:

$$\mathrm{Y}_{1a} = \frac{\dfrac{\mathrm{C}}{\mathrm{Y}_L} + \mathrm{D}}{\dfrac{\mathrm{A}}{\mathrm{Y}_L} + \mathrm{B}} = \frac{\mathrm{C} + \mathrm{DY}_L}{\mathrm{A} + \mathrm{BY}_L}. \tag{27d}$$

Performing the indicated division and using

$$\mathrm{AD} - \mathrm{BC} = 1 \tag{43a}$$

give

$$\mathrm{Y}_{1a} = \frac{\mathrm{D}}{\mathrm{B}} + \frac{\mathrm{C} - \dfrac{\mathrm{AD}}{\mathrm{B}}}{\mathrm{A} + \mathrm{BY}_L} = \frac{\mathrm{D}}{\mathrm{B}} - \frac{1}{\mathrm{B}^2}\,\frac{1}{\mathrm{Y}_L + \dfrac{\mathrm{A}}{\mathrm{B}}}, \tag{27e}$$

or

$$\mathrm{Y}_{1a} = \frac{\mathrm{D}}{\mathrm{B}} - \frac{1}{j\omega\mathrm{B}^2}\,\frac{1}{C_1 + \dfrac{\mathrm{A}}{j\omega\mathrm{B}}}. \tag{27f}$$

Equation 27f is now in the form of Eq. 27c, with ρ representing the real variable C_1, and with

$$\frac{\mathrm{C}}{\mathrm{A}} \text{ of Eq. 27c} = \frac{\mathrm{D}}{\mathrm{B}} \text{ of Eq. 27f}, \tag{44}$$

$$\frac{1}{A^2} \text{ of Eq. 27c} = \frac{j}{\omega B^2} \text{ of Eq. 27f,} \quad [45]$$

$$\text{P of Eq. 27c} = \frac{-jA}{\omega B} \text{ of Eq. 27f.} \quad [46]$$

Values which can be used for laying out the desired Y_{1a} locus by the method of Art. 4 can now be determined. Using the symbols of Eq. 27c gives

$$\frac{C}{A} = \frac{1}{10{,}750 \underline{/59.2°}} = 0.931 \times 10^{-4} \underline{/-59.2°} = (0.476 - j0.801)10^{-4}, \quad [47]$$

$$\left.\begin{aligned} \frac{1}{A^2} &= \frac{j}{5{,}000(10{,}750 \underline{/59.2°})^2} = j1.725 \times 10^{-12} \underline{/-118.4°} \\ &= 1.725 \times 10^{-12} \underline{/-28.4°}, \end{aligned}\right\} \quad [48]$$

$$H = 1.725 \times 10^{-12}, \quad [49]$$

$$\eta = -28.4°, \quad [50]$$

$$P = \frac{-j2.04 \underline{/34.4°}}{5{,}000 \times 10{,}750 \underline{/59.2°}} = 3.80 \times 10^{-8} \underline{/-114.8°}, \quad [51]$$

$$P_j = \mathcal{J}[P] = -3.46 \times 10^{-8}, \quad [52]$$

$$D_F = \frac{H}{P_j} = \frac{1.725 \times 10^{-12}}{3.46 \times 10^{-8}} = 0.500 \times 10^{-4}. \quad [53]$$

The angle of the diameter going out from the point $F(\infty)$ or the tip of C/A with respect to the negative imaginary axis along which P_j lies is

$$\eta + 180° = -28.4° + 180° = 151.6°. \quad [54]$$

Measured from the axis of reals, this is an angle of

$$151.6° - 90° = 61.6°. \quad [55]$$

For location of the C_1 scale

$$P_k \underline{/\eta} = 3.80 \times 10^{-8} \underline{/114.8° - 28.4°} = 3.80 \times 10^{-8} \underline{/86.4°}. \quad [56]$$

The straight line through the terminus of the $P_k \underline{/\eta}$ vector which carries the linear scale for C_1 is, of course, normal to the diameter of the circle through $F(\infty)$.

It is now possible to plot the circle, which should coincide with the one obtained by geometrical construction. In order to show the coincidence, $10^4 Y_{1a}$ is plotted as before, instead of Y_{1a}. This is shown in Fig. 20.

A problem occurs because of the small size of $10^4 P_k \underline{/\eta}$ as compared to the other dimensions which are not far from unity. An examination of Fig. 20 shows, however, that the $P_k \underline{/\eta}$ vector can be increased in size without altering the scale for C_1 on the circle, provided the linear scale for C_1 is opened out by the same factor that $P_k \underline{/\eta}$ is increased. Therefore $10^7 P_k \underline{/\eta}$ is plotted instead of $10^4 P_k \underline{/\eta}$ as would correspond to the other quantities on the diagram. Unit distance on the C_1 scale then represents 10^{-7} farad or 0.1 microfarad because this C_1 scale has been magnified by 10^7.

Comparison shows that the final diagrams of Figs. 12 and 20 are identical, except that Fig. 20 is provided with a convenient scale for C_1.

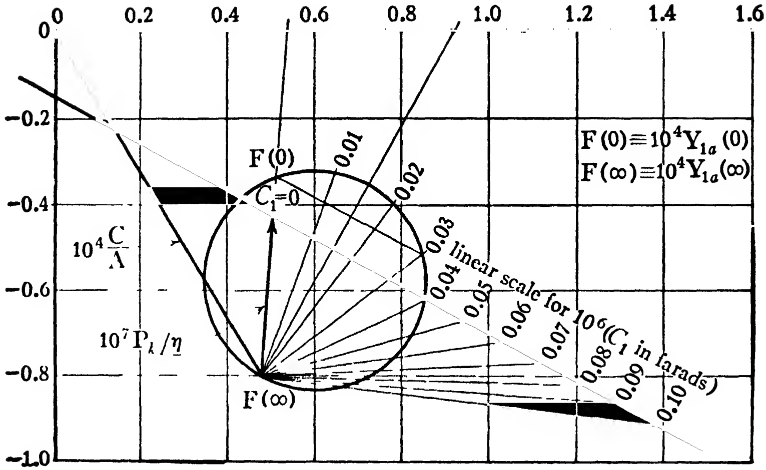


FIG. 20. Locus of circuit admittance, including a scale for C_1.

6. SPECIAL CONSIDERATIONS IN CIRCLE-DIAGRAM THEORY

Extensions of the theory of circle diagrams frequently prove useful in practical applications. First, when the equation for a function whose locus is desired can by any method be put in the form of Eq. 21,

$$F(\rho) = \frac{M\rho + N}{T\rho + U}, \qquad \blacktriangleright[21]$$

this locus is known to be a circle. The $F(\rho)$ locus can be determined from the complex values of $F(\rho)$ for any three values of ρ, since any three points determine a circle. Often the values of $F(\rho)$ corresponding to ρ equal to zero, ρ equal to infinity, and one other value are convenient. These are sufficient also to fix the linear scale for ρ, since ρ varies linearly along any line perpendicular to the diameter through $F(\infty)$.

The second extension relates to determining the scale of ρ when the linear-scale method gives flat intersections on the circle, as, for example, for large values of ρ. It frequently happens in circle diagrams for electrical machinery that this region of the diagram is the most useful. The value of ρ for any point on the circle can be obtained as follows: Solving Eq. 21 for ρ gives

$$\rho = \frac{N - UF(\rho)}{TF(\rho) - M} = \frac{U\left[\frac{N}{U} - F(\rho)\right]}{T\left[F(\rho) - \frac{M}{T}\right]} = P\,\frac{F(0) - F(\rho)}{F(\rho) - F(\infty)}, \qquad [21a]$$

in which, from Eq. 21,

$$F(0) = \frac{N}{U} \tag{57}$$

and

$$F(\infty) = \frac{M}{T}, \tag{58}$$

and, from inspection of Eqs. 27b and 27c,

$$P = \frac{U}{T}. \tag{59}$$

It can be seen from Fig. 21 that $F(0) - F(\rho)$ and $F(\rho) - F(\infty)$ are chords from $F(\rho)$ to $F(0)$ and from $F(\infty)$ to $F(\rho)$, respectively. Furthermore, ρ is real and therefore is equal to the magnitude of Eq. 21a. If P is known, ρ can be determined from the lengths of the chords $F(0) - F(\rho)$ and $F(\rho) - F(\infty)$. The algebraic sign to be used with ρ is evident from the diagram.

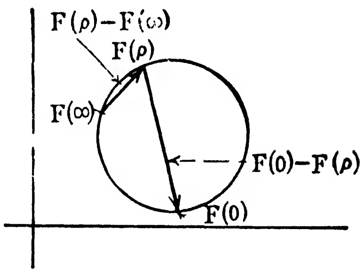


FIG. 21. Chords used to determine ρ.

If P is not known, as, for example, when the circle is determined from three points or from experimental data, it can be obtained by solving Eq. 21a for ρ and substituting in the result a value ρ_1, for which the location on the circle is known. Thus,

$$P = \rho_1 \frac{F(\rho_1) - F(\infty)}{F(0) - F(\rho_1)}. \tag{21b}$$

From ρ_1 and measurements of the two chords $F(\rho_1) - F(\infty)$ and $F(0) - F(\rho_1)$, the magnitude P is readily determined.

As an illustration of the use of Eq. 21b, P is determined from the point corresponding to 0.03 microfarad or 3.00×10^{-8} farad in Fig. 20 (C_1 corresponds to the ρ of the general theory). From this diagram the magnitudes of the chords are

$$F(\rho_1) \text{ to } F(\infty) = 0.472, \tag{60}$$

and

$$F(0) \text{ to } F(\rho_1) = 0.375, \tag{61}$$

whence

$$P = 3.00 \times 10^{-8} \frac{0.472}{0.375} = 3.78 \times 10^{-8}, \tag{62}$$

which practically agrees with the value obtained in Eq. 51.

7. Graphical representation of related complex variables

Graphical methods of representing functions in the complex plane are useful when one complex variable is a linear function of another. An illustration which is very simple and yet important in practice is provided by the two-terminal-pair network, or coupling network, as shown in Fig. 22.

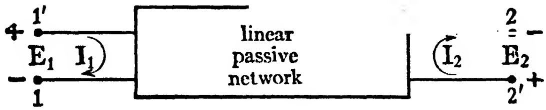


Fig. 22. Two-terminal-pair network.

The following equations, identical with Eqs. 58b and 59a, p. 449, apply to this coupling network:

$$I_1 = y_{11}E_1 + y_{12}E_2, \tag{63}$$

$$I_2 = y_{21}E_1 + y_{22}E_2. \tag{64}$$

It is assumed that y_{11}, y_{12}, y_{22}, and E_2 are known complex constants. Then, by Eq. 63, I_1 is the sum of two terms. The first is the complex constant $y_{12}E_2$; the second is E_1 multiplied by the complex constant y_{11}. This correspondence between E_1 and I_1 can be shown by plotting each

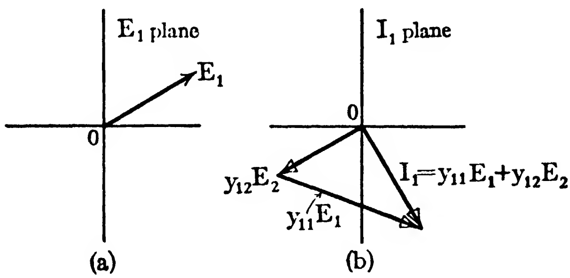


Fig. 23. Relation between E_1 and I_1 shown by plots in complex plane.

in its own complex plane, as shown in Figs. 23a and 23b for an arbitrary value of E_1. When numerous values of I_1 are wanted, the calculations are greatly facilitated by the following construction in the I_1 plane as shown in Fig. 24: The vector $y_{12}E_2$ is plotted, and from its tip as an origin a set of axes is laid out which may be called E_1 axes rotated with respect to the I_1 axes by the angle of y_{11}. Along these E_1 axes linear scales with y_{11} as a unit are laid off, each of these y_{11} units representing a unit of E_1. Then at any point on the plane the complex value of E_1 can be read on the E_1

co-ordinates and the corresponding complex value of $\mathbf{I}_1$ can be read on the $\mathbf{I}_1$ co-ordinates. For power-transmission networks of the two-terminal-pair type, the $\mathbf{E}_1$ co-ordinates are usually laid off in polar form so that $\mathbf{E}_1$ is read as $E_1\underline{/\psi}$, whereas the $\mathbf{I}_1$ co-ordinates are rectangular so that $\mathbf{I}_1$ is read as $I_{r1} + jI_{j1}$ where I_{r1} and I_{j1} are the real and imaginary parts respectively of $\mathbf{I}_1$.

As another example of a linear vector relation, the vector power $P_2 + jQ_2$ at the terminals 2–2′ of Fig. 22 is considered as a function of the vector voltage $\mathbf{E}_1$. Multiplying Eq. 64 by the conjugate $\overline{\mathbf{E}}_2$ of $\mathbf{E}_2$ gives

$$P_2 + jQ_2 = \overline{\mathbf{E}}_2\mathbf{I}_2 = y_{12}\overline{\mathbf{E}}_2\mathbf{E}_1 + y_{22}\overline{\mathbf{E}}_2\mathbf{E}_2. \qquad [65]$$

The $\mathbf{E}_1$ co-ordinates can be superposed upon the $P_2 + jQ_2$ axes in a way similar to that used in Fig. 24, the displacement of the $\mathbf{E}_1$ origin from the $P_2 + jQ_2$ origin being $y_{22}\overline{\mathbf{E}}_2\mathbf{E}_2$ or $y_{22}E_2^2$. These axes are then rotated by the angle of $y_{12}\overline{\mathbf{E}}_2$, and a scale is placed on them such that a unit of E_1 occupies a distance $y_{12}E_2$.

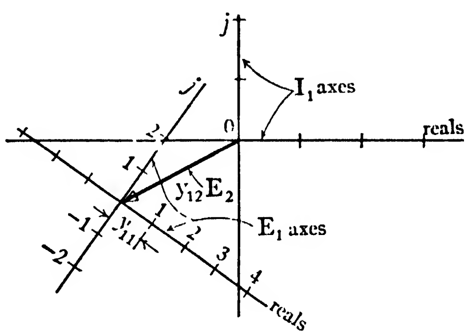


FIG. 24. $\mathbf{E}_1$ plane and $\mathbf{I}_1$ plane superimposed.

From such a diagram the vector power at the terminals 2–2′ can be read directly as a function of the vector $\mathbf{E}_1$. As in the preceding example, the use of polar co-ordinates for $\mathbf{E}_1$ and rectangular co-ordinates for $P_2 + jQ_2$ is usually convenient in practice. This form or a modification obtained by dividing Eq. 65 by E_2^2 to make the resulting "chart," as it is often called, independent of E_2 is widely used in power-transmission studies.

In the usual derivation of this type of chart, or diagram, it is commonly demonstrated that circles in the $\mathbf{E}_1$ plane become circles in the $\mathbf{I}_2$, the $\mathbf{I}_1$, and the $P_2 + jQ_2$ planes. From the foregoing linear vector theory, consideration of any particular form of locus is evidently unnecessary, for the entire $\mathbf{E}_1$ plane is merely superposed with a shift in zero point, a rotation, and a change in scale, upon the plane of the dependent complex variable.

8. Illustrative example of use of vector-power chart

To illustrate the numerical application of such a relation as Eq. 65, the corresponding linear relation in terms of the general circuit constants A, B, C, and D is used for a transmission circuit representing a 126-mile,

220-kilovolt, 2-circuit, 60-cycles-per-second transmission line. For this system, in the notation of Fig. 10, p. 453,

$$\mathbf{A} = 0.967\underline{/2.9°} = \mathbf{D}, \tag{66}$$

$$\mathbf{B} = 49.2\underline{/81.4°}\text{ ohms}, \tag{67}$$

$$\mathbf{C} = 0.0013\underline{/90.0°}\text{ mho}, \tag{68}$$

$$\mathbf{E}_2' = 220{,}000\underline{/0.0°}\text{ v.} \tag{69}$$

Solution: To obtain $\mathbf{I}_2$ in terms of $\mathbf{E}_1$ and $\mathbf{E}_2'$, Eq. 85, p. 454, is solved for $\mathbf{I}_2$ with the result

$$\mathbf{I}_2 = \frac{1}{\mathbf{B}}\mathbf{E}_1 - \frac{\mathbf{A}}{\mathbf{B}}\mathbf{E}_2'. \tag{70}$$

Multiplying by $\overline{\mathbf{E}}_2'$ gives

$$P_2 + jQ_2 = \overline{\mathbf{E}}_2'\mathbf{I}_2 = \frac{\overline{\mathbf{E}}_2'}{\mathbf{B}}\mathbf{E}_1 - \frac{\mathbf{A}\overline{\mathbf{E}}_2'\mathbf{E}_2'}{\mathbf{B}}, \tag{71}$$

which is of the same form as Eq. 65 with general circuit constants substituted for the y's.

Taking $\mathbf{E}_2'$ along the axis of reals and of magnitude 220,000 v gives

$$\left.\begin{aligned} -\frac{\mathbf{A}\overline{\mathbf{E}}_2'\mathbf{E}_2'}{\mathbf{B}} &= -\frac{0.967\underline{/2.9°}(2.20 \times 10^5\underline{/0.0°})^2}{49.2\underline{/81.4°}} = 9.55 \times 10^8\underline{/101.5°} \\ &= (-190.2 + j936)10^6\text{ va.}\end{aligned}\right\} \tag{72}$$

This is the location of the origin of the $\mathbf{E}_1$ plane on the $P_2 + jQ_2$ plane.

Next is found the complex factor by which the $\mathbf{E}_1$ plane is multiplied to obtain its contribution to $P_2 + jQ_2$. This factor is, from Eq. 71,

$$\frac{\overline{\mathbf{E}}_2'}{\mathbf{B}} = \frac{220{,}000\underline{/0°}}{49.2\underline{/81.4°}} = 4{,}470\underline{/-81.4°} = 668 - j4{,}400\text{ amp.} \tag{73}$$

The units of $\overline{\mathbf{E}}_2'/\mathbf{B}$ can also be expressed as volt-amperes of $P_2 + jQ_2$ per volt of $\mathbf{E}_1$.

In Fig. 25 are shown the principal dimensions of the chart for $P_2 + jQ_2$ as a function of $\mathbf{E}_1$ for $\mathbf{E}_2'$ equal to 220,000 v. The scale for $P_2 + jQ_2$ is selected arbitrarily to give a chart of convenient size, all the other scales being determined in terms of $P_2 + jQ_2$ or vector volt-ampere units. Figure 26 represents a chart of the sort often used in practice except for the fine subdivision of the co-ordinate scales which is omitted here in order that the major features of the chart may stand out more clearly.

Such a chart as that of Fig. 26 is used to answer questions such as this: What must be the magnitude and angle of $\mathbf{E}_1$ with respect to $\mathbf{E}_2'$ when the transmission line is delivering $(800 + j200)10^3$ kilovolt-amperes?

Locating this point on the $P + jQ$ axes and reading the same point on the E_1 co-ordinates give approximately

$$E_1 = 270{,}000\underline{/45°} \text{ v.} \tag{74}$$

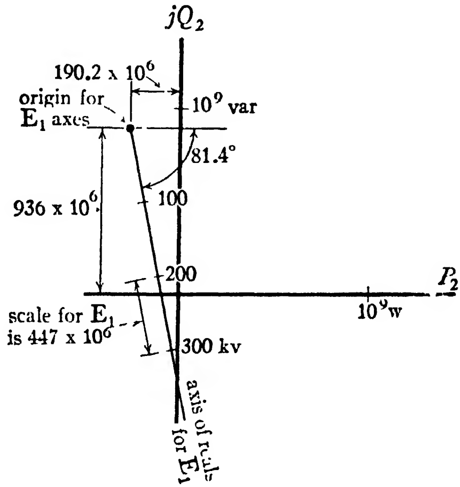


FIG. 25. Principal dimension of vector-power chart.

A chart may be used hundreds of times in this manner during the study of a transmission problem.

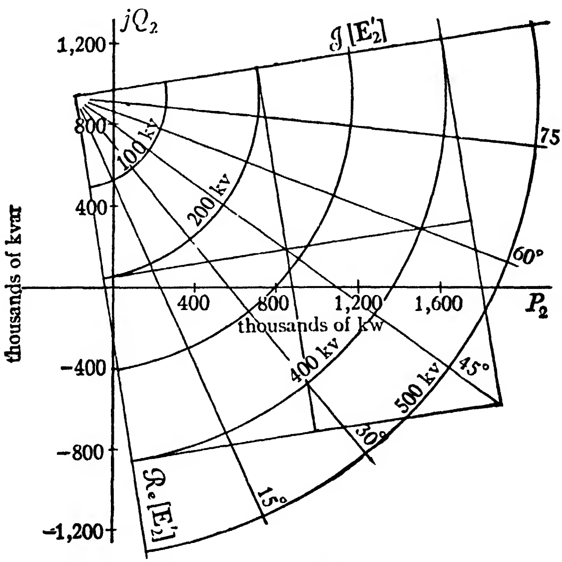


FIG. 26. Vector-power chart.

9. MAXIMA AND MINIMA OF COMPLEX FUNCTIONS IN GENERAL

In dealing with complex functions, a frequently recurring problem is to find the value or values of the independent variable which give certain maximum or minimum values of the function. Such a problem is given in Art. 2. When the locus of the complex function is a circle, the circle-diagram methods are often effective. In other cases the locus may not be a circle, or an analytical solution may be desired, so that other methods are desired. One such method, of rather wide usefulness, follows.[3]

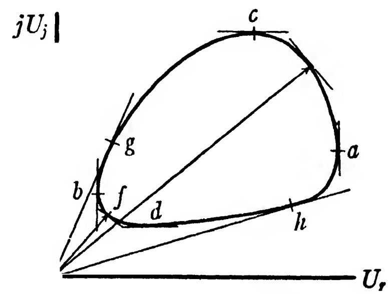


FIG. 27. Locus of complex function U(ρ).

If a complex function U(ρ) of the real variable ρ has in the U plane a locus such as that shown in Fig. 27, a number of maximum and minimum conditions may be of interest. These conditions are maximum and minimum values of:

(a) the real part of the function, as at a and b;
(b) the imaginary part of the function, as at c and d;
(c) the absolute value of the function, as at e and f;
(d) the angle or argument of the function, as at g and h.

The function U(ρ) may become stationary or independent of ρ at one or more points as ρ approaches certain values. This condition is not illustrated in Fig. 27 but may occur.

Before these five cases are considered individually, certain ideas must be established which relate to the geometrical interpretation of derivatives and differentials of a complex function of a real variable. The function U(ρ) is written in rectangular form as the complex sum of the real part $U_r(\rho)$ and the imaginary part $U_j(\rho)$:

$$\mathrm{U}(\rho) \equiv U_r(\rho) + jU_j(\rho); \tag{75}$$

or, by dropping the functional notation for brevity, it is written thus:

$$\mathrm{U} \equiv U_r + jU_j. \tag{75a}$$

The derivative of U is

$$\frac{d\mathrm{U}}{d\rho} = \frac{dU_r}{d\rho} + j\frac{dU_j}{d\rho} \tag{76}$$

[3] W. Van B. Roberts, "A Method for Maximization in Circuit Calculations," *I.R.E. Proc.*, XIV (1926), 689–694; Roberts, "Maximization Methods for Functions of a Complex Variable," *ibid.*, XV (1927), 519–524.

since U_r and U_j are both *real* functions of ρ. The differential of U is

$$dU = \frac{dU_r}{d\rho}\,d\rho + j\frac{dU_j}{d\rho}\,d\rho = dU_r + j\,dU_j. \qquad [77]$$

In Fig. 28 these differentials are indicated geometrically for an arbitrary case.

The various maxima and minima of U are seen to be characterized by particular orientations of the dU vector with respect to the axes or with respect to the U vector. The cases stated at the beginning of this article are now considered in order.

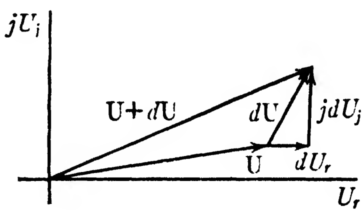


Fig. 28. Differentials used in studying the properties of the complex function U(ρ)

(a) Inspection of Fig. 27 shows that in general when the dU vector lies parallel to the imaginary axis, the real part of U has a maximum or minimum value. The analytical condition for this case is evidently that dU is a pure imaginary, or that dU_r and thus $dU_r/d\rho$ are zero. This represents the conditions at *a* and at *b* in Fig. 27.

(b) For maximum and minimum values of the imaginary part of U, evidently dU must be real; hence $dU_j/d\rho$ must be zero, which is the desired criterion.

(c) A consideration of Fig. 27 shows that the U vector has a maximum or minimum length at those positions where the locus lies at right angles to the vector, that is, where dU is at right angles to U. The criterion for perpendicularity of two directions is, from analytical geometry, that their slopes be negative reciprocals. The slopes of dU and U are as follows:

$$\text{slope of } dU = \frac{\dfrac{dU_j}{d\rho}}{\dfrac{dU_r}{d\rho}}, \qquad [78]$$

$$\text{slope of U} = \frac{U_j}{U_r}. \qquad [79]$$

Hence the criterion for a maximum or minimum of U is

$$\frac{\dfrac{dU_j}{d\rho}}{\dfrac{dU_r}{d\rho}} = -\frac{U_r}{U_j}. \qquad [80]$$

At times it may be easier to apply this form directly than to take the more obvious course of maximizing the real functions

$$U = \sqrt{U_r^2 + U_j^2} \qquad [81]$$

or

$$U^2 = U_r^2 + U_j^2 \qquad [82]$$

by the usual methods for real functions.

(d) From Fig. 27 maximum and minimum values of the angle or argument of **U** can be seen to occur when d**U** and **U** have the same direction or slope, or when

$$\frac{U_j}{U_r} = \frac{\dfrac{dU_j}{d\rho}}{\dfrac{dU_r}{d\rho}}. \qquad [83]$$

Sometimes the point at which the function **U** ceases to change as ρ is changed may be of interest. In such instances d**U** is zero, as it is only when dU_r and dU_j are both zero; that is, when

$$\frac{dU_r}{d\rho} = 0 \qquad [84]$$

and

$$\frac{dU_j}{d\rho} = 0. \qquad [85]$$

Sometimes the algebraic work can be reduced by using the fact that the real part of the derivative of **U** is equal to the derivative of the real part of **U**, and likewise for the imaginary parts. In symbols,

$$\Re e\left[\frac{d\mathbf{U}}{d\rho}\right] = \frac{dU_r}{d\rho} \qquad [86]$$

and

$$\mathcal{I}\left[\frac{d\mathbf{U}}{d\rho}\right] = \frac{dU_j}{d\rho}. \qquad [87]$$

Inspection of any particular problem ordinarily indicates whether to differentiate first or to separate the real and imaginary parts first.

PROBLEMS

1. A series circuit consisting of a coil and a variable condenser of capacitance C is connected across a voltage source 100 cos 1,000t v, in which t is in seconds. The capacitance C can be varied from 1.0 to 5 μf. The coil has a Q of 2 and an impedance of 750 ohms (both measured at source frequency).

(a) A circle diagram is to be drawn to scale for the current I in the circuit as C is varied slowly.
(b) The end values of I and the sense in which I varies as C is increased are to be marked on the diagram
(c) Values of I corresponding to the resonance and to half-power conditions are to be marked on the diagram.
(d) What is the value of the total circuit impedance at half-power value of current?
(e) What are the minimum and the maximum values of the power dissipated in this circuit?
(f) How would the diagram change if the frequency dropped to half its value?

All scales should be given on the diagram.

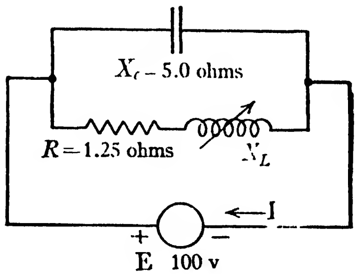


FIG. 29. Parallel-resonant circuit for Prob. 2.

2. In the circuit of Fig. 29, X_L may have any value from 0 to ∞. With the aid of a graphical construction the following are to be determined:

(a) I for parallel resonance (E and I in phase),
(b) minimum I,
(c) maximum I,
(d) the values of X_L corresponding to each value of I.

3. In the circuit of Fig. 30 the source is able to deliver the same current I for all values of capacitance C. The angular frequency ω and the parameters R and L are constant.

(a) An analytical proof is to be given to show that as C is slowly varied, the locus of the vector V is a circle.
(b) For what value of C is the voltage across the condenser a maximum?

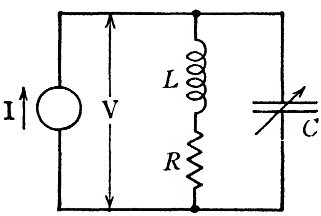


FIG. 30. Constant-current circuit for Prob. 3.

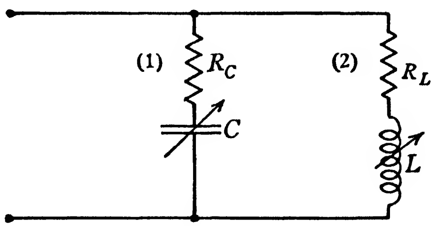


FIG. 31. Parallel branches for Prob. 4.

4. Two branches, (1) and (2) of Fig. 31, are in parallel. Branch (1) is equivalent to a constant resistance of 8.00 ohms in series with an adjustable capacitance C.

Branch (2) is equivalent to a constant resistance of 10.0 ohms and an adjustable inductance L in series. A circle diagram is to be drawn showing

(a) the circular locus of the admittance of branch (1),

(b) the circular locus of the admittance of branch (2).

If the inductance of branch (2) is fixed at a value which makes its reactance 10.0 ohms, is the locus of the resultant admittance of branches (1) and (2) in parallel a circle?

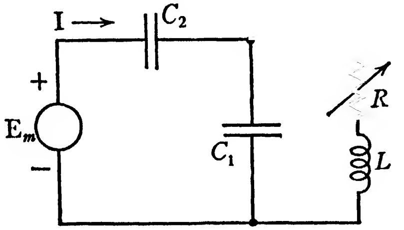
When X_L is fixed, two values of the condenser capacitance produce unity power factor for the two circuits in parallel, and neither of these values of capacitance produces minimum current. These facts are to be shown by the diagram.

What are the values of the conductance and susceptance of branch (1), found from the circle diagram, which produce unity power factor in the circuit as a whole when X_L is 10.0 ohms?

What are the values of the capacitive reactance which causes unity power factor for the circuit when X_L is 10.0 ohms?

What are the values of the conductance and susceptance of branch (1) which make the resultant current a minimum when X_L is 10.0 ohms?

Scales of 1 ohm /in. and 0.02 mho /in. are to be used.



$$\omega L = 2.00 \text{ ohms}$$
$$\frac{1}{\omega C_1} = 2.00 \text{ ohms}$$
$$\frac{1}{\omega C_2} = 1.00 \text{ ohm}$$
$$R = 2.00 \longrightarrow \infty \text{ ohms}$$
$$E_m = 10 \text{ v}$$

FIG. 32. Circuit for Prob. 5.

5. In the circuit of Fig. 32 what are

(a) the locus of the impedance connected to the generator for values of R between 2.00 and ∞ ohms?

(b) the maximum and minimum values of I?

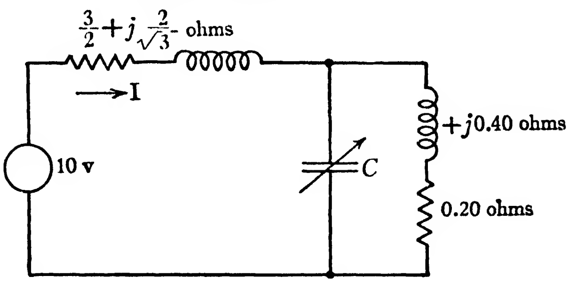


FIG. 33. Circuit for Prob. 6.

6. In the circuit of Fig. 33 the capacitance C is adjustable over the range from 0 to ∞.

(a) What are the maximum and minimum possible values of the current I?

(b) What is the power factor of the circuit viewed from the source for each of the two conditions in (a)?

7. In the design of electrical apparatus for control or other special purposes, it is sometimes desirable to obtain a voltage whose phase relative to a given source reference may be continuously varied while the magnitude of this voltage remains constant. If no current is drawn from the output terminals, the network in Fig. 34 can be

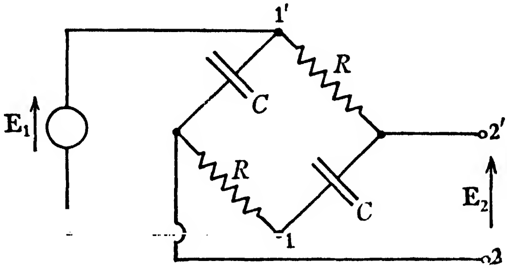


FIG. 34. Phase-shifting circuit for Prob. 7.

used to accomplish this purpose by varying either both resistances R or both capacitances C, keeping the two R's or C's at all times equal. With this device the magnitudes E_1 and E_2 are at all times alike, and the circuit behaves in the desired manner no matter what the frequency of the source is. These statements are to be verified

8. The circuit of Fig. 35 is to be considered as an alternative for accomplishing the same purpose as the circuit of Fig. 34. The voltage E_1 and the parameters L and C remain fixed; the phase angle of E_2 is shifted by varying R. By using the impedance-locus method of analysis, the following are to be determined:

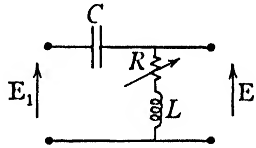


FIG. 35. Phase-shifting circuit for Prob. 8.

(a) the relation between C, L, and ω necessary so that E_2 may be varied in phase, but held constant in magnitude;
(b) the amount of total phase shift obtainable as compared with the circuit of Fig. 34;
(c) the method of adjusting the constant magnitude of E_2 to various desired values;
(d) the effect of source frequency on the parameter relations found in (a).

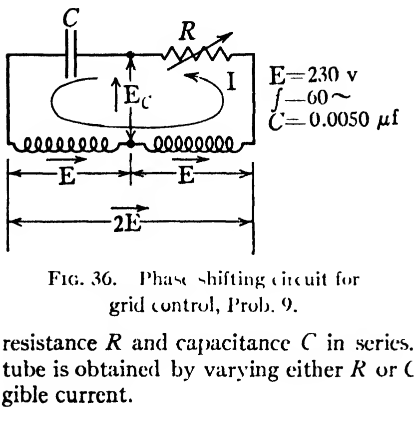


FIG. 36. Phase shifting circuit for grid control, Prob. 9.

9. A circuit frequently used to control the action of grid controlled gaseous-discharge tubes, known commercially by such trade names as Grid-Glow Tubes or Thyratrons, is shown in Fig. 36. The coil indicates the secondary winding of a transformer which may be considered as an impedanceless source of alternating voltage 2E. A center connection is available so that each of the two halves of the coil supply vector voltages E equal in magnitude and identical in phase. Across the source of voltage is connected a circuit of resistance R and capacitance C in series. Control of the voltage E_c applied to the tube is obtained by varying either R or C. The tube can be assumed to draw negligible current.

For this circuit what variation of E_r is possible (both magnitude and phase) as R is varied over its entire range? The results are to be plotted as a function of R.

10. The circuit in Fig. 37 represents an impedance bridge for measuring inductance. The following loci are to be drawn, using V_{ab} as a reference axis:

(a) the locus of the voltage V_{ac} when R_s is varied;
(b) the locus of the voltage V_{ad} when N is varied;
(c) the loci of (a) and (b) under the conditions that Q_x is 0.2 and ωL_s equals M.

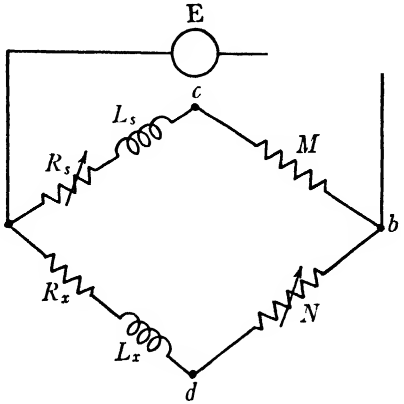


FIG. 37. Impedance bridge for Prob. 10.

What conclusions can be drawn from the results of (c) concerning the ease of balancing the bridge, that is, making V_{cd} equal to 0, by successive adjustments of R_s and N? Can a better choice of variables be made?

11. A coupled circuit used in radio-frequency apparatus consists of two coils with adjustable mutual inductance. In series with each coil is a condenser. The constants of the circuits are as follows:

Primary Circuit		*Secondary Circuit*
L	C	L
0.150 mh	160 $\mu\mu$f	0.150 mh

The resistances of the two circuits are such that at a frequency of 10^6 ~, $\omega L/R$ is 100. A voltage source is connected in series with the primary circuit. The secondary circuit is closed.

(a) What is the value of mutual inductance for which the apparent impedance of the primary, that is, its impedance when measured in the presence of the secondary circuit, appears as a pure resistance if the secondary capacitance is 160 $\mu\mu$f? The primary voltage has a frequency of 10^6 ~.
(b) If the mutual inductance is made twice that found in (a), what values of secondary capacitance make the apparent primary impedance a pure resistance at the same frequency?

12. In Fig. 38, M represents an induction motor having a rating of 40.0 kw at 80% power factor, lagging; S represents a synchronous motor having a rating of 40.0 kva, the power factor of which can be varied from 10% lagging to 10% leading. The induction motor is operated under the constant-rated load conditions specified. The syn-

chronous motor is controlled so that it operates at all times at full kilovolt-ampere rating within the specified power-factor limits. The locus of the kilovolt-amperes

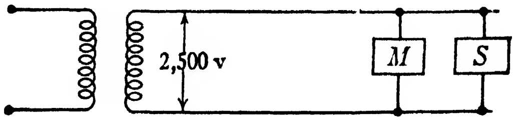


FIG. 38. Parallel motor loads, Prob 12.

taken by the combination of the two machines is to be drawn, indicating the following on the diagram:

(a) vector proportional to the maximum kilovolt-amperes,
(b) vector proportional to the maximum kilowatts,
(c) vector proportional to kilovolt-amperes at unity power factor,
(d) vector proportional to combined load at 90% power factor.

CHAPTER X

Polyphase Systems

1. Introduction

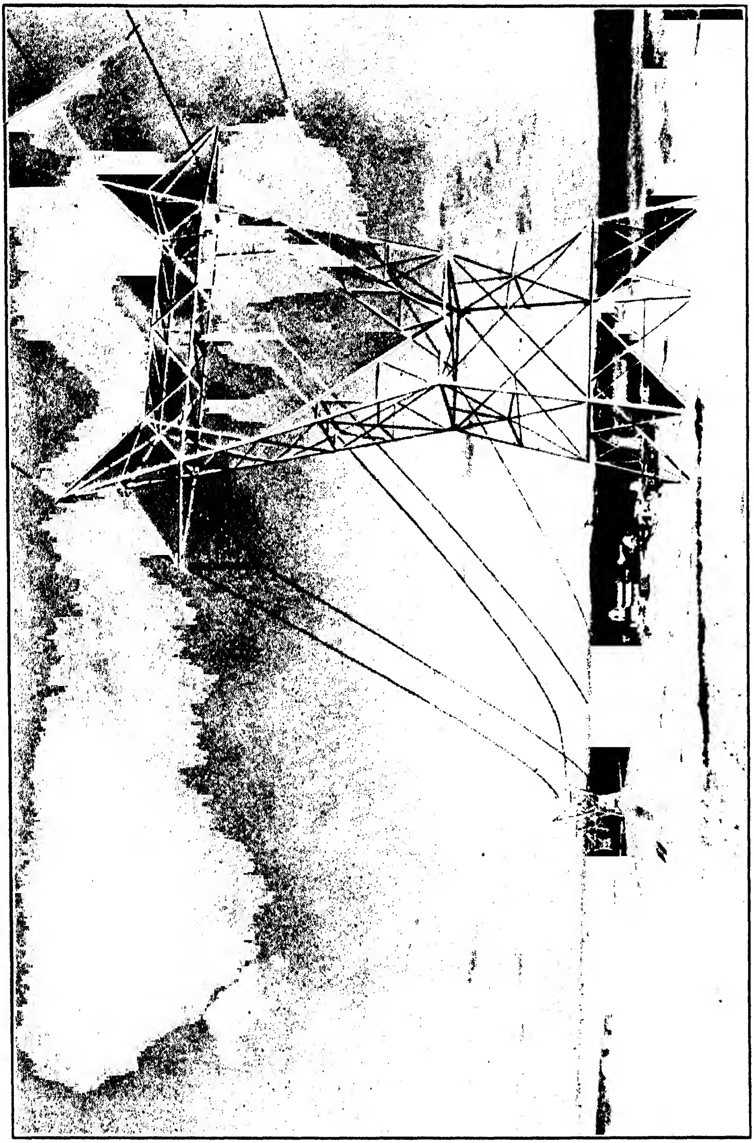
This chapter develops an important special application of the general theory of linear alternating-current networks to circuits whose voltage sources are interrelated in both magnitude and phase angle. Every electrical engineer, regardless of his particular field of interest, should be familiar with the fundamental properties of circuits of this type, because such circuits are almost universally used in electric-power generation, transmission, and distribution.

An alternating voltage source of the type used in electric-power generation on a commercial scale almost invariably consists of a group of voltages having related phase angles and magnitudes. If one or more groups of n associated voltages having substantially equal magnitudes and definitely related phase angles are impressed into a network, the electrical system resulting is called a *polyphase system.* If the corresponding individual voltages of each of the source groups are impressed into an electrically separate circuit, the conductors, loads, and portions of the generating apparatus associated with this separate circuit are said to constitute a *phase* of the system, and each separate circuit is called a *single-phase circuit.* As is demonstrated presently, however, a polyphase system can be utilized much more economically if electrical connections are made among the various phases to create a *polyphase circuit* which can serve substantially all the functions of these separate single-phase circuits, but which requires less conductor material. It is usually impossible to identify certain elements of such a polyphase circuit as belonging definitely in one or another phase, though, for the purpose of analysis, conversions can be made which result in a fictitious equivalent circuit whose elements can be definitely assigned to one phase or another.

In the field of electrical engineering, a very widely used meaning of the word " phase," standing alone, is that defined in the preceding paragraph; this meaning is used hereafter in this chapter. It should be pointed out, however, that the American Institute of Electrical Engineers has tentatively defined *phase* to express, as a fraction of a period, the same thing that is meant by *phase angle* expressed in radians or degrees, so that *phase* and *phase angle* are synonymous terms. Little difficulty should arise from these differences of definition of phase, for the context in which the word appears should make evident the intended meaning. In this chapter, the word *circuit* has its customary meaning, a circuit being the entire collec-

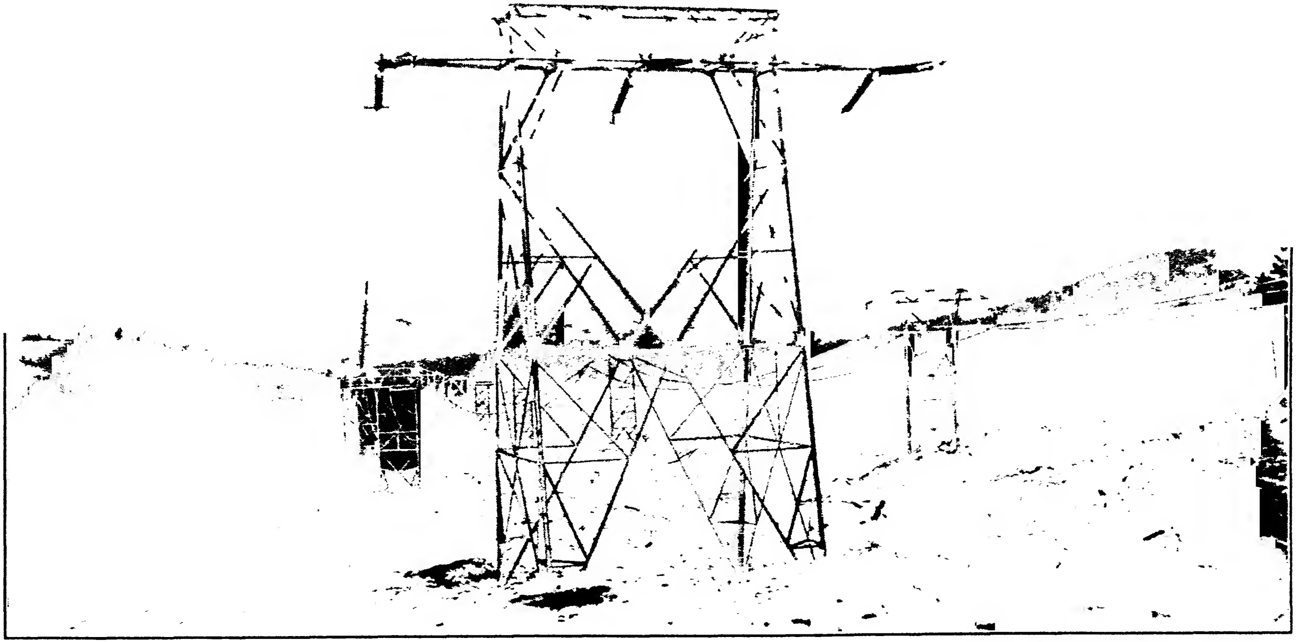
Courtesy Aluminum Company of America

Famous Big Creek Line of the Southern California Edison Co. It was constructed in 1913 and operated for 10 years at 150,000 v. The voltage then was raised to 220,000 v — the first line to be operated at that voltage. The line is three-phase, double-circuit. Each conductor is 605,000 cir mil A.C.S.R. Each circuit has one ½-in steel ground wire strung above it for lightning protection. The line is 240 miles long. At low altitudes the normal span is 660 ft. At altitudes over 2000 feet the normal span is 550 ft The maximum span is 2870 feet. The conductors of each circuit are normally spaced horizontally 17 ft 3 in apart suspended from cross arms 43 ft above ground.



Courtesy Aluminum Company of America

Line of the Pennsylvania Water and Power Co. between Ellicott City, Md., and Washington, D. C., 21.8 mi. It was constructed in 1933 for operation at 220,000 v. The line is three-phase, single-circuit. Each conductor is 795,000 cir mil, A.C.S.R. There are two ground wires, each 203,200 cir mil. The normal span is 943 ft. The maximum span is 1525 ft. The conductors are



Courtesy Aluminum Company of America

Line of the New England Power Association System from the Comerford Station near East Monroe, N. H., to Tewksbury, Mass., 126.4 mi. The line was constructed in 1930 for operation at 220,000 v. It is three phase double circuit. Each conductor is 795,000 cir mil. There are two ground wires above each circuit, ½ in steel, not visible in the picture. The normal span is 593 ft. The maximum span is 1176 ft. The conductors of each circuit are normally spaced 23 ft 6 in apart, suspended from cross arms 55 ft above ground. The tower in the foreground is

tion of elements, electrically or magnetically interrelated, used to transmit or transform electric energy in the performance of useful functions. Circuits may be classified by the number of phases used in their operation. The word *system* refers to the general classification in which the circuit under consideration belongs, implying, for example, that a certain network is operated by a three-phase method rather than by a two-phase method.

The use of double-subscript notation to identify the directions of complex voltages and currents with respect to points on the circuit diagram is especially pertinent in the analysis of polyphase circuits. Thus, as explained in Ch. IV, the expression E_{ab} means an electromotive force or *voltage rise* of complex value E from point *a* to point *b*. It is obvious that E_{ab} equals $-E_{ba}$. Also, the expression V_{ab} means a *voltage drop* of vector value V from point *a* to point *b*. In Fig. 1a, the voltage rise E_{ab} equals the

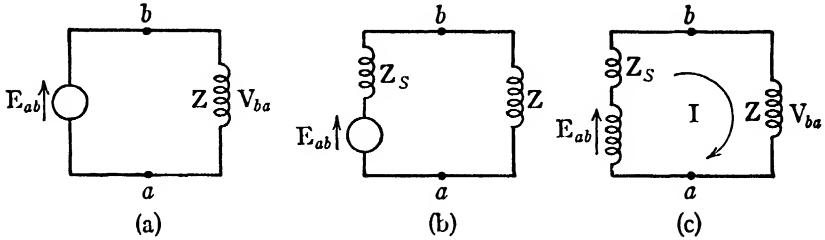


FIG. 1. Double-subscript notation for source-voltage rises and circuit-voltage drops.

voltage drop V_{ba}. In Fig. 1b, in which there is a source impedance Z_s, E_{ab} represents the electromotive force between *a* and *b*, which is not equal to V_{ba} except on open circuit, when no load Z is connected. In Fig. 1c, an induced voltage rise E_{ab} in a transformer output winding is indicated; E_{ab} is the voltage rise which actually appears at the transformer terminals under no-load conditions. When the transformer is loaded by connection of the impedance Z between *a* and *b*, the available terminal-voltage drop V_{ba} is the vector difference E_{ab} minus IZ_s. A transformer is not actually a source and hence has not a true electromotive force. In many circuit problems, however, common practice is to regard a transformer as an equivalent source and to start the circuit analysis at the secondary terminals.

2. SINGLE-PHASE SYSTEM

When electric power has to be transmitted at low voltages, a considerable proportion of the power may be lost by conversion into heat in the line wires unless the wires are made unreasonably large. The recognition of this fact early in the development of the electric-power industry led to the adoption of the three-wire system for distributing power to ordinary

loads. Figure 2b shows how, by use of a single three-wire voltage source, loads may be supplied with greater efficiency over transmission-line wires, each of which has a resistance R, than if the same total load is supplied over two wires having resistances R as in Fig. 2a.

In Fig. 2a the load of power P takes a current I at voltage V with a corresponding heat loss in the line of $2I^2R$. In the three-wire system (Fig. 2b) the load is divided into two equal parts, each of value $P/2$ and each having an applied voltage V. A wire called the neutral conductor connects the mid-point of the source voltage to the common connection

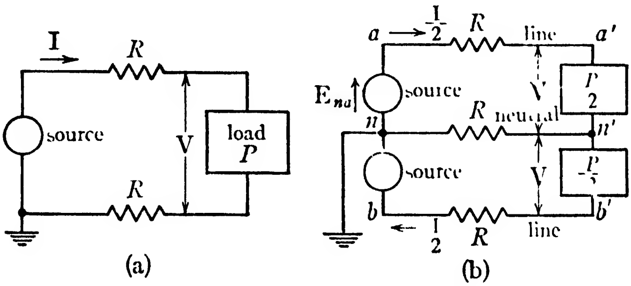


FIG. 2. Two-wire and three-wire single phase circuits.

joining the two halves of the load. Under this condition the load is said to be *balanced*, and no current is in the neutral conductor. Conductors each of the same resistance as in Fig. 2a are used, the current is I/2, and the corresponding line loss is $(I/2)^2(2R_1)$, or $I^2R/2$. The line losses are thus reduced to one-quarter of the value corresponding to the two-wire system without any reduction in the amount of power supplied to the load. To interpret the result another way, the balanced three-wire circuit can operate at the same transmission efficiency as the two-wire circuit if designed so that each wire of the three-wire circuit has four times the resistance, or one-fourth of the cross-section area, of a wire of the two-wire system. Thus the balanced three-wire system needs only three-eighths the amount of copper required by the two-wire system to transmit the same amount of power with the same efficiency. As the neutral conductor is indispensable in any case where an unbalanced condition may occur, the three-wire system requires three conductors each having one-fourth of the copper of the two conductors of the two-wire system. The neutral conductor is operated without a fuse or other protective device, so that it provides at all times a path for current returning to the source. The load voltages are thus kept approximately balanced in case the loads become unbalanced. The neutral is usually connected to ground, thus limiting the potential of the circuit above ground to V.

In practice the exact theoretical ratios derived in the preceding para-

graphs cannot usually be achieved because of the necessity of using standard sizes of wire and of adhering to fire underwriters' regulations, and because the load is rarely exactly balanced. If the load P represents ten 115-volt lamps, each consuming 500 watts at that voltage (such as used in lighting a portion of a factory), V of Figs. 2a and 2b equals 115 volts. For a two-wire system, as in Fig. 2a, the current is 43.5 amperes, requiring 6 AWG rubber-covered copper wire as a minimum safe size in accordance with the rules of the National Board of Fire Underwriters. This wire has a resistance of 0.395 ohm per thousand feet and a weight of 129 pounds per thousand feet. If the longest distance from source to lamps is 300 feet, 600 feet of rubber-covered wire having a resistance of 0.237 ohm and weighing 77.5 pounds are required. The loss of power in this resistance, represented by the two resistances R in series, is 448 watts if the load is considered to be concentrated at the end of the circuit. This loss is nearly the equivalent of the power required by one lamp and represents a definite operating expense.

If the circuit is installed in accordance with Fig. 2b, representing the three-wire single-phase system, economies result in comparison with the two-wire system. Under the conditions of the problem, the current in the line conductors of the three-wire circuit is only 21.8 amperes, so that 10 AWG rubber-covered wire having a resistance of 1.0 ohm per thousand feet and a weight of 54 pounds per thousand feet can be used safely for these conductors. As long as the load is balanced, no current is in the neutral wire, but, if one of the five-lamp groups is disconnected, the other lamps cause the neutral conductor to carry the same current that a line wire carries. Hence the neutral conductor must be of the same size as the line conductors, and a total length of 900 feet of 10 AWG rubber-covered wire is required, having a weight of 48.7 pounds, or 37.3 per cent less than in the two-wire system. The resistance $2R_1$ of the two line conductors in series is 0.60 ohm, with a corresponding loss of 285 watts under balanced full-load conditions, a saving of 36.2 per cent over the two-wire system.

Under the worst possible conditions of unbalance, the entire lamp group between n' and b' is out of service, while full load remains on the group a'-n'. The source voltage E_{na} from neutral to one line wire under balanced conditions is $V + R_2(I/2)$, or in this case $115 + 0.3(21.8)$ or 121.5 volts if the reactance of the line conductors is negligible. The current in the connected group a'–n', calculated on the assumption that the lamp resistance is unchanged, is 20.7 amperes. The resulting voltage across the operating lamps is 108.8 volts, and the line loss equals $(20.7)^2 \times 0.6$, or 258 watts, almost as much as the total line loss with the full 5,000-watt load operating under balanced conditions. Thus for most satisfactory operation of a three-wire system, the load evidently should be kept as nearly balanced as possible.

Figure 3 illustrates a three-wire single-phase supply from a utility-company's transformer to a residence having modern three-wire circuits, the usual switches, fuses, and branches being omitted for simplicity. The power supply ordinarily comes from a medium-voltage (such as 2,300 or 4,600 volts) distribution system to the transformer primary winding at a–b. The secondary windings are connected so that their polarities are in the same direction, as shown by the arrows $E_{b'n}$ and $E_{na'}$. In this way a potential difference of 230 volts or some value in that range is delivered across the line wires at a'-b', and one-half of this voltage is available from the neutral terminal to either line wire. The system is a single-phase

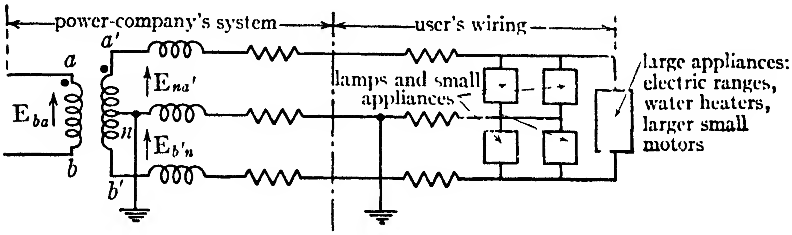


FIG. 3. Three-wire secondary circuit from power-company transformer to user's load.

system, having the vector relationship given in Fig 4. Service conditions fix the allowable voltage to ground and the allowable difference between no-load and full-load voltage which results at the user's appliances from the impedance of the transformer windings and of the conductors between the transformer secondary terminals a' b' and the appliances. Within these limitations, and with a given size of conductor, a three-wire system delivers to a balanced load four times the power that a two-wire system delivers. The great increase in use of electrical appliances in recent years has emphasized the practical importance of the efficiency of the three-wire system. An additional advantage of the three-wire system is that loads requiring two different voltages, such as 115-volt lamps and 230-volt heating appliances, can be operated from the same circuit.

FIG. 4. Vector diagram of transformer voltages of Fig 3 (The scale of E_{ba} is approximately $2a$ times the scale of $E_{b'n}$ and $E_{na'}$, where a is the ratio of primary to total secondary turns of the transformer.)

The three-wire single-phase alternating-current system is an outgrowth of the three-wire direct-current system which Thomas A. Edison developed for reasons identical with those discussed for alternating current. The three-wire direct-current system is extensively used in industrial plants where direct current offers certain advantages over alternating current. There are also several densely populated urban areas where three-wire direct-current systems are in use for the general distribution

of electricity. These direct-current systems have remained in service in some places because they provide, through the use of storage batteries, a reliable emergency supply and an economical means of handling peak loads of short duration; another reason is that the utility company has been unable to justify the cost of changing from direct-current to alternating-current equipment, the direct-current system having been expanded over many years from a small start made in the earliest days of the commercial use of electric service.

If a certain voltage is being given preliminary consideration for use in a two-wire circuit to deliver power to a certain load, the reasoning developed in previous paragraphs indicates that if the load can be operated at a higher voltage than the one under consideration, the two-wire system can still be used, requiring a correspondingly smaller cross section of copper to deliver the same power at the same loss. Specifically, if, in the example used, ten 230-volt 500-watt lamps are used, power can be supplied to them over two 10 AWG wires instead of the three of the three-wire system without any increase in voltage to ground. The use of still higher voltages might be considered, but the economies resulting in conductor material might be offset by considerations of safety and insulation expense. The fact that lamps of higher voltage classes are more costly and less efficient than 115-volt lamps is an additional difficulty. For high-voltage power-transmission lines, the minimum size of wire commensurate with a certain voltage is also limited by corona discharge, which is extremely harmful to insulation in cables and always represents an energy loss; the wire size is also limited in open wire lines by the necessary mechanical strength. In general, however, the reductions in heat loss and in weight of conductor which accompany an increase in circuit voltage explain the widespread use of alternating current for power transmission and distribution. The alternating-current transformer makes practicable the transmission of electricity at high voltages, no commercially acceptable substitute direct-current method having been developed as yet. If the conductors of long-distance transmission lines had to be of sufficiently large cross section to transmit large blocks of power at the direct-current voltages commonly utilized, these lines could not have been economically justified and would not have been built. In fact, the whole central-station industry is made possible by the fact that power can be economically transmitted and distributed at high voltages.

3. Generation of polyphase voltages

An alternating single-phase voltage can be generated by revolving two magnetic poles of opposite polarity in such a way that most of their field flux links a stationary winding as they rotate. If the pole faces are properly shaped, and the speed of rotation is held constant, it is possible to

make the flux linkages with this winding vary as a sinusoidal function of time, and there occurs a sinusoidally varying voltage e_{xa} across the winding terminals x and a shown in Fig. 5. This is, in principle, the method used for generating sinusoidal alternating voltages in commercial single-phase generators. If it is desired to produce voltages having phase angles differing by 90 degrees, it is necessary only to place another winding at such a position that the induced voltage in the winding is a quarter of a cycle out of phase with e_{xa}. This method is illustrated by the rudimentary two-phase generator of Fig. 6, in which e_{xa} is 90 degrees ahead of e_{yb} for the direction of field rotation indicated.

Generators having more than one phase are called polyphase generators or polyphase alternators. Within practical limitations, as many phases as desired may be obtained by spacing a winding for each phase at the

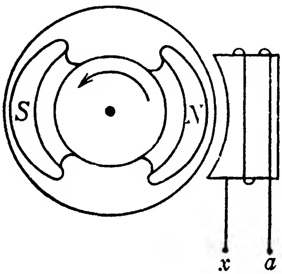


FIG. 5. Rudimentary single-phase generator.

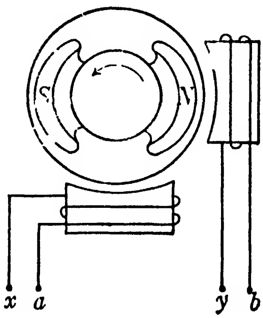


FIG. 6. Rudimentary two-phase generator.

proper angular position around the periphery of an alternator. Though the fundamental principle of generation of polyphase voltages as outlined here is the method employed in actual alternators, many complications and special considerations arise in applying the theory, so that the detailed treatment of alternator theory is presented as a special study in this series in the volume on rotating electric machinery. Polyphase voltages can also be generated by mounting the field windings of one separate single-phase generator for each phase on a common shaft, spacing the field windings of the machines at proper relative angles. As this method is impractical from the standpoint of cost and wasted space, it is not used in practice.

If the two voltages e_{xa} and e_{yb} are plotted as functions of time, the two sine waves shown in Fig. 7 are obtained. These voltages compose a *two-phase* source in accordance with accepted usage, since a two-phase system is one whose source voltages are two in number, equal in magnitude, and have a difference in phase angle of 90 degrees. Stated in terms

of time, the maximum values of the sinusoidal voltages of the two phases occur one-quarter of a cycle apart. Two two-phase systems are shown vectorially in Figs. 8a and 8b. In these figures, the currents are assumed

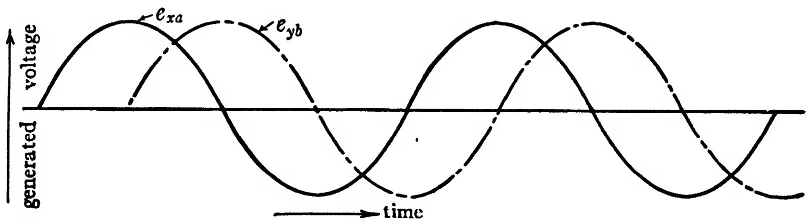


FIG. 7. Sinusoidal source voltages of a two-phase system, plotted as a function of time.

to result from equal load impedances connected to each of the individual phases of the system.

The arrangement represented in Fig. 8a is called a *four-wire two-phase system* because four conductors are necessary to connect the source to the load. There is no electrical connection between the two phases, and each phase can have its own single-phase loads. Many such systems are still in

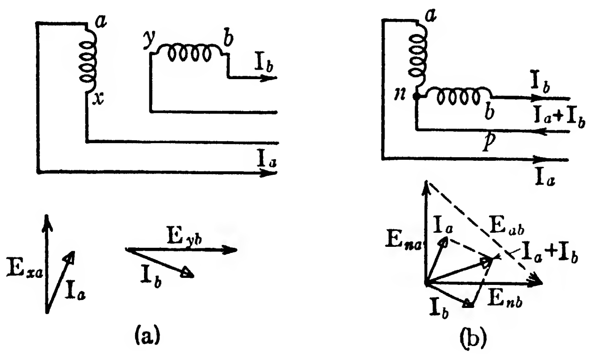


FIG. 8. Diagrams for four-wire and three-wire two-phase sources.

service, although the trend is clearly toward the three-phase system described in later articles of this chapter. If equal impedances are connected across each of the two phases, equal currents, 90 degrees apart in phase relationship, result in the two phases. Figure 8b represents a *three-wire two-phase system*, occasionally employed but undesirable because the neutral conductor *np* carries a current equal in magnitude to $\sqrt{2}$ times the current in each of the two line conductors when equal loads are connected from each line conductor to the neutral conductor, in contrast with the neutral of a three-wire single-phase system, which

carries no current when the loads are equal. The neutral of the three-wire two-phase system requires different sizes of wire and protective equipment than used in the line wires, which is an objectionable constructional feature. In this system a voltage of magnitude

$$E_{ab} = \sqrt{2}E_{na} = \sqrt{2}E_{nb} \tag{1}$$

is available from terminal a to terminal b. In rare cases the voltage E_{ab} may have some practical value.

Another polyphase system, called a *five-wire two-phase system* is illustrated in Fig. 9a. This system is called two phase because it can be considered to consist of two three-wire single-phase systems whose neutrals are connected. From a more scientific standpoint, however, it should properly be termed a four-phase system. If the common-origin type of vector diagram is used, the directions of the arrows and vector subscripts assigned in Fig. 9a to $\mathbf{E}_{dn}$ and $\mathbf{E}_{cn}$ are reversed, and Fig. 9b is obtained. The system of Fig. 9b, obviously identical with the one of Fig. 9a, consists of four separate phase voltages, $\mathbf{E}_{na}$, $\mathbf{E}_{nb}$, $\mathbf{E}_{nc}$, and $\mathbf{E}_{nd}$, which are equal in magnitude and have phase angles of 90 degrees between any two adjacent vectors. When developed in this manner, the system becomes a symmetrical *four-phase system*. It is also evident from Fig. 9b that the three-wire scheme of Fig. 8b could be properly termed one-half of a four-phase system. However, as the expression *two-phase* is generally accepted for all these systems having 90-degree phase angles between voltages, it is employed in this chapter.

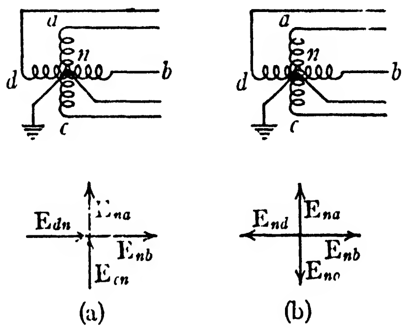


FIG. 9. Diagrams for five-wire two-phase source.

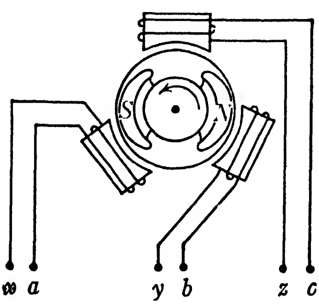


FIG. 10. Rudimentary three-phase generator.

If three identical windings are placed symmetrically around the periphery of the rudimentary generator as in Fig. 10, there result the three voltages e_{xa}, e_{yb}, and e_{zc}, equal in magnitude and having a phase-angle difference of 120 degrees between any two voltages. If these voltages are plotted against time, the three separate sine waves shown in Fig. 11 are obtained. These voltages constitute a *three-phase* voltage

source, by far the most common type of voltage source found in commercial power generation.

In a very carefully designed and constructed machine, it is possible theoretically to generate voltages which are pure sinusoidal functions of time, as shown in Fig. 11. It is, however, almost impossible to build a commercial alternator which generates pure sine waves of fundamental frequency alone, for there are usually present odd harmonic frequencies of

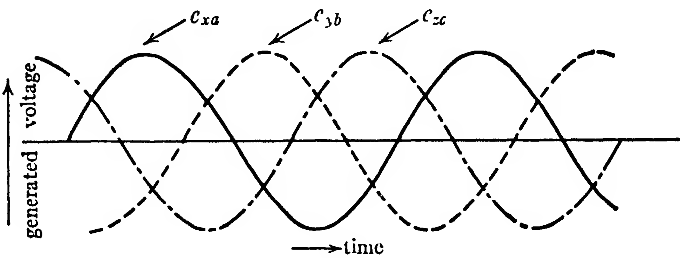


FIG. 11. Sinusoidal source voltages of a three-phase system, plotted as functions of time.

third, fifth, and sometimes higher order which cause the wave form of voltage to be nonsinusoidal to a certain extent. These harmonic voltages result from slight irregularities in the reluctances of the various paths of magnetic flux, from slight deviations in the shape of the field pole faces from that desired, and from other factors. Such nonsinusoidal voltages can be expressed by a Fourier analysis[1] in terms of sine waves of harmonic frequencies, each of which acts to produce a current of the same frequency whose magnitude and phase angle depend on the impedance of the circuit to that frequency.

4. Phase order and symmetry

In dealing with the alternating currents and voltages in three-phase circuits, the idea of *phase order* or *phase sequence* must be established. If sinusoidal voltages of a given frequency are considered, the voltage of one generator phase reaches a given point in its cycle — for example, the positive maximum — at a given instant. At some later instant, the voltage of another phase reaches the same point in its cycle, and similarly for the third phase. If the phase-a voltage maximum is followed in time by the phase-b maximum, and this in turn by the phase-c maximum, the phase order of these voltages is said to be abc. Conversely, if the phase-a voltage maximum is followed in time by the phase-c maximum, and this

[1] R. R. Lawrence, *Principles of Alternating Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1935), Ch. iv; P. Franklin, *Differential Equations for Electrical Engineers* (New York: John Wiley & Sons, 1933), Ch. viii; also subsequent volumes of this series.

in turn by the phase-b maximum, the phase order of these voltages is said to be *acb*. Three-phase voltages of phase order *abc* and *acb* are illustrated in Figs. 12a and 12b, respectively. In this chapter, the phase order *abc* is used as standard except where otherwise specified.

The direction of *rotation* of *time* vectors is in all cases counterclockwise. This direction has been established as an international standard, though clockwise vector rotation was used in the technical literature of some of

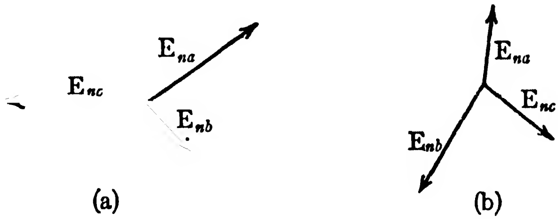
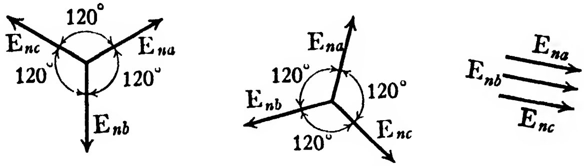


FIG. 12. Three-phase voltages of phase order *abc* and *acb*.

the earlier writers before the adoption of the counterclockwise standard by the International Electrotechnical Commission in 1911. Phase order depends on the direction of rotation, construction and winding connections of the generator and the lettering of the terminals. Since rotation of all time vectors is taken to be counterclockwise, the phase order or sequence of a system can be determined conveniently from the vector diagram by observation of the order in which the vectors pass any reference point as they rotate in a counterclockwise direction. The commonly used diagram of stationary vectors, such as Fig. 12 has of course the same phase sequence as the rotating vectors from which it is derived. The expression *phase rotation*, having the same meaning as phase order or phase sequence, is used a great deal in practice. The difference in meaning between phase rotation and vector rotation should be clearly understood. Instruments called phase-rotation indicators are used for determining phase order when the order is not known.

If the three-phase voltages or currents of a given frequency are equal in magnitude and if each differs from each of the other two by the same magnitude of phase angle, the voltages or currents are said to form a *symmetrical*, or *balanced*, *system*. Evidently only three symmetrical three-phase systems are possible. These are shown in Fig. 13 as *positive-sequence symmetrical*, having the phase order *abc*; *negative-sequence symmetrical*, having the phase order *acb*; and *zero-sequence symmetrical*, so called because the voltages in all three phases are represented by equal vectors having no phase order. Balanced currents result from connecting a balanced load to a balanced voltage source. The sum of the vectors of a symmetrical positive- or negative-sequence system is zero.

The concept of balance as used here has the connotation of symmetry and must be distinguished from the idea of balanced forces as employed in the study of free bodies in mechanics. In the latter, a body is said to be in equilibrium if the vector sum of all the forces acting on it is zero, but the individual forces need not necessarily be equal in magnitude and symmetrically spaced in their angles of application. In electrical terminology, however, it is possible for the phase voltages of a system to add vectorially



symmetrical positive-sequence system — symmetrical negative-sequence system — symmetrical zero-sequence system

FIG. 13. Symmetrical sets of three-phase voltage vectors.

to zero even if the voltages of the system are not balanced. Thus the vectors $6 + j4$, $-3 + j2$, and $-3 - j6$ add up to zero, but a system composed of these voltages is electrically unbalanced. On the other hand, the sum of the vectors of a zero-sequence system never can be zero, yet electrically the system is balanced. Furthermore, a balanced load is not three impedances whose sum is zero but is a load of equal impedance in each phase.

5. BALANCED THREE-PHASE CIRCUIT, *Y* CONNECTION

In Fig. 14 is shown a group of three simple single-loop circuits, each of which composes one phase of the system. The three voltages E_{xa}, E_{yb}, and E_{zc} are the terminal voltages of the three generator windings. The

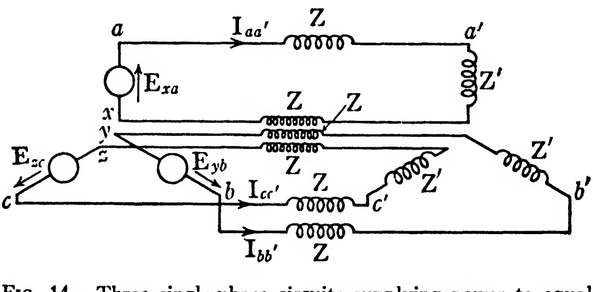


FIG. 14. Three single-phase circuits supplying power to equal loads.

impedances Z represent the impedances of transmission-line conductors, whereas the impedances Z' represent the balanced impedance of a load to which power is delivered. This particular circuit configuration consist-

ing of generator, line, and load is used in much that follows because it is characteristic of simple power-transmission circuits. It is evidently possible to use a common conductor as a return path for all three circuits as is done in the two circuits of the two-phase three-wire system. The resulting configuration of Fig. 15 may be regarded as a single circuit,

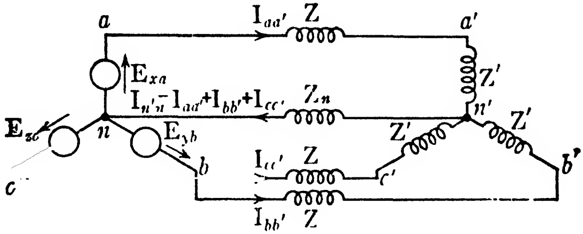


FIG. 15. Three-phase Y-connected circuit supplying power to a balanced load

universally known as a *Y-connected* three-phase circuit. This three-phase circuit may also be looked upon as three separate circuits with one conductor common to parts of the three. The common conductor is ordinarily known as the *neutral conductor*. It joins the common point n of the three generator phase windings, or *generator neutral*, with the common point n' of the load phase impedances, or *load neutral*.

Figure 16a shows the method of connecting the generator terminals when a Y system is employed. If the phase order is abc and the generator voltages are symmetrical (positive-sequence symmetrical voltages), the sine waves of Fig. 11 and the connections of Fig. 16a establish the form of the vector diagram Fig. 16b, in which E_{na} leads E_{nb} by 120 degrees, and E_{nb} leads E_{nc} by 120 degrees. In determining from this diagram a line-to-line voltage such as E_{ab}, it is necessary to add E_{an} and E_{nb}. Thus E_{an} (the negative of the phase voltage E_{na}) must be added vectorially to the adjacent phase voltage E_{nb} to obtain the voltage E_{ab}. In other words, it is necessary to *subtract* the phase voltage rise E_{na} from the phase voltage rise E_{nb} in order to obtain the resultant line-to-line voltage rise E_{ab}. If each line-to-neutral voltage has a magnitude E_y, the line-to-line voltages E_{ab}, E_{bc}, and E_{ca} may be calculated. If E_{na} is taken as the axis of reference,

$$E_{ab} = E_{an} + E_{nb}, \tag{2}$$

$$E_{ab} = -E_{na} + E_{nb}, \tag{2a}$$

$$E_{ab} = E_y\underline{/180°} + E_y\underline{/-120°}, \tag{2b}$$

$$E_{ab} = \left(-1 - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)E_y = \left(-\frac{3}{2} - j\frac{\sqrt{3}}{2}\right)E_y, \tag{2c}$$

$$\mathrm{E}_{ab} = \sqrt{3}E_y\underline{/-150^\circ};\ \mathrm{E}_{ba} = \sqrt{3}E_y\underline{/30^\circ} = \sqrt{3}\mathrm{E}_{na}\underline{/30^\circ}. \qquad \blacktriangleright[2d]$$

Similarly,

$$\mathrm{E}_{bc} = \sqrt{3}E_y\underline{/90^\circ};\qquad \mathrm{E}_{cb} = \sqrt{3}E_y\underline{/-90^\circ} = \sqrt{3}\mathrm{E}_{na}\underline{/-90^\circ}, \qquad \blacktriangleright[3]$$

$$\mathrm{E}_{ca} = \sqrt{3}E_y\underline{/-30^\circ};\ \mathrm{E}_{ac} = \sqrt{3}E_y\underline{/150^\circ} = \sqrt{3}\mathrm{E}_{na}\underline{/150^\circ}. \qquad \blacktriangleright[4]$$

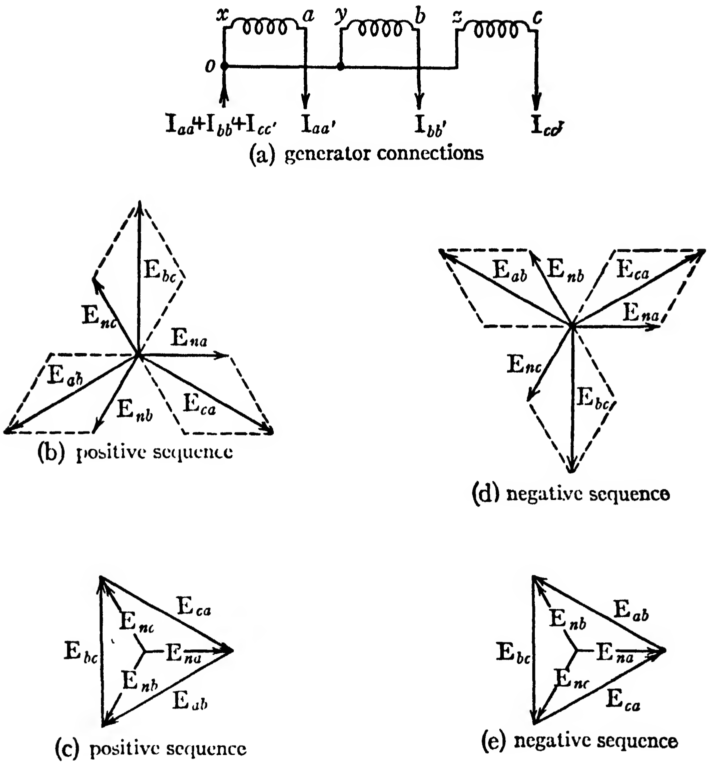


FIG. 16. Generator connections and source-voltage vector diagrams for three-phase Y-connected circuit.

If, however, the phase order is *acb* and the generator voltages are symmetrical (negative-sequence symmetrical voltages), Fig. 16d applies, in which E_{na} leads E_{nc} by 120 degrees and E_{nc} leads E_{nb} by 120 degrees. For the negative sequence, therefore, the relations of the phase angles are changed. If E_{na} is taken as the axis of reference for the negative sequence,

$$\mathrm{E}_{ab} = -\mathrm{E}_{na} + \mathrm{E}_{nb}, \qquad [5]$$

$$\mathrm{E}_{ab} = E_y\underline{/180^\circ} + E_y\underline{/120^\circ}, \qquad [5a]$$

$$\mathrm{E}_{ab} = \left(-\frac{3}{2} + j\frac{\sqrt{3}}{2}\right) E_y, \qquad [5b]$$

$$\mathrm{E}_{ab} = \sqrt{3}E_y \underline{/150^\circ}, \quad \mathrm{E}_{ba} = \sqrt{3}E_y \underline{/-30^\circ} = \sqrt{3}\mathrm{E}_{na}\underline{/-30^\circ}, \qquad [5c]$$

$$\mathrm{E}_{bc} = \sqrt{3}E_y \underline{/-90^\circ}, \quad \mathrm{E}_{cb} = \sqrt{3}E_y \underline{/90^\circ} = \sqrt{3}\mathrm{E}_{na}\underline{/90^\circ}, \qquad [6]$$

$$\mathrm{E}_{ca} = \sqrt{3}E_y \underline{/30^\circ}. \quad \mathrm{E}_{ac} = \sqrt{3}E_y \underline{/-150^\circ} = \sqrt{3}\mathrm{E}_{na}\underline{/-150^\circ}. \qquad [7]$$

In Figs. 16c and 16e the line-to-line voltage vector arrows are rearranged in a manner analogous to the actual electrical connections of the Y connected system. These diagrams correctly indicate relative vector magnitudes and phase angles, by the method outlined in Art. 4, Ch. IV. Figures 16c and 16e illustrate for the positive- and negative-sequence systems, respectively, the useful fact that the phase voltages of a balanced Y-connected system may be represented by the rays connecting the geometric center to the apexes of an equilateral triangle whose sides represent the line-to-line voltages.

In the Y-connected three-phase circuit, no current is present in the neutral conductor if the voltages and loads are balanced, as can be shown from the three equations expressing the Kirchhoff law for voltages around each of the loops $naa'n'n$, $nbb'n'n$, and $ncc'n'n$ of Fig 15:

$$\mathrm{E}_{na} = \mathrm{I}_{aa'}(Z + Z') + (\mathrm{I}_{aa'} + \mathrm{I}_{bb'} + \mathrm{I}_{cc'})Z_n, \qquad [8]$$

$$\mathrm{E}_{nb} = \mathrm{I}_{bb'}(Z + Z') + (\mathrm{I}_{aa'} + \mathrm{I}_{bb'} + \mathrm{I}_{cc'})Z_n, \qquad [9]$$

$$\mathrm{E}_{nc} = \mathrm{I}_{cc'}(Z + Z') + (\mathrm{I}_{aa'} + \mathrm{I}_{bb'} + \mathrm{I}_{cc'})Z_n. \qquad [10]$$

If these three equations are added, the left-hand member is zero, since the sum of the three balanced source voltages is zero:

$$\left.\begin{aligned} 0 = \mathrm{I}_{aa'}(Z + Z' + 3Z_n) + \mathrm{I}_{bb'}(Z + Z' + 3Z_n) \\ + \mathrm{I}_{cc'}(Z + Z' + 3Z_n). \end{aligned}\right\} \qquad [11]$$

The common factor $(Z + Z' + 3Z_n)$ is now divided out, leaving the expression

$$\mathrm{I}_{aa'} + \mathrm{I}_{bb'} + \mathrm{I}_{cc'} = \mathrm{I}_{n'n} = 0. \qquad [11a]$$

Hence in a symmetrical system having a Y-connected generator and load and sinusoidal voltages and currents, the load neutral is at the same potential as the generator neutral. The neutral conductor, therefore, carries no current and may be omitted under these special conditions. The current in each line conductor is now easily determined by setting

up a single-loop equation for each of the currents in Fig. 15:

$$\mathbf{I}_{aa'} = \frac{\mathbf{E}_{na}}{\mathbf{Z}' + \mathbf{Z}} = \frac{E_y/\underline{0}}{\mathbf{Z}' + \mathbf{Z}}, \tag{12}$$

$$\mathbf{I}_{bb'} = \frac{\mathbf{E}_{nb}}{\mathbf{Z}' + \mathbf{Z}} = \frac{E_y/\underline{-120^\circ}}{\mathbf{Z}' + \mathbf{Z}}, \tag{13}$$

$$\mathbf{I}_{cc'} = \frac{\mathbf{E}_{nc}}{\mathbf{Z}' + \mathbf{Z}} = \frac{E_y/\underline{+120^\circ}}{\mathbf{Z}' + \mathbf{Z}}. \tag{14}$$

Figure 17 is a vector diagram showing the current and voltage relations that exist in a positive-sequence balanced *Y*-connected system, the angle θ being the power-factor angle corresponding to the single-phase impedance $\mathbf{Z}' + \mathbf{Z}$. The angle between current such as $\mathbf{I}_{aa'}$ and line-to-line voltage such as $\mathbf{E}_{ba}$ is not the power-factor angle. Since there is no voltage drop in the neutral conductor, evidently problems in a symmetrical *Y*-connected circuit may be solved per phase as if the circuit were single phase, by use of the phase voltage, phase impedance, and phase current which is the same as the line current. After conditions in phase *a* have been ascertained, for example, the current and voltage drops in phases *b* and *c* are at once determined by multiplying the corresponding values in phase *a* by $1/\underline{-120}$ degrees and $1/\underline{+120}$ degrees, respectively, as indicated in Eqs. 13 and 14.

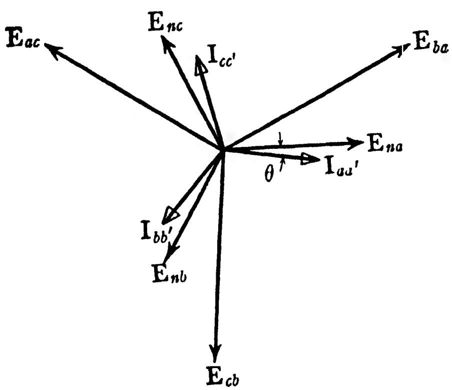


FIG. 17. Vector diagram of voltages and currents in *Y*-connected source.

6. Illustrative example of balanced *Y*-connected system

In Fig. 15 the generator terminal voltage (line to line) is 230 volts and the load and line impedances are, respectively, $15 + j10$ ohms and $2.0 + j4.0$ ohms in each phase.* The problem is to find the line currents and load voltages.

* In this chapter, line impedances are used as if they were series impedances independent of the mutual effects of the lines of a polyphase transmission system. In this example, the line drop is exaggerated to make it apparent on the vector diagram.

Solution: The phase voltage E_{na} in the generator taken along the axis of reals is

$$E_{na} = \frac{230}{\sqrt{3}} + j0 = 133 + j0 \text{ v.} \qquad [15]$$

Hence,

$$I_{aa} = \frac{133}{17 + j14} = 6.04\underline{/-39.5°} \text{ amp,} \qquad [16]$$

$$V_{a'n'} = I_{aa'}Z' = (6.04\underline{/-39.5°})(15 + j10) = 109\underline{/-5.8°} \text{ v.} \qquad [17]$$

The vector relations are shown in Fig. 18 for phase *a*. This diagram, revolved 120 degrees clockwise, shows the voltages and currents in phase *b* and, revolved 120 degrees counterclockwise, shows the conditions in phase *c*, the phase order being assumed as *abc*.

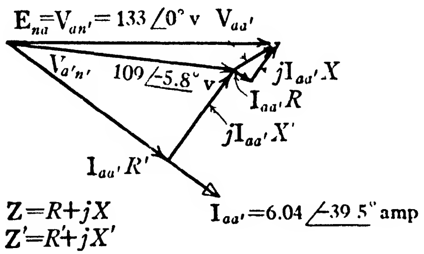


FIG. 18. Vector diagrams of voltages and currents in phase *a* of Fig. 15.

7. Balanced Three-Phase Circuit, Δ Connection

The sum of the voltages of the three sine waves of Fig. 11 at any instant is zero. This means that, if the generator winding terminals are connected together in series in the proper order, a closed-loop circuit is obtained in which there is no current in the absence of an external load, provided that all the voltages are sinusoidal, are of equal magnitude, and are equally spaced in phase. The winding connections and the corresponding vector diagram are presented in Fig. 19, which corresponds to the terminal notation of Fig. 10. In the vector diagram, E_{xa} leads E_{yb} by 120 degrees, E_{yb} leads E_{zc} by 120 degrees, and E_{zc} leads E_{xa} by 120 degrees, which arrangement agrees with the generator construction and the sine waves of Fig. 11. The symmetrical voltages which are present across the terminals of the three windings may be used as sources to deliver

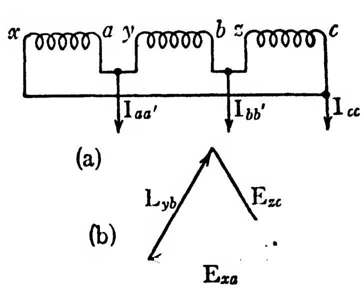


FIG. 19. Generator connections and source-voltage vector diagram for three phase Δ connected circuit.

power over three external conductors to the load. If the load is also Δ connected, as in Fig. 20, the current in each conductor is the difference of the load currents in the two impedances connected to it. If the three impedances Z are equal, and the three impedances Z′ are equal, the resulting phase voltages across the load impedances are equal and displaced

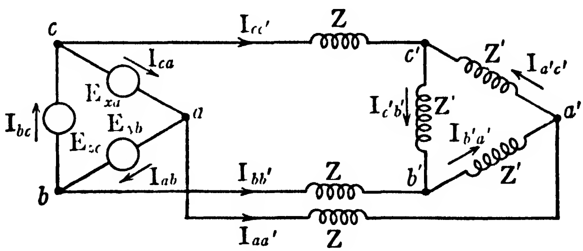


FIG. 20. Three-phase Δ-connected circuit supplying power to a balanced load.

ing phase voltages across the load impedances are equal and displaced from each other by 120 degrees. Calling the magnitude of these phase voltage drops V_Δ, using the voltage drop $V_{b'a'}$ as the reference axis, and assuming positive sequence give

$$I_{b'a'} = \frac{V_{b'a'}}{Z'} = \frac{V_\Delta \angle 0}{Z'}, \qquad [18]$$

$$I_{c'b'} = \frac{V_{c'b'}}{Z'} = \frac{V_\Delta \angle -120^\circ}{Z'}, \qquad [19]$$

$$I_{a'c'} = \frac{V_{a'c'}}{Z'} = \frac{V_\Delta \angle +120^\circ}{Z'}, \qquad [20]$$

$$\left.\begin{aligned} I_{aa'} = I_{a'c'} - I_{b'a'} &= \frac{\sqrt{3}V_\Delta}{Z'} \angle 150^\circ = \sqrt{3} I_{a'c'} \angle 30^\circ \\ &= \sqrt{3} I_{b'a'} \angle 150^\circ, \end{aligned}\right\} \qquad ▶[21]$$

$$\left.\begin{aligned} I_{bb'} = I_{b'a'} - I_{c'b'} &= \frac{\sqrt{3}V_\Delta}{Z'} \angle 30^\circ = \sqrt{3} I_{b'a'} \angle 30^\circ \\ &= \sqrt{3} I_{c'b'} \angle 150^\circ, \end{aligned}\right\} \qquad ▶[22]$$

$$\left.\begin{aligned} I_{cc'} = I_{c'b'} - I_{a'c'} &= \frac{\sqrt{3}V_\Delta}{Z'} \angle -90^\circ = \sqrt{3} I_{c'b'} \angle 30^\circ \\ &= \sqrt{3} I_{a'c'} \angle 150^\circ. \end{aligned}\right\} \qquad ▶[23]$$

Figure 21a shows the relations expressed by Eqs. 18 to 23. The angle θ is the phase angle between load phase voltage and load phase current resulting from the characteristics of the load impedance Z' in each phase. It should be carefully noted that the angle between a line current, such as

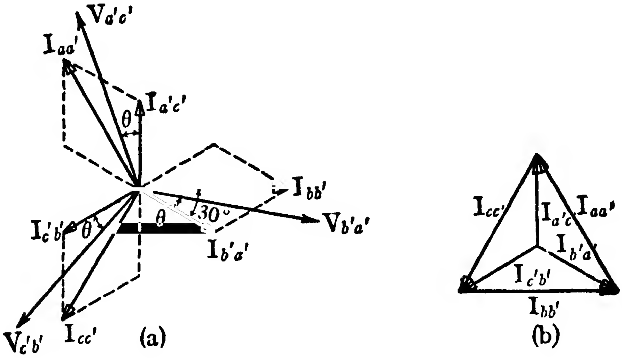


FIG. 21. Vector diagrams of voltages and currents in balanced Δ-connected load, positive-sequence system.

$I_{aa'}$ and any line voltage is not the power-factor angle. The angle between phase current such as $I_{a'c'}$ and phase voltage such as $V_{a'c'}$ must be taken to gain any physical concept of power factor.

If the vectors of the load-current diagrams are rearranged as in Fig. 21b, the sides of an equilateral triangle represent the line currents, and the

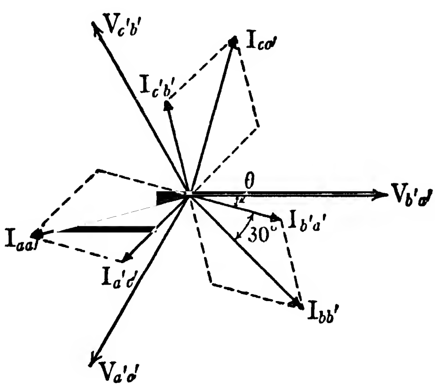


FIG. 22. Vector diagram of voltages and currents in balanced Δ-connected load, negative-sequence system.

rays connecting the apexes to the geometrical center represent the phase currents. Thus the currents in a balanced Δ connection have the same relation as the voltages have in a Y connection.

Relations corresponding to Eqs. 18 to 23 are presented in Eqs. 24 to 29 for a negative-sequence system and illustrated vectorially in Fig. 22:

$$\mathbf{I}_{b'a'} = \frac{\mathbf{V}_{b'a'}}{Z'} = \frac{V_\Delta\underline{/0}}{Z'}, \qquad [24]$$

$$\mathbf{I}_{c'b'} = \frac{\mathbf{V}_{c'b'}}{Z'} = \frac{V_\Delta\underline{/+120^\circ}}{Z'}, \qquad [25]$$

$$\mathbf{I}_{a'c'} = \frac{\mathbf{V}_{a'c'}}{Z'} = \frac{V_\Delta\underline{/-120^\circ}}{Z'}, \qquad [26]$$

$$\left.\begin{aligned}\mathbf{I}_{aa'} = \mathbf{I}_{a'c'} - \mathbf{I}_{b'a'} &= \frac{\sqrt{3}V_\Delta}{Z'}\underline{/-150^\circ} = \sqrt{3}\mathbf{I}_{a'c'}\underline{/-30^\circ}\\ &= \sqrt{3}\mathbf{I}_{b'a'}\underline{/-150^\circ},\end{aligned}\right\} \qquad ▶[27]$$

$$\left.\begin{aligned}\mathbf{I}_{bb'} = \mathbf{I}_{b'a'} - \mathbf{I}_{c'b'} &= \frac{\sqrt{3}V_\Delta}{Z'}\underline{/-30^\circ} = \sqrt{3}\mathbf{I}_{b'a'}\underline{/-30^\circ}\\ &= \sqrt{3}\mathbf{I}_{c'b'}\underline{/-150^\circ},\end{aligned}\right\} \qquad ▶[28]$$

$$\left.\begin{aligned}\mathbf{I}_{cc'} = \mathbf{I}_{c'b'} - \mathbf{I}_{a'c'} &= \frac{\sqrt{3}V_\Delta}{Z'}\underline{/90^\circ} = \sqrt{3}\mathbf{I}_{c'b'}\underline{/-30^\circ}\\ &= \sqrt{3}\mathbf{I}_{a'c'}\underline{/-150^\circ}.\end{aligned}\right\} \qquad ▶[29]$$

For the negative sequence, the angles between phase-current and line-current vectors are the negatives of the corresponding angles in the positive sequence, as is readily seen by a comparison of Eqs. 21 to 23 with Eqs. 27 to 29.

From inspection of the generator connections of Fig. 20, it is evident that

$$\mathbf{I}_{aa'} = \mathbf{I}_{ca} - \mathbf{I}_{ab}, \qquad [30]$$

$$\mathbf{I}_{bb'} = \mathbf{I}_{ab} - \mathbf{I}_{bc}, \qquad [31]$$

$$\mathbf{I}_{cc'} = \mathbf{I}_{bc} - \mathbf{I}_{ca}, \qquad [32]$$

which equations are represented vectorially in Fig. 23 for positive-sequence currents that result from positive-sequence voltages. From these diagrams the relations between line currents and generator phase currents are obtained:

$$\mathbf{I}_{aa'} = \sqrt{3}\mathbf{I}_{ca}\underline{/30^\circ} = \sqrt{3}\mathbf{I}_{ab}\underline{/150^\circ}, \qquad ▶[33]$$

$$\mathbf{I}_{bb'} = \sqrt{3}\mathbf{I}_{ab}\underline{/30^\circ} = \sqrt{3}\mathbf{I}_{bc}\underline{/150^\circ}, \qquad ▶[34]$$

$$\mathbf{I}_{cc'} = \sqrt{3}\mathbf{I}_{bc}\underline{/30^\circ} = \sqrt{3}\mathbf{I}_{ca}\underline{/150^\circ}. \qquad ▶[35]$$

The line currents $I_{aa'}$, $I_{bb'}$, and $I_{cc'}$ are the same in Figs. 21 and 23, which represent load and generator conditions, respectively. However, the phase currents are oppositely directed; for example, $I_{a'c'}$ and I_{ca} point in the

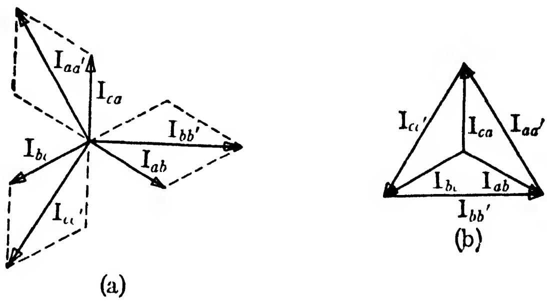


FIG. 23. Vector diagrams of generator phase and line currents in symmetrical positive sequence Δ-connected source.

same direction, a fact which means that the currents in phase windings *a–c* of the generator and *a′–c′* of the load are oppositely directed. This relation is a perfectly natural one, as is shown from an inspection of the circuit diagram of Fig. 20.

For a negative-sequence system, Eqs. 30 to 32 also apply, but the

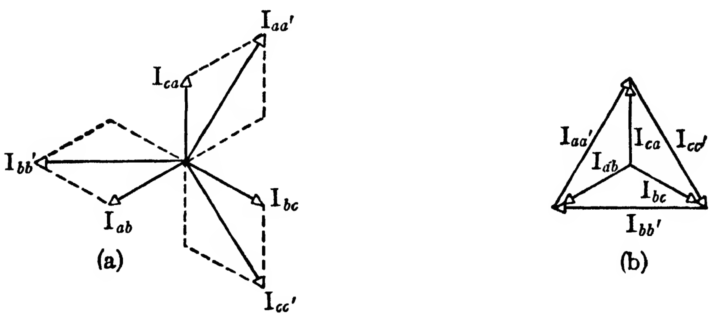


FIG. 24. Vector diagrams of generator phase currents and line currents in symmetrical negative-sequence Δ-connected source.

vector diagrams appear as in Fig. 24 and the relations among the currents are expressed by Eqs. 36 to 38:

$$I_{aa'} = \sqrt{3}I_{ca}\underline{/-30^\circ} = \sqrt{3}I_{ab}\underline{/-150^\circ}, \quad ▶[36]$$

$$I_{bb'} = \sqrt{3}I_{ab}\underline{/-30^\circ} = \sqrt{3}I_{bc}\underline{/-150^\circ}, \quad ▶[37]$$

$$I_{cc'} = \sqrt{3}I_{bc}\underline{/-30^\circ} = \sqrt{3}I_{ca}\underline{/-150^\circ}. \quad ▶[38]$$

To find the relation of load current to source voltage, Kirchhoff's law for loop voltage can be applied to the loop $abb'a'$ of Fig. 20:

$$\mathbf{E}_{ab} = \mathbf{I}_{bb'}\mathbf{Z} + \mathbf{I}_{b'a'}\mathbf{Z}' - \mathbf{I}_{aa'}\mathbf{Z}; \tag{39}$$

but from Eqs. 22 and 21,

$$\mathbf{I}_{bb'} = \sqrt{3}\mathbf{I}_{b'a'}\underline{/30^\circ}, \tag{40}$$

$$\mathbf{I}_{aa'} = \sqrt{3}\mathbf{I}_{b'a'}\underline{/150^\circ}. \tag{41}$$

Substituting these values in Eq. 39 and collecting coefficients give

$$\mathbf{I}_{b'a'} = \frac{\mathbf{E}_{ab}}{\mathbf{Z}' + 3\mathbf{Z}}. \tag{▶42}$$

Similarly,

$$\mathbf{I}_{c'b'} = \frac{\mathbf{E}_{ab}\underline{/-120^\circ}}{\mathbf{Z}' + 3\mathbf{Z}}, \tag{▶43}$$

$$\mathbf{I}_{a'c'} = \frac{\mathbf{E}_{ab}\underline{/+120^\circ}}{\mathbf{Z}' + 3\mathbf{Z}}. \tag{▶44}$$

Equations 42 to 44 show that the Δ-connected circuit, like the Y-connected circuit, can be solved as a single-phase problem provided the line impedance is considered to be $3Z$, instead of Z or $2Z$ as a casual inspection of the circuit might suggest.

Often Δ-connected circuits are solved in terms of the equivalent Y circuit. In this equivalent Y circuit, the line currents and line-to-line voltages are the same as in the Δ circuit. The Δ-connected generator is replaced by one that is Y connected, the two being indistinguishable by any tests made at their three terminals. Similarly, the Δ-connected set of load impedances is replaced by a Y-connected set which, when viewed from its three terminals at a given frequency, is indistinguishable from the Δ-connected set.

The equivalent impedance relationships* for use in converting a balanced Δ load to a balanced Y load and for the reverse operation are readily determined. In Fig. 25 the Δ-connected phase impedances are each Z_Δ and the Y-connected phase impedances are each Z_y. The impedance viewed from terminals b' and c' is $2Z_\Delta/3$ in the Δ system and $2Z_y$ in the Y system. Hence,

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_y, \tag{45}$$

$$\mathbf{Z}_y = \tfrac{1}{3}\mathbf{Z}_\Delta. \tag{45a}$$

* Article 11c, Ch. VIII; Art. 10, Ch. II.

Since the load is symmetrical, Eqs. 45 and 45a are also true when the networks are viewed from the terminals a'–c' or a'-b'.

As is shown subsequently, all connections in one three-phase system need not be either Y or Δ. In practice both are used. The choice in any given situation is based on practical exigencies or on effects that may be regarded as unimportant in this elementary discussion. In given practical situations, however, one may be much more desirable than the other.

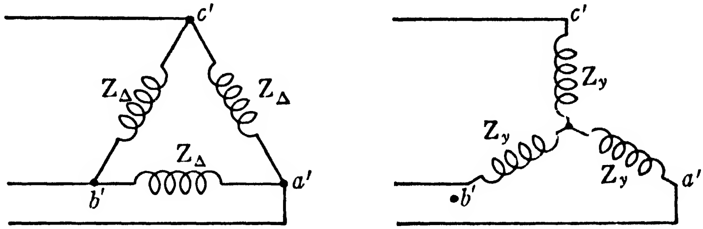


FIG. 25. Equivalent balanced Δ- and Y-connected impedances.

Problems involving Y- and Δ-connected loads in parallel can be solved by changing the Δ to its equivalent Y or vice versa, or by dealing with each load directly. Since the neutrals of two balanced Y-connected loads are at the same potential, the branches can be combined in parallel.

8. Unbalanced three-phase circuits with passive loads

When voltages, impedances, and currents in a three-phase circuit are known to be balanced, problems associated with this circuit can be solved by treatment of a single phase, because the knowledge of the voltage in phase a, together with the specification that the voltages are balanced, determines the voltages in the other two phases in both magnitude and angle. In other words, a balanced set of three-phase sinusoidal voltages or currents has only two degrees of freedom. If the magnitude and position of the vector representing the voltage in phase a are known, the magnitude and position of all three voltage vectors are thereby established.

If the line-to-neutral voltages are unbalanced, the three vectors are known only when the magnitude and angle of each are known. Thus six numbers are required to describe three independent vectors. In other words, a set of three independent plane vectors has six degrees of freedom. In general, three times as many equations have to be written to solve for unbalanced conditions as for balanced conditions, since all three phase currents and all three phase voltages must be included.

Two methods of solving three-phase circuits for unbalanced currents and voltages are in use. The first, presented here, is the analysis in terms

of actual phase currents and voltages. The elements of the second, known as the method of symmetrical components, are presented in Ch. XI.

A three-phase circuit having unbalanced loads or unbalanced source voltages is a special case of the multibranch network treated in Ch. VIII, and involves no new ideas. Figure 26 represents the general case of an unbalanced Y-connected circuit in which the source voltages are unbalanced and neither the line nor the load impedances are balanced. The circuit is a simple three-loop network to which three known voltages are applied.

Although the general procedure of solving for the currents in such a circuit is familiar, an example is given to illustrate the procedure. The

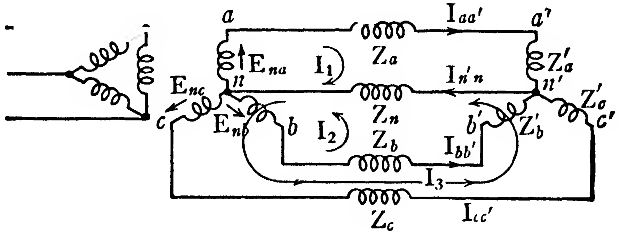


FIG. 26. Unbalanced Y-connected three-phase circuit.

source voltages E_{na}, E_{nb}, and E_{nc} are known unbalanced three-phase voltages supplied at the secondary terminals of a set of transformers* connected with secondaries in Y. The load impedances are unbalanced and are equal to Z'_a, Z'_b, and Z'_c ohms. The line wires have impedances Z_a, Z_b, and Z_c ohms, and the neutral impedance is Z_n ohms. A solution is required for the three currents I_1, I_2, and I_3. The load impedances Z'_b and Z'_c are reversed from the locations used in preceding diagrams in this chapter in order to make clearer the elements in the three loops selected for analysis. Also the loop currents are given numerical, rather than literal, subscripts in order to conform to the systematic method of solution presented in Ch. VIII.

The self- and mutual impedances of the three loops are:

$$Z_{11} = Z_a + Z'_a + Z_n, \qquad [46]$$

$$Z_{12} = Z_{21} = Z_n, \qquad [47]$$

$$Z_{22} = Z_b + Z'_b + Z_n, \qquad [48]$$

$$Z_{23} = Z_{32} = Z_n, \qquad [49]$$

$$Z_{33} = Z_c + Z'_c + Z_n, \qquad [50]$$

$$Z_{13} = Z_{31} = Z_n. \qquad [51]$$

* The equivalent internal source impedances are neglected; they can be assumed to be included in the line impedances.

The three equations for this unbalanced circuit corresponding to the one equation for the balanced circuit are

$$Z_{11}I_1 + Z_{12}I_2 + Z_{13}I_3 = E_{na}, \tag{52}$$

$$Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 = E_{nb}, \tag{53}$$

$$Z_{31}I_1 + Z_{32}I_2 + Z_{33}I_3 = E_{nc}. \tag{54}$$

These equations are readily solved either by ordinary simultaneous methods or by means of determinants to find the three loop currents I_1, I_2, and I_3. Each of these loop currents is a line conductor current, and the sum of the loop currents is the return current in the neutral conductor. Voltage drops across any circuit elements are readily found, once the currents I_1, I_2, and I_3 are known.

Another method of solution which should be borne in mind for possible application is the node method as described in Ch. VIII. In this method, the source neutral is usually called the reference node, and the load neutral is either the only other node or one of the other nodes whose voltages are to be determined.

On account of the special configuration of the Y connected three-phase circuit, an alternative method of solution, expressed by Eqs. 62 and 64 to 66, is widely used. In deriving these equations, three simultaneous equations are first written, each expressing the Kirchhoff voltage law for one of the loops, as follows:

loop $naa'n'n$: $\quad I_{aa'}(Z_a + Z_a') + (I_{aa'} + I_{bb'} + I_{cc'})Z_n = E_{na}, \qquad [55]$

loop $nbb'n'n$: $\quad I_{bb'}(Z_b + Z_b') + (I_{aa'} + I_{bb'} + I_{cc'})Z_n = E_{nb}, \qquad [56]$

loop $ncc'n'n$: $\quad I_{cc'}(Z_c + Z_c') + (I_{aa'} + I_{bb'} + I_{cc'})Z_n = E_{nc}. \qquad [57]$

If

$$Y_a = \frac{1}{Z_a + Z_a'}, \tag{58}$$

$$Y_b = \frac{1}{Z_b + Z_b'}, \tag{59}$$

$$Y_c = \frac{1}{Z_c + Z_c'}, \tag{60}$$

then multiplying Eq. 55 by Y_a, Eq. 56 by Y_b, Eq. 57 by Y_c, and adding the results give

$$\begin{aligned} I_{aa'} + I_{n'n}Y_aZ_n + I_{bb'} + I_{n'n}Y_bZ_n + I_{cc'} + I_{n'n}Y_cZ_n \\ = I_{n'n}[1 + Z_n(Y_a + Y_b + Y_c)] = E_{na}Y_a + E_{nb}Y_b + E_{nc}Y_c. \end{aligned} \tag{61}$$

Therefore,

$$\mathbf{I}_{n'n} = \frac{\mathbf{E}_{na}\mathbf{Y}_a + \mathbf{E}_{nb}\mathbf{Y}_b + \mathbf{E}_{nc}\mathbf{Y}_c}{1 + \mathbf{Z}_n(\mathbf{Y}_a + \mathbf{Y}_b + \mathbf{Y}_c)}. \quad ▶[62]$$

But from Eq. 55,

$$\mathbf{I}_{aa'} = \mathbf{E}_{na}\mathbf{Y}_a - \mathbf{I}_{n'n}\mathbf{Z}_n\mathbf{Y}_a; \quad [63]$$

hence

$$\mathbf{I}_{aa'} = (\mathbf{E}_{na} - \mathbf{I}_{n'n}\mathbf{Z}_n)\mathbf{Y}_a, \quad ▶[64]$$

$$\mathbf{I}_{bb'} = (\mathbf{E}_{nb} - \mathbf{I}_{n'n}\mathbf{Z}_n)\mathbf{Y}_b, \quad ▶[65]$$

$$\mathbf{I}_{cc'} = (\mathbf{E}_{nc} - \mathbf{I}_{n'n}\mathbf{Z}_n)\mathbf{Y}_c. \quad ▶[66]$$

Equations 64 to 66 may be solved directly for the three line currents by substituting the value of the neutral current $\mathbf{I}_{n'n}$ determined in Eq. 62. The values of the individual load phase voltage drops, if these are desired, are easily determined by calculating for each phase the product of its load impedance and its phase current:

$$\mathbf{V}_{a'n'} = \mathbf{I}_{aa'}\mathbf{Z}_a'; \quad [67]$$

$$\mathbf{V}_{b'n'} = \mathbf{I}_{bb'}\mathbf{Z}_b'; \quad [68]$$

$$\mathbf{V}_{c'n'} = \mathbf{I}_{cc'}\mathbf{Z}_c'. \quad [69]$$

The effect of phase order upon the currents and voltages in an unbalanced three-phase circuit should be understood. Equations 62 to 66 show that the neutral current and the three line currents are all functions of the individual source voltages and the unbalanced admittances or impedances of the circuit. Different phase order of the source voltages leads to different values of the product terms such as $\mathbf{E}_{na}\mathbf{Y}_a$ in Eq. 62, and in an unbalanced circuit the corresponding currents are, in general, different for the two different phase orders. Since the currents change as a function of phase order, so also do the circuit voltages, such as the neutral voltage $\mathbf{V}_{nn'}$, change with the phase order. An interesting application of this property of unbalanced circuits is the phase-rotation indicator, which uses the change in circuit voltages to furnish a visual indication of the phase order of a circuit.

Next the unbalanced Δ-connected circuit of Fig. 27 is considered. Three loop-voltage equations and one current equation for this circuit are easily written, but the arithmetical labor of solution in any given problem is possibly greater by this method than by an alternative method in which the Δ-connected load is changed to an equivalent *Y*-connected load. The relations between the Δ impedance and its equivalent *Y* impedance are

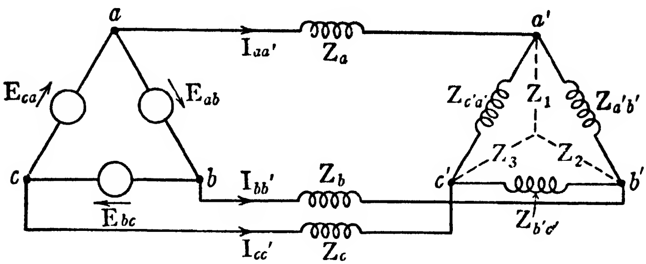


FIG. 27. Unbalanced Δ-connected three-phase circuit with equivalent Y connected load indicated by broken lines.

presented in Fig. 28 and Eqs. 70 to 75 for ready reference, the primes previously associated with load impedance nomenclature being omitted for the sake of simplicity.

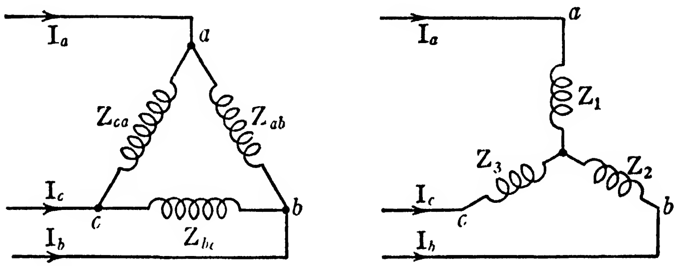


FIG. 28. Equivalent unbalanced Δ and Y-connected impedances.

Y to Δ:

$$Z_{ab} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}, \tag{70}$$

$$Z_{bc} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}, \tag{71}$$

$$Z_{ca} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}. \tag{72}$$

Δ *to Y*:

$$Z_1 = \frac{Z_{ab}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}}, \tag{73}$$

$$Z_2 = \frac{Z_{bc}Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}}, \tag{74}$$

$$Z_3 = \frac{Z_{ca}Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}}. \tag{75}$$

The transformation from Δ-connected to Y-connected impedances replaces the Δ-connected load of Fig. 27 with the Y-connected load shown by the dotted lines Z_1, Z_2, and Z_3. The circuit resulting from this change may be analyzed by means of the following two voltage equations and one current equation:

$$\mathbf{I}_{aa'}(\mathbf{Z}_a + \mathbf{Z}_1) - \mathbf{I}_{bb'}(\mathbf{Z}_b + \mathbf{Z}_2) = -\mathbf{E}_{ab}, \qquad [76]$$

$$\mathbf{I}_{bb'}(\mathbf{Z}_b + \mathbf{Z}_2) - \mathbf{I}_{cc'}(\mathbf{Z}_c + \mathbf{Z}_3) = -\mathbf{E}_{bc}, \qquad [77]$$

$$\mathbf{I}_{aa'} + \mathbf{I}_{bb'} + \mathbf{I}_{cc'} = 0. \qquad [78]$$

In connection with this problem, it should be carefully noted that Eq. 78 must be used rather than the third loop equation,

$$\mathbf{I}_{cc'}(\mathbf{Z}_c + \mathbf{Z}_3) - \mathbf{I}_{aa'}(\mathbf{Z}_a + \mathbf{Z}_1) = -\mathbf{E}_{ca}, \qquad [79]$$

since Eq. 79 is obtainable from Eq. 76 and Eq. 77 and therefore is not independent.

The discussion of this problem covers also the problem of the unbalanced Y-connected load without neutral.

As a result of tests made with a voltmeter, the line-to-line voltages of a three-phase system are often known in magnitude, but not in phase relation. These values are, however, sufficient to determine the phase relations if the phase order is known. At a given place in a three-phase system, the vector sum of the three line-to-line voltages must be zero; oth-rwise one conductor would operate at two different potentials, a physical absurdity. In geometrical terms, as shown in Fig. 29, the vectors representing the line voltages form a closed triangle, the angles of which are completely determined when the phase order and length of the sides are known. This relation is frequently of great value in solving three-phase circuit problems. The usual methods of plane geometry or trigonometry may be applied to the determination of these angles, or the following useful method involving inphase and quadrature components may be used.

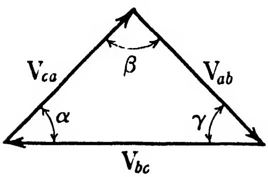


FIG. 29. Unbalanced voltage vectors in a three-phase Δ-connected circuit.

One of the three sides of a given line-to-line voltage triangle such as $\mathbf{V}_{bc}$ is called the reference axis. Then the sum of the components of the other two sides in phase with the reference vector equals the reference vector, and the sum of their quadrature components equals zero. Expressed in equations, these relations are as follows:

$$V_{ca} \cos \alpha + V_{ab} \cos \gamma = V_{bc} \qquad [80]$$

and

$$V_{ca} \sin \alpha - V_{ab} \sin \gamma = 0, \qquad [81]$$

in which

$$\alpha + \beta + \gamma = 180°. \qquad [82]$$

Since all the voltages in Eqs. 80 and 81 are known, the values of α, β, and γ are readily determined by simultaneous solution. Angles α, β, and γ are not the relative phase angles of the three vector voltages of Fig. 29. The actual phase angles are readily determined by inspection, as follows: phase angle of $\mathbf{V}_{bc}$ is 180 degrees; phase angle of $\mathbf{V}_{ca}$ is α; phase angle of $\mathbf{V}_{ab}$ is $-\gamma$.

If it is desired to determine the phase currents of an unbalanced Δ-connected load, they can be obtained by solving for the phase currents in the delta in terms of the line currents. From Fig. 27 the following equations may be written:

$$\mathbf{I}_{aa'} = \mathbf{I}_{a'b'} - \mathbf{I}_{c'a'}, \qquad [83]$$

$$\mathbf{I}_{bb'} = -\mathbf{I}_{a'b'} + \mathbf{I}_{b'c'}, \qquad [84]$$

$$0 = Z_{a'b'}\mathbf{I}_{a'b'} + Z_{b'c'}\mathbf{I}_{b'c'} + Z_{c'a'}\mathbf{I}_{c'a'}, \qquad [85]$$

which with Eq. 78 give for the desired phase currents the values

$$\mathbf{I}_{a'b'} = \frac{Z_{c'a'}\mathbf{I}_{aa'} - Z_{b'c'}\mathbf{I}_{bb'}}{Z_{a'b'} + Z_{b'c'} + Z_{c'a'}}, \qquad [86]$$

$$\mathbf{I}_{b'c'} = \frac{Z_{a'b'}\mathbf{I}_{bb'} - Z_{c'a'}\mathbf{I}_{cc'}}{Z_{a'b'} + Z_{b'c'} + Z_{c'a'}}, \qquad [87]$$

$$\mathbf{I}_{c'a'} = \frac{Z_{b'c'}\mathbf{I}_{cc'} - Z_{a'b'}\mathbf{I}_{aa'}}{Z_{a'b'} + Z_{b'c'} + Z_{c'a'}}. \qquad [88]$$

9. Effect of source impedance

In all the foregoing examples, the generator terminal voltages are assumed to be known and to be independent of the generator currents. This assumption is often true practically; yet, the source impedance must often be taken into account also. Then the solution of the problem is greatly complicated, because of the interaction of the currents in the different phases within the machine, which necessitates the use of self- and mutual reactances of the windings or the use of the method of symmetrical components as shown in this series in the volume on rotating electric machinery. If the three-phase source consists of three single-phase transformers whose internal impedances are known, a circuit problem may be readily solved by whichever one of the various methods pre-

sented here applies to the best advantage. Series impedances located in the phases of a Y-connected source present no special problem in circuit analysis, by either the loop or node method, provided these impedances can be independently determined. For series impedances in the phases of a Δ-connected source, one method is to set up three equations similar to Eqs. 95 to 97 for voltage drops around the loops. For this purpose, it is convenient to convert the circuit diagram of Fig. 30a to the electrically

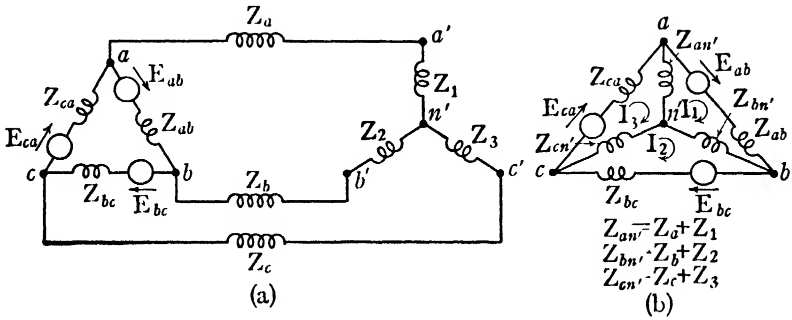


FIG. 30. Circuit diagrams of unbalanced three-phase circuit having Δ-connected generator with internal impedance.

identical diagram of Fig. 30b, whose geometric symmetry facilitates the setting up of the equations. These mathematically symmetrical equations are readily solved by the methods outlined in Ch. VIII. A Y-connected load is assumed. If the load is Δ connected, possibly a transformation to a Y connection is advisable before the solution is undertaken.

The self- and mutual impedances for the loops of Fig. 30b are

$$Z_{11} = Z_{ab} + Z_{bn'} + Z_{an'}, \qquad [89]$$

$$Z_{22} = Z_{bc} + Z_{cn'} + Z_{bn'}, \qquad [90]$$

$$Z_{33} = Z_{ca} + Z_{an'} + Z_{cn'}, \qquad [91]$$

$$Z_{12} = Z_{21} = -Z_{bn'}, \qquad [92]$$

$$Z_{23} = Z_{32} = -Z_{cn'}, \qquad [93]$$

$$Z_{31} = Z_{13} = -Z_{an'}; \qquad [94]$$

and the loop equations are

$$Z_{11}I_1 + Z_{12}I_2 + Z_{13}I_3 = E_{ab}, \qquad [95]$$

$$Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 = E_{bc}, \qquad [96]$$

$$Z_{31}I_1 + Z_{32}I_2 + Z_{33}I_3 = E_{ca}. \qquad [97]$$

An alternative method of taking into account the unbalanced impedances of a Δ-connected voltage source is to convert the source into an equivalent Y-connected source, and then solve the circuit problem as an unbalanced Y-connected circuit by the methods previously outlined. When the source impedances all have the same impedance angle, this may be a quicker method of procedure. The necessary relations for the conversion of a Δ-connected source are given in Eqs. 98 to 100, which are derived in the following manner. Figure 31 represents the unbalanced Δ-connected source whose voltage rises $\mathbf{E}_{ab}$, $\mathbf{E}_{bc}$, and $\mathbf{E}_{ca}$ are the induced voltages in the corresponding source phase windings; they are *not* the terminal voltages. Each source voltage with its associated series impedance is converted to an equivalent current source, as in Fig. 32. Next, the Δ-connected impedances of Fig. 32 are converted to equivalent Y-connected impedances, as in Fig. 33, using Eqs. 73 to 75. At the same time, each current source is broken into two parts as shown, so that the same source current leaves and approaches the terminals a, b, and c as before, while the current-source currents at the fork of the Y still total zero. Next, the resultant current sources connected across each Y impedance are combined into one current source, as in Fig. 34. The circuit of Fig. 34 can now be converted into equivalent voltage sources with series impedances, giving the configuration of Fig. 35, which is the desired equivalent unbalanced Y voltage source. The new voltages $\mathbf{E}_{na}$, $\mathbf{E}_{nb}$, and $\mathbf{E}_{nc}$ are induced voltage rises in each phase of the Y-connected source; they are *not* the voltages actually available at the generator terminals, except on open circuit. By final conversion from current sources to voltage sources and substitution of the proper values from Eqs. 73 to 75, the final relations are obtained:

$$\mathbf{E}_{na} = \mathbf{Z}_1\left(\frac{\mathbf{E}_{ca}}{\mathbf{Z}_{ca}} - \frac{\mathbf{E}_{ab}}{\mathbf{Z}_{ab}}\right) = \frac{\mathbf{E}_{ca}\mathbf{Z}_{ab} - \mathbf{E}_{ab}\mathbf{Z}_{ca}}{\mathbf{Z}_{ab} + \mathbf{Z}_{bc} + \mathbf{Z}_{ca}}, \qquad [98]$$

$$\mathbf{E}_{nb} = \mathbf{Z}_2\left(\frac{\mathbf{E}_{ab}}{\mathbf{Z}_{ab}} - \frac{\mathbf{E}_{bc}}{\mathbf{Z}_{bc}}\right) = \frac{\mathbf{E}_{ab}\mathbf{Z}_{bc} - \mathbf{E}_{bc}\mathbf{Z}_{ab}}{\mathbf{Z}_{ab} + \mathbf{Z}_{bc} + \mathbf{Z}_{ca}}, \qquad [99]$$

$$\mathbf{E}_{nc} = \mathbf{Z}_3\left(\frac{\mathbf{E}_{bc}}{\mathbf{Z}_{bc}} - \frac{\mathbf{E}_{ca}}{\mathbf{Z}_{ca}}\right) = \frac{\mathbf{E}_{bc}\mathbf{Z}_{ca} - \mathbf{E}_{ca}\mathbf{Z}_{bc}}{\mathbf{Z}_{ab} + \mathbf{Z}_{bc} + \mathbf{Z}_{ca}}. \qquad [100]$$

It should be noticed that, unlike the method used in all previous conversions of voltage sources to current sources, the one used in this solution employs impedances rather than admittances. There is no value here in finding admittances, because no node equations are used. Likewise, it should be pointed out that each of the equivalent Y source voltages

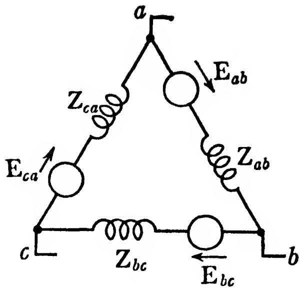


FIG. 31. Unbalanced Δ-connected generator with internal impedance.

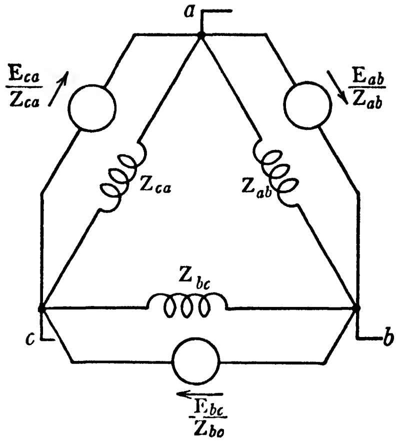


FIG. 32. Unbalanced Δ-connected current sources equivalent to voltage sources of Fig 31.

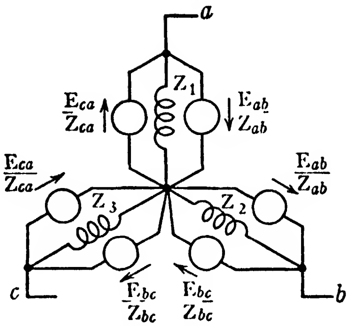


FIG. 33. Transformation of Δ-connected current sources to Y-connected current sources.

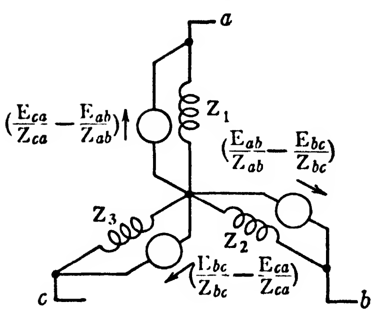


FIG 34 Union of parallel Y-connected current sources of Fig 33.

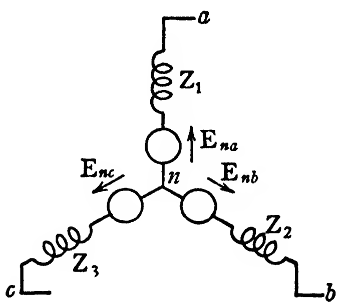


FIG. 35. Unbalanced Y-connected voltage source equivalent to Δ-connected source of Fig 31.

determined in Eqs. 98 to 100 may have added to it an arbitrary constant vector voltage without affecting the line-to-line voltages. Since these three arbitrary voltages form a zero-sequence symmetrical set which has no effect on the results, as is shown in Ch. XI, they may arbitrarily be called zero and omitted. In most practical cases, the Δ source voltages and impedances are balanced, under which conditions Eqs. 98 to 100 become

$$E_y = \frac{E_\Delta}{\sqrt{3}}, \qquad [101]$$

in which the phase angles are determined as in Eqs. 2d, 3, and 4, or Eqs. 5c, 6, and 7, and

$$Z_y = \frac{Z_\Delta}{3}. \qquad [45a]$$

At this point, an interesting generalization can be made. If Eqs. 12 to 14, 18 to 38, 42 to 44, 64 to 66, 70 to 75, and 86 to 100 are examined, it is seen that for *geometrically symmetrical* three-phase circuit diagrams, it is necessary only to determine one equation expressing a network relation from inspection of the diagram in order to obtain immediately the other two similar equations. When one equation has been obtained, substituting for its subscripts the next subscripts in cyclic order *abca* for positive sequence and *acba* for negative sequence gives the other two. For example, Eq. 87 is obtained from Eq. 86 by substituting $Z_{a'b'}$ for $Z_{c'a'}$, $\mathbf{I}_{cc'}$ for $\mathbf{I}_{bb'}$, and so on. This fact is also of value as a check on the accuracy of equations as set down.

10. Power in three-phase circuits and its measurement

The power relations in a three-phase circuit require special analysis. The vector power* absorbed by one phase, such as phase a of the Y-connected load of Fig. 36, is given by

$$P + jQ = \overline{\mathbf{V}}_{an}\mathbf{I}_{an} = \overline{\mathbf{Z}}\overline{\mathbf{I}}_{an}\mathbf{I}_{an} = \overline{\mathbf{Z}}I_y^2 = \overline{\mathbf{V}}_{an}\mathbf{V}_{an}\mathbf{Y} = \mathbf{Y}V_y^2, \qquad [102]$$

where V_y and I_y are the phase voltage and current, respectively, for the phase in question. The average power is the real part of the vector power, or

$$\left.\begin{aligned} P &= V_yI_y\cos \measuredangle_{\mathbf{V}_y}^{\mathbf{I}_y} \\ &= RI_y^2 = GV_y^2 \\ &= \mathcal{R}_e[\mathbf{V}_y]\mathcal{R}_e[\mathbf{I}_y] + \mathcal{I}[\mathbf{V}_y]\mathcal{I}[\mathbf{I}_y]. \end{aligned}\right\} \qquad [103]$$

* Article 21, Ch. IV.

This power is easily measured by any one of the three wattmeters W_a, W_b, or W_c connected as shown in Fig. 36, for each reads the power absorbed in one phase of the load. The total average power absorbed by

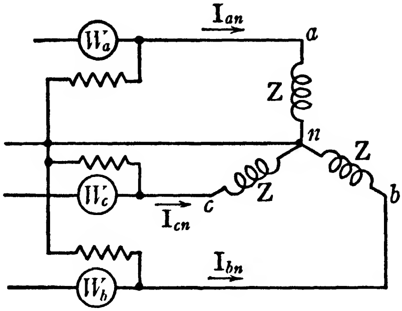


FIG. 36. Three-wattmeter connection for measurement of power delivered to Y-connected load with neutral.

the load is given by the sum of the three wattmeter readings, as in Eq. 104:

$$\left.\begin{aligned}\Sigma \text{ readings} = V_{an}I_{an} \cos \measuredangle_{\mathbf{V}_{an}}^{\mathbf{I}_{an}} + V_{bn}I_{bn} \cos \measuredangle_{\mathbf{V}_{bn}}^{\mathbf{I}_{bn}} \\ + V_{cn}I_{cn} \cos \measuredangle_{\mathbf{V}_{cn}}^{\mathbf{I}_{cn}}.\end{aligned}\right\} \quad [104]$$

If conditions are balanced, Eq. 104 becomes

$$P = 3V_yI_y \cos \measuredangle_{\mathbf{V}_y}^{\mathbf{I}_y}. \quad ▶[104a]$$

If now the common point of the three wattmeter potential circuits is removed from the load neutral as shown in Fig. 37 and instantaneous values of current and voltage are considered, the derivation which follows is made simpler and at the same time more general. If the currents in the wattmeter potential coils are neglected, the wattmeters read as follows:

wattmeter W_a reads the time average of $(v_{an} + v_{nn'})i_{an}$,
wattmeter W_b reads the time average of $(v_{bn} + v_{nn'})i_{bn}$,
wattmeter W_c reads the time average of $(v_{cn} + v_{nn'})i_{cn}$.

The sum of the three wattmeter readings is

$$\left.\begin{aligned}\Sigma \text{ readings} &= \text{time average of } v_{an}i_{an} + v_{bn}i_{bn} + v_{cn}i_{cn} \\ &\quad + v_{nn'}(i_{an} + i_{bn} + i_{cn}) \\ &= \text{time average of } v_{an}i_{an} + v_{bn}i_{bn} + v_{cn}i_{cn} \\ &= \text{average power delivered to load,}\end{aligned}\right\} \quad [105]$$

since by Kirchhoff's current law the sum of the currents is zero. Hence

the potential of n' is immaterial as far as the *sum* of the three readings is concerned. Of course the individual readings in general have no significance unless n' is connected to n. Nothing given in this argument requires that the wattmeters be similar in design. In other words, all the foregoing theory applies whether the individual instruments have different potential-circuit resistances or are used on different ranges.

If n' is connected to one line — for example, c — the sum of the three readings is still the average power, since n' can be at any potential, but wattmeter W_c reads zero and is therefore not needed. This scheme is called the *two-wattmeter method*, which is in general use for three-phase systems without neutrals.

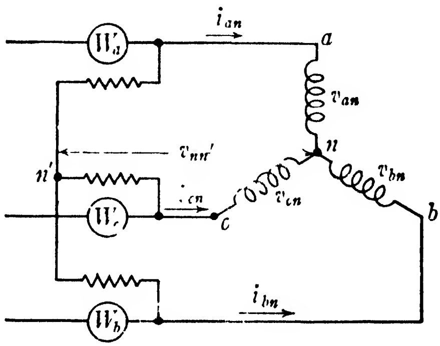


FIG. 37. Three wattmeter connection for measurement of power delivered to Y-connected load without neutral.

A caution should be mentioned here regarding the algebraic sign of wattmeter readings, especially important in the two-wattmeter method. Since a wattmeter is so constructed that it can distinguish the direction of energy flow but reads only in one direction from zero, this instrument is always connected in the circuit so that it reads upscale. The reading may, however, represent either a positive or negative value of power in a given direction. It is necessary to know which sign to attach to the reading before using it.

The following paragraphs show that, if the load power factor of a balanced three-phase load is less than 0.5, the reading of one wattmeter in the two-wattmeter scheme is negative. It is, therefore, important, in connecting wattmeters in a circuit, to know that power in a given direction causes an upscale deflection. This fact can be tested in any of several ways, the simplest of which is often to measure the power delivered to a small resistance load, and note the wattmeter connections. An alternative test consists of connecting the current coils in series and the potential circuits in parallel, and noting the corresponding terminals for upscale deflections. This test can be made on a suitable circuit of any power factor. If, after both wattmeters have been connected symmetrically in the circuit, the current-coil connections or the potential-coil reversing switch of one instrument must be reversed to obtain an upscale deflection, the wattmeter readings have opposite signs.

The individual readings obtained when the two-wattmeter method is used to measure the power input to a balanced Y-connected impedance

load without neutral connection to the source are now considered in more detail. This circuit arrangement is shown in Fig. 38a and the various vector voltages and currents in Fig. 39. If wattmeter W_a is so connected that it reads upscale when the Y load is removed and a simple resistance load R

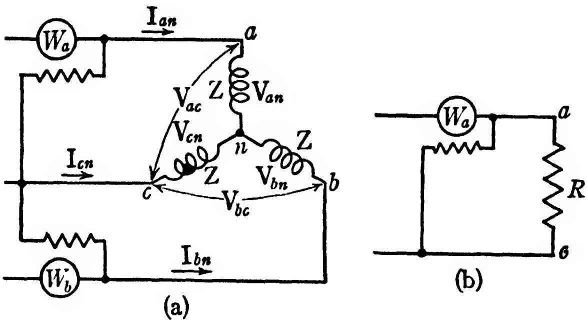


FIG. 38. (a) Two-wattmeter connection for measurement of power in a three-wire three-phase circuit; (b) Connections for testing polarity of wattmeters.

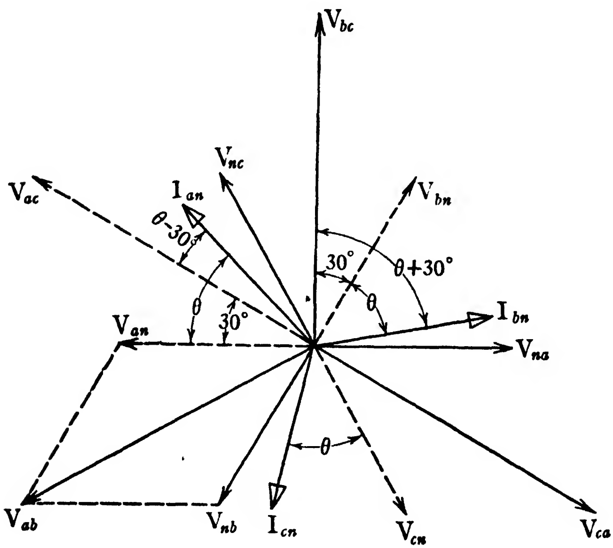


FIG. 39. Vector diagram for computation of power as measured by the two-wattmeter method in a three-phase circuit.

is connected between a and c as shown in Fig. 38b, an upscale reading means an energy flow from left to right. Such a reading is called positive. In the circuit of Fig. 38a wattmeter W_a reads the product of the current I_{an}, the voltage V_{ac}, and the cosine of the angle between their vectors (the

potential-coil current being assumed as negligible). This wattmeter reading is*

$$P_a = I_{an}V_{ac} \cos \measuredangle_{V_{ac}}^{I_{an}}. \quad [106]$$

From Fig. 39 it is seen that

$$P_a = I_{an}V_{ac} \cos (\theta - 30°), \quad [106a]$$

where θ is the angle of the phase impedance Z.

In a similar way it is seen that wattmeter W_b reads

$$P_b = I_{bn}V_{bc} \cos \measuredangle_{V_{bc}}^{I_{bn}}, \quad [107]$$

or

$$P_b = I_{bn}V_{bc} \cos (\theta + 30°). \quad [107a]$$

Equations 106a and 107a show that if θ is negatively greater than -60 degrees, wattmeter W_a reads negative, and if θ is positively greater than $+60$ degrees, wattmeter W_b reads negative.

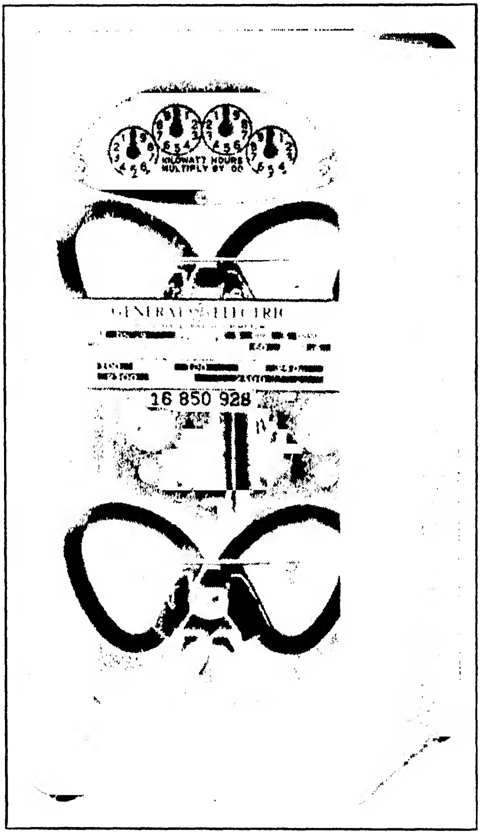
In interpreting the meaning of wattmeter readings, the use of vector diagrams of the type illustrated in Fig. 39 is very effective. In fact it is difficult to interpret power measurements, other than the sum of wattmeter readings, by any other method. The student should therefore develop facility in this use of three-phase vector diagrams.

Although this discussion of the two-wattmeter method indicates that two separate instruments are used, it is customary in practice to construct three-phase wattmeters in one unit, the two elements being mounted on a common shaft so that their total torque rotates the indicating needle to a scale reading representing total power. By far the most common application of this principle is the polyphase watthour meter, in which two rotating disks mounted on a common shaft furnish the driving torque to operate the energy-integrating mechanism.

In the foregoing discussion the use of two wattmeter elements in a three-wire, three-phase circuit has been demonstrated. This is but a special case of a more general method of power measurement that can be stated as follows: The average electric power transfer along any n conductors whatsoever in which the sum of the currents is zero can be measured with $n - 1$ wattmeter elements. The various current coils are connected in any $n - 1$ of the conductors. One terminal of each potential coil is connected to the same conductor as its corresponding current coil.

* Though the symbol P is used for wattmeter readings, it should be remembered that wattmeter readings frequently may not represent actual power.

The other ends of the potential coils are connected together and to the conductor in which there is no current coil. The proof is exactly the same as that preceding Eq 105, but contains n terms instead of three A simple example is the use of three wattmeters to measure the power in an



Courtesy General Electric Co

Polyphase watthour meter, which integrates the power of a three-wire system and shows total energy on its dials.

unbalanced three-phase Y-connected system having a neutral, as illustrated in part (e) of the illustrative example, Art 14

Thus far in this article, only Y-connected circuits have been considered. For Δ-connected circuits the usual method is the two-wattmeter method, since no neutral is available The two-wattmeter method merely gives the total power and indicates nothing about the individual phase powers unless the circuit is entirely balanced. If the circuit is balanced, of course,

the three individual phase powers are equal, and each is equal to one-third of the total power. An individual phase power can be measured by a wattmeter only if its current coil is put in series with the phase and its potential coil across the phase.

The $n - 1$ wattmeter method is perfectly general, regardless of the wave forms of voltages and currents. However, a word of caution concerning the Y-connected system with neutral is advisable at this juncture. Even with balanced loads, currents of third harmonic frequency, or multiples thereof, can exist in the neutral and make necessary the use of the three-wattmeter method for the correct measurement of total power. The subject of harmonics is discussed in subsequent volumes of this series.

11. Three-phase power factor and reactive power

An understanding of the limitations of the meaning of power factor and its measurement is important in the theory of three-phase circuits. The power factor of a two-terminal circuit, across which there is a voltage drop of effective value V and in which there is a current of effective value I, is defined by

$$\text{power factor} = \frac{P}{VI}, \tag{108}$$

in which P is the average power input to the circuit. If the voltage and current are sinusoidal and have the same frequency, the power factor is also equal to the cosine of the angular displacement between the voltage and current. The cosine definition of power factor has significance only for a two-terminal network and for a single frequency of current and voltage. As applied to three-phase circuits, the same definition of power factor has significance if the wave forms of voltages and currents are sinusoidal and conditions are completely balanced. In this case the power factor of one phase is said to be the power factor of the circuit. It is readily determined from the line currents and voltages or phase currents and voltages and the total power. Thus

$$P_a + P_b = 3V_y I_L \cos\theta = 3V_\Delta I_\Delta \cos\theta = \sqrt{3}V_\Delta I_L \cos\theta, \qquad \blacktriangleright[109]$$

where V_y is the voltage to neutral, V_Δ is the line-to-line voltage, I_L is the line current, and I_Δ is the delta current. One of these relations provides the usual means for determining the power factor $\cos\theta$.

Another method of determining the power factor involves the sum and differences of the two two-wattmeter readings P_a and P_b. By applying the trigonometric identities

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \tag{110}$$

to Eqs. 106a and 107a, and taking the sum and the difference of P_a and P_b one may easily show that, for balanced circuit conditions,

$$\tan\theta = \sqrt{3}\,\frac{P_a - P_b}{P_a + P_b}. \qquad [111]$$

It is to be emphasized that power factor has the usual physical significance in a three-phase, or any polyphase, circuit only when conditions are completely balanced and wave forms of current and voltage are sinusoidal.[2] However, the term power factor also has wide commercial application to unbalanced loads as found in practice. For such loads, power factor is defined as

$$\text{power factor} = \frac{\text{total active power}}{\sqrt{(\text{total active power})^2 + (\text{total reactive power})^2}}. \qquad [112]$$

This definition of power factor is important in that it establishes a means of measuring the quality of the load in any part of a power system. The reactive-power term represents an undesirable element in the operation of power circuits, as explained in Art. 20, Ch. IV; in contracts for the sale of power, therefore, power factor is often used as a basis for encouraging the purchasers of wholesale power to keep to a satisfactorily small fraction the ratio of their use of reactive power to their use of active power. The total active power in Eq. 112 is measured by any appropriate method; the total reactive power is measured by one of the following methods.

If wave forms in the circuit are known to be sinusoidal, the reactive power of each phase is the imaginary part of the vector power of that phase, or in Fig. 40 for phase a,

$$Q_a = V_{an}I_{an}\sin \measuredangle^{\mathbf{I}_{an}}_{\mathbf{V}_{an}}, \qquad [113]$$

and the total reactive power of the circuit is, for balanced conditions,

$$\left.\begin{aligned} Q = V_{an}I_{an}\sin \measuredangle^{\mathbf{I}_{an}}_{\mathbf{V}_{an}} &+ V_{bn}I_{bn}\sin \measuredangle^{\mathbf{I}_{bn}}_{\mathbf{V}_{bn}} \\ &+ V_{cn}I_{cn}\sin \measuredangle^{\mathbf{I}_{cn}}_{\mathbf{V}_{cn}} \end{aligned}\right\} \qquad [114]$$

$$Q = 3V_{an}I_{an}\sin \measuredangle^{\mathbf{I}_{an}}_{\mathbf{V}_{an}}. \qquad [114a]$$

[2] A. E. Knowlton, "Reactive Power Concepts in Need of Clarification," *A.I.E.E. Trans.*, LII (1933), 744–747; V. G. Smith, "Reactive and Fictitious Power," *id.*, 748–751; J. A. Johnson, "Operating Aspects of Reactive Power," *id.*, 752–757; C. L. Fortescue, "Power, Reactive Volt-Amperes, Power Factor," *id.*, 758–762; W. V. Lyon, "Reactive Power and Power Factor," *id.*, 763–770; W. H. Pratt, "Notes on the Measurement of Reactive Volt-Amperes," *id.*, 771–779; discussion of these articles, *id.*, 779–781.

For the connection indicated by Fig. 40a, it is seen from the vector diagram of Fig. 40b that the reading of the wattmeter is

$$P_a = V_{bc}I_{an} \cos \measuredangle^{I_{an}}_{V_{bc}} \tag{115}$$

$$= V_{bc}I_{an} \cos (90° - \theta) \tag{115a}$$

$$= \sqrt{3}V_{an}I_{an} \sin \theta = \sqrt{3}V_{an}I_{an} \sin \measuredangle^{I_{an}}_{V_{an}}. \tag{115b}$$

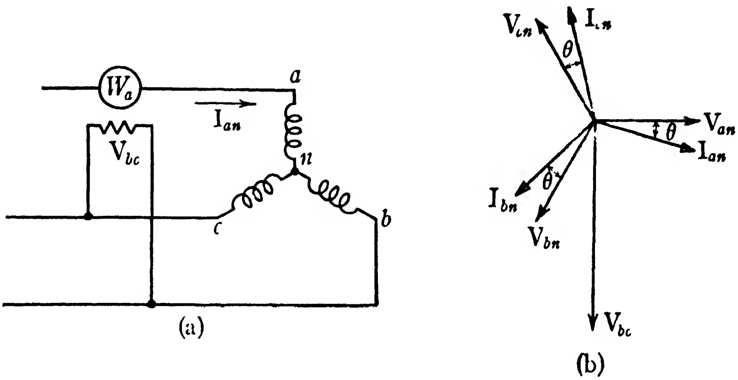


FIG. 40. Circuit connections and vector diagram for use of a single-phase wattmeter to indicate total reactive power in a balanced three-phase circuit.

Hence

$$Q = \sqrt{3}P_a. \tag{116}$$

Equation 116 therefore permits the statement:

▶In any balanced three-phase linear circuit whose source voltages are sinusoidal, the total reactive power is equal to $\sqrt{3}$ times the reading of a single-phase wattmeter whose current coil is connected in one line conductor and whose potential coil is connected across the other two line conductors.◀

If conditions are unbalanced and the circuit has a neutral, total reactive power may be measured by adding the readings of three reactive-power meters connected to measure the reactive power of each of the three phases. However, where the wave-forms of the circuit are sinusoidal and there is no neutral conductor, it is usual to measure reactive power by the use of two reactive-power meters or a two-element polyphase reactive-power meter connected like the two wattmeters of the two-wattmeter method shown in Fig. 41. Each of these instruments reads $VI \sin \measuredangle^{I}_{V}$, as mentioned in Art. 20, Ch. IV. By a procedure similar

to the development of Eq. 105, the total vector power can be shown to be

$$\overline{\mathbf{V}}_{an}\mathbf{I}_{an} + \overline{\mathbf{V}}_{bn}\mathbf{I}_{bn} + \overline{\mathbf{V}}_{cn}\mathbf{I}_{cn} = \overline{\mathbf{V}}_{ac}\mathbf{I}_{an} + \overline{\mathbf{V}}_{bc}\mathbf{I}_{bn}, \qquad [117]$$

or

$$\left.\begin{aligned} P + jQ = {} & V_{ac}I_{an} \cos \measuredangle^{\mathbf{I}_{an}}_{\mathbf{V}_{ac}} + V_{bc}I_{bc} \cos \measuredangle^{\mathbf{I}_{bc}}_{\mathbf{V}_{bc}} \\ & + j\left(V_{ac}I_{an} \sin \measuredangle^{\mathbf{I}_{an}}_{\mathbf{V}_{ac}} + V_{bc}I_{bn} \sin \measuredangle^{\mathbf{I}_{bn}}_{\mathbf{V}_{bc}}\right), \end{aligned}\right\} \qquad [117a]$$

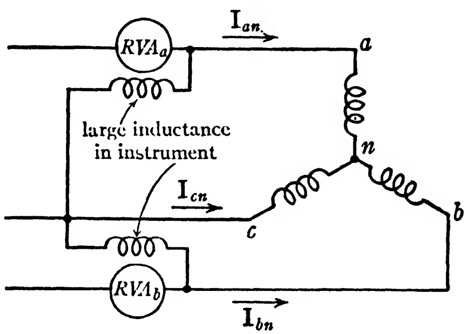


FIG. 41. Two reactive voltammeters connected to indicate reactive power in a three-phase three-wire circuit.

from which the total reactive power is

$$\left.\begin{aligned} Q & = V_{ac}I_{an} \sin \measuredangle^{\mathbf{I}_{an}}_{\mathbf{V}_{ac}} + V_{bc}I_{bn} \sin \measuredangle^{\mathbf{I}_{bn}}_{\mathbf{V}_{bc}} \\ & = Q_a + Q_b. \end{aligned}\right\} \qquad [118]$$

The two terms of the right-hand member of Eq. 118 represent the respective readings of the reactive-power meters of Fig. 41; their sum hence is the total reactive power of the load. For circuits having more than three wires, the $n - 1$ reactive-power meter method, analogous to the $n - 1$ wattmeter method, can be used.

12. Reasons for the use of three-phase systems

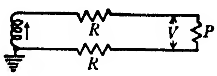
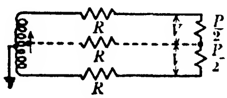
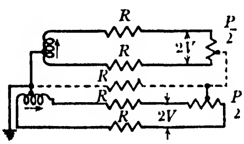
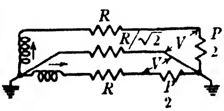
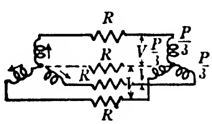
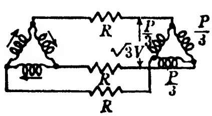
Since three-phase systems are so widely used, it is natural that the question be asked: " What advantages has the three-phase system which have led to its wide adoption by the central-station industry?" One line of investigation is to determine whether three-phase systems have any superiority from the standpoint of economy. In the table on page 560 a comparison is worked out among the various systems most commonly used, the results being very helpful in answering the question of relative

economy. This table is based upon the assumptions that a certain maximum voltage V above ground potential is permissible and that the load P is divisible into any number of equal elements, which latter assumption is valid for large total loads divided among many users. The column headed *Relative Power Loss* expresses the relative heat loss in the circuit conductors of a balanced system, referred to the two-wire single-phase system with grounded source neutral as a base, and the column headed *Relative Weight of Conductors* refers to the same base and uses for comparison the conductor material itself, uninsulated. The last column, headed *Relative Utilization*, is the most interesting one; it expresses an apparent relative economy, assigning equal value to losses and to conductor weight of all the systems, again referred to the two-wire single-phase system with grounded source neutral as a base. It is recommended that the student check the derived results in all columns as an exercise. The entire table is based on conditions of unity power factor, for such an assumption is valid for a study of relative performance among the systems.

The column headed *Relative Utilization*, while important as a rough comparison of the economic efficiencies of the various systems, should be used with discretion. In arriving at the ratios presented in this column, it is assumed that line losses and conductor weight have the same value in a cost comparison, an assumption that may be far from the truth. For example, the two three-phase three-wire systems both have the same relative utilization as the single-phase two-wire system with grounded source neutral. The three-phase line losses are 66.7 per cent of the losses in the single-phase system, however, so that if the cost of losses is more important in a particular case than are the fixed charges on the conductor installation, the three-phase system is definitely superior from the standpoint of cost. In another very important sense the three-phase systems are superior, because the lower line loss involved in delivering power to a certain load at a given voltage to ground means that, for a constant source voltage, the voltage at the load is subject to less fluctuation with the three-phase system than with the apparently equivalent single-phase system. This is a very strong argument in favor of the three-phase system, for the voltage available at a load should be as nearly uniform as possible to give satisfactory operation.

The comparative results in the column headed *Relative Utilization* indicate that economy cannot in general account for the advantages of three-phase systems over others, as any savings they may cause result from a particular, rather than general, comparison between the value of conductor loss and the value of fixed charges on conductor installation. Polyphase systems of all classes are used principally because they provide a means of transmitting energy in a form which is readily adaptable to

COMPARATIVE ECONOMY OF TRANSMISSION SYSTEMS

System	Diagram	I Line Current	Power Loss in Line	Relative Power Loss	Number of Conductors	Relative Weight of Conductors	Relative Utilization
1ϕ*		$\frac{P}{V}$	$\frac{2RP^2}{V^2}$	400%	2	100%	25%
1ϕ		$\frac{P}{2V}$	$\frac{RP^2}{2V^2}$	100%	2 3	100% 150%	100% 67%
2ϕ		$\frac{P}{4V}$	$\frac{RP^2}{4V^2}$	50%	4 5	200% 250%	100% 80%
2ϕ		line $\frac{P}{2V}$ neutral $\frac{P}{\sqrt{2}V}$	$0.85\,\frac{RP^2}{V^2}$	171%	3	171%	34%
3ϕ Y		$\frac{P}{3V}$	$\frac{RP^2}{3V^2}$	66.7%	3 4	150% 200%	100% 75%
3ϕ Δ		$\frac{P}{3V}$	$\frac{RP^2}{3V^2}$	66.7%	3	150%	100%

* Here ϕ means *phase*, a common usage.

the power requirements of electric motors. Polyphase motors have better starting torque characteristics and more uniform running torque than single-phase alternating -current motors.

The instantaneous power flow in a balanced polyphase circuit is constant, in contrast to the pulsating instantaneous power in a single-phase circuit. In a balanced three-phase system, if p_a, p_b, and p_c are the instantaneous powers in phases a, b, and c, respectively,

$$e_a = E_m \cos \omega t, \tag{119}$$

$$e_b = E_m \cos (\omega t + 120°), \tag{120}$$

$$e_c = E_m \cos (\omega t + 240°), \tag{121}$$

are the instantaneous Y-connected source voltages in phases a, b, and c, respectively, and

$$i_a = I_m \cos (\omega t + \theta), \tag{122}$$

$$i_b = I_m \cos (\omega t + \theta + 120°), \tag{123}$$

$$i_c = I_m \cos (\omega t + \theta + 240°) \tag{124}$$

are the instantaneous source currents in phases a, b, and c, respectively; then

$$p_a = e_a i_a = \frac{E_m I_m}{2} [\cos (2\omega t + \theta) + \cos \theta], \tag{125}$$

$$p_b = e_b i_b = \frac{E_m I_m}{2} [\cos (2\omega t + \theta + 240°) + \cos \theta], \tag{126}$$

$$p_c = e_c i_c = \frac{E_m I_m}{2} [\cos (2\omega t + \theta + 480°) + \cos \theta], \tag{127}$$

and

$$p_a + p_b + p_c = \tfrac{3}{2} E_m I_m \cos \theta = 3E_p I_p \cos \angle^{I_p}_{E_p} = p, \qquad \blacktriangleright[128]$$

since the sum of the first cosine terms in Eqs. 125 to 127 is zero. Equation 128 shows that the instantaneous total power in a balanced three-phase system is constant and hence equal to the average power, a fact which can also be shown for all symmetrical polyphase systems.

A polyphase motor, though not a passive load, is a balanced load. The constant flow of power implies a constant torque which gives polyphase motors a minimum of vibration. Polyphase motors are rugged in service, simple in construction, and operate in dust-laden atmospheres with less trouble and maintenance expense than do motors which, unlike most polyphase motors, have commutators. These reasons explain the use of

polyphase systems for operating electric motors, but they do not show why the three-phase system in particular is in such wide use. The experience of electrical machine designers shows that the symmetry of the three-phase system permits more efficient use of material and space in electric motors and generators than does any other polyphase system not requiring the complication and expense of more connections (and probably auxiliary transformers). In common with this view, electrical engineers in the public-utility field find certain advantages in the three-phase system for transmission and distribution. The three-phase system has the least number of conductors possible in a symmetrical polyphase system; hence the smallest possible number of insulating supports are required in open-wire lines, with the minimum resultant exposure to the elements, flash-overs, and mechanical damage. The same reasoning indicates an advantage of three-phase systems in underground cables, where the possibilities of insulation breakdown are in proportion to the number of conductors required.

13. Other polyphase systems

Certain special types of equipment when built in large sizes can show sufficient economy to justify having more than three phases. One example is the synchronous converter. Theoretically this operates most efficiently with a large number of phases, but the economic balance between energy cost and machine cost indicates that six phases are best. Another example is the steel-tank mercury-arc rectifier, which is built in commercial sizes in six-, twelve-, and eighteen-phase types. In the communications field, a twelve-phase supply to the vacuum-tube oscillators in radio transmitting stations is sometimes employed. All the systems mentioned here have numbers of phases which are even multiples of three, because by means of special transformer connections these systems can receive their energy from a standard three-phase source. A set of balanced six-phase voltages and currents with their phase angles is shown in Fig. 42. It is also possible to obtain two-phase or four-phase outputs from three-phase lines by use of the proper kind of transformer connections. All these polyphase systems can be analyzed *per phase* for balanced conditions, and for unbalanced conditions can be analyzed by methods similar to those used in three-phase analysis.

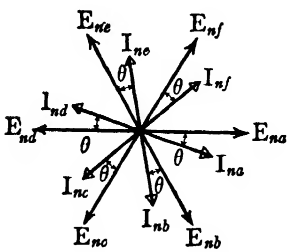


Fig. 42. Vector diagram of phase currents and voltages in a balanced six-phase system.

14. Illustrative example of three-phase power line

Figure 43 represents a four-wire, three-phase rural electric line operated by a public-utility company for distributing power from a substation at S along a state highway through a prosperous farming district to the point W. Most of the loads are single phase and are served through transformers whose connections to the supply circuit are arranged to result under normal conditions in a balanced load along the line from S to W. The substation is of capacity so large that, in calculating the characteristics of the distribution line, the substation may be considered a pure voltage source without internal impedance. At P are located sectionalizing cutouts which are hand-operated switches incorporating fuses that blow when accidental grounds or short circuits occur along the line between P

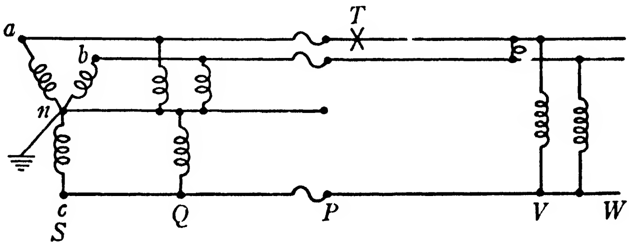


Fig. 43. Circuit diagram of three-phase rural power-distribution line.

and W. Typical transformer connections are indicated at Q and V. Actually many such transformers are distributed along the line, for serving the loads at secondary voltage. The section of line from S to P is quite old and operates under a Y system, all load transformer primary windings being rated 2,300 volts and connected from line to neutral. From P to W is a new extension operated as a Δ system without neutral, the transformers having primary rating of 4,000 volts. Under normal conditions the line-to-line voltage measured at P is 4,000 volts, and the load measured at P which results from the line PW is 150 kilowatts balanced, with a power factor of 90 per cent lagging. The distributed load between S and P may be represented by an equivalent Y-connected balanced load of 135 kilowatts, with a power factor of 85 per cent lagging, concentrated at P. The impedance Z of each line conductor from S to P is $5.10 + j2.60$ ohms, and the impedance Z_n of the neutral conductor is $12.0 + j3.00$ ohms.

The problem is divided into the following parts, each of which illustrates an important general category of three-phase circuit calculations:

(a) The determination of power input, power factor at the substation, and voltage drop in the line under balanced conditions.

(b) The calculation of load voltage with balanced source voltage and unbalanced load resulting from a break in the line. While load conditions are normal, the a-phase conductor is broken during a storm by a falling tree limb at a point T just beyond point P, and in falling makes contact with the b-phase conductor, blowing the a-phase fuse at P. After this momentary short circuit, the a-phase wire comes to rest on a stone wall, in a clear and ungrounded condition. It is desired to calculate the voltages from each line wire to neutral at the point P under these conditions if the voltage at the substation is maintained at the value found in (a).

(c) Discussion of the physical situation resulting from broken line conductor.

(d) The construction of a complete vector diagram of circuit conditions.

(e) The calculation of vector power and power factor at the source under unbalanced conditions. At the substation S, are a three-element wattmeter and a three-element reactive-power meter for indicating and integrating the load of the distribution line. The indications of the elements are to be calculated.

Solutions: (a) *Power and Voltage Drop, Balanced Conditions.* The equivalent circuit diagram is represented in Fig. 44. The load from P to W is represented by the Δ-connected impedances $Z_{a'b'}$, $Z_{b'c'}$, and $Z_{a'c'}$, and the concentrated load equivalent

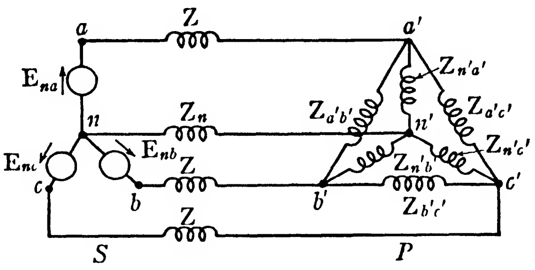


Fig. 44. Equivalent circuit of rural power-distribution line.

to the load from S to P is represented by the Y-connected impedances $Z_{n'a'}$, $Z_{n'b'}$, and $Z_{n'c'}$. The total load vector-power value may be determined by adding the Y- and Δ-connected vector powers. In Fig. 45, which represents load vector power,

$$\alpha = \cos^{-1} 0.90 = -26°, \quad [129]$$

$$\beta = \cos^{-1} 0.85 = -31.8°, \quad [130]$$

$$Q_\Delta = 150 \tan(-26°) = -73.0 \text{ kvar}, \quad [131]$$

$$Q_Y = 135 \tan(-31.8°) = -83.6 \text{ kvar}; \quad [132]$$

$$\text{total load } Q = -156.6 \text{ kvar}; \quad [133]$$

$$\text{total load kva} = \sqrt{P^2 + Q^2} = \sqrt{285^2 + 156.6^2} = 325 \text{ kva}; \quad [134]$$

$$\text{combined load power factor} = \frac{285}{325} = 0.878, \text{ lagging,} \qquad [135]$$

$$\theta = \cos^{-1} 0.878 = -28.6°. \qquad [136]$$

The load power of one phase is

$$I_{aa'}V_{a'n'} \cos\theta = \frac{285{,}000}{3} = I_{aa'}\left(\frac{4{,}000}{\sqrt{3}}\right)(0.878), \qquad [137]$$

whence

$$\mathbf{I}_{aa'} = 46.8\underline{/-28.6°} \text{ amp}, \qquad [138]$$

using $\mathbf{V}_{a'n'}$ as reference.

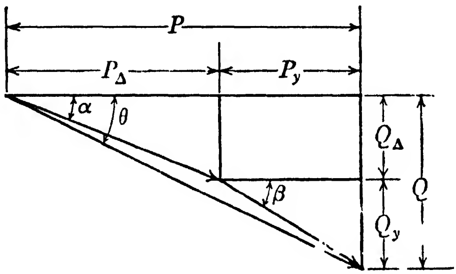


FIG. 45. Vector-power diagram of rural power-distribution line.

The substation voltage to neutral is

$$\left.\begin{aligned} \mathbf{E}_{na} &= \mathbf{I}_{aa'}\mathbf{Z} + \mathbf{V}_{a'n'} = (46.8\underline{/-28.6})(5.10 + j2.60) + 2{,}310 \\ &= (46.8\underline{/-28.6})(5.73\underline{/27°}) + 2{,}310 = 268\underline{/-1.6°} + 2{,}310 \\ &= 2{,}580\underline{/-0.2°} \text{ v.} \end{aligned}\right\} \qquad [139]$$

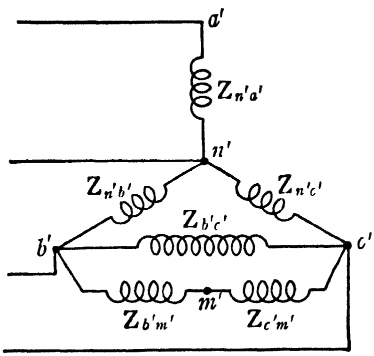


FIG. 46. Circuit diagram of load on power line beyond break in phase-a conductor.

Since $\mathbf{I}_{aa'}$ lags $\mathbf{E}_{na}$ by 28.4 degrees, the power factor at the substation is 0.880, lagging. The total substation power output P_s is

$$P_s = 3(2{,}580 \times 46.8 \times 0.880) = 318{,}000 \text{ w.} \qquad [140]$$

(b) *Load Voltage, Load Unbalanced.* The solution of this part of the example involves the application of some judgment. The impedances $Z_{n'c'}$ and $Z_{n'b'}$ represent the loads of customers along the line. When the fuse in phase a at P blows, it is evident that these impedances are now placed in series across the voltage $V_{b'c'}$, as shown at $b'm'c'$ of Fig. 46. Hence the customers' lights and appliances all receive voltages well below their rating. Some users may disconnect their loads, whereas others leave their loads connected. Some of the load is in motors and, since it is an active rather than a passive load, it changes its ap-

parent impedance. As a result of these factors the connected impedances $Z_{a'c'}$ and $Z_{a'b'}$ may change markedly. Calculations can, of course, be made for any reasonable assumption, but to illustrate the method only one such calculation is carried out here, based on the assumptions that half the load is disconnected by the users or by automatic protective devices, and that the remaining half retains its original impedance. Before the storm, under normal conditions,

$$I_\Delta = \frac{P_\Delta}{V_\Delta \cos\theta} = \frac{50{,}000}{4{,}000(0.90)} = 13.9 \text{ amp;} \qquad [141]$$

$$\mathbf{I}_\Delta = 13.9\underline{/-26^\circ} \text{ amp,} \qquad [141a]$$

$$\mathbf{Z}_\Delta = \frac{\mathbf{V}_\Delta}{\mathbf{I}_\Delta} = \frac{4{,}000}{13.9\underline{/-26^\circ}} = 288\underline{/26^\circ} \text{ ohms,} \qquad [142]$$

in which the Δ's indicate values for one phase of the Δ connection. With half of the load disconnected, the impedances representing the load are doubled.

$$\mathbf{Z}_{b'm'} = \mathbf{Z}_{c'm'} = 2(288\underline{/26^\circ}) = 576\underline{/26^\circ} \text{ ohms.} \qquad [143]$$

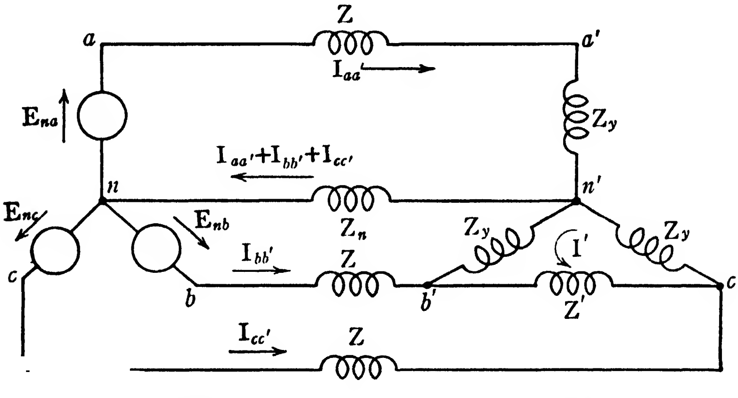


FIG. 47. Equivalent circuit of power line after break in phase-a conductor.

The combined circuit load is now represented by Fig. 47. The combined impedance $\mathbf{Z}'$ connected line to line between b' and c' is now

$$\mathbf{Z}' = \frac{(288\underline{/26^\circ})(1{,}152\underline{/26^\circ})}{288\underline{/26^\circ} + 1{,}152\underline{/26^\circ}} = 230\underline{/26^\circ} \text{ ohms.} \qquad [144]$$

The Y-connected impedances, which are not changed by the storm, are now determined, from the original data:

$$I_y = \frac{P_y}{V_y \cos\theta_y} = \frac{45{,}000}{2{,}310(0.85)} = 22.9 \text{ amp,} \qquad [145]$$

$$\mathbf{I}_y = 22.9\underline{/-31.8^\circ} \text{ amp,} \qquad [145a]$$

$$\mathbf{Z}_y = \frac{\mathbf{V}_y}{\mathbf{I}_y} = \frac{2{,}310}{22.9\underline{/-31.8^\circ}} = 101.0\underline{/31.8^\circ} = 86 + j53.3 \text{ ohms,} \qquad [146]$$

in which the subscript y's indicate values for one phase of the Y connection.

Applying Kirchhoff's voltage laws to the loops of Fig. 47 gives

$$I_{aa'}(Z + Z_y) + (I_{aa'} + I_{bb'} + I_{cc'})Z_n = E_{na}, \quad [147]$$

$$I_{bb'}Z + (I_{bb'} - I')Z_y + (I_{aa'} + I_{bb'} + I_{cc'})Z_n = E_{nb}, \quad [148]$$

$$I_{cc'}Z + (I_{cc'} + I')Z_y + (I_{aa'} + I_{bb'} + I_{cc'})Z_n = E_{nc}. \quad [149]$$

Equations 147 to 149 may now be added:

$$(I_{aa'} + I_{bb'} + I_{cc'})(Z + Z_y + 3Z_n) = E_{na} + E_{nb} + E_{nc} = 0. \quad [150]$$

Hence

$$I_{aa'} + I_{bb'} + I_{cc'} = 0. \quad [151]$$

Equation 151 shows that no current is in the neutral conductor; hence the points n' and n are at the same potential. Therefore it is possible to omit the neutral conductor from the circuit diagram. At the same time the load may be converted into the simple form of Fig. 48 by means of the Δ-Y transformations given in eqs. 73 to 75. The calculations follow:

$$\left.\begin{aligned} Z_{k'n'} &= \frac{(101\angle 31.8^\circ)^2}{2(101\angle 31.8^\circ) + 230\angle 26^\circ} = \frac{10{,}200\angle 63.6^\circ}{172 + j106.5 + 207 + j101} \\ &= \frac{(10{,}200\angle 63.6^\circ)}{379 + j207.5} = \frac{10{,}200\angle 63.6^\circ}{432\angle 28.7^\circ} \\ &= 23.6\angle 34.9^\circ = 19.3 + j13.5 \text{ ohms,} \end{aligned}\right\} \quad [152]$$

$$\left.\begin{aligned} Z_{k'b'} &= Z_{k'c'} = \frac{(101\angle 31.8^\circ)(230\angle 26^\circ)}{432\angle 28.7^\circ} = 53.8\angle 29.1^\circ \\ &= 47.0 + j26.1 \text{ ohms.} \end{aligned}\right\} \quad [153]$$

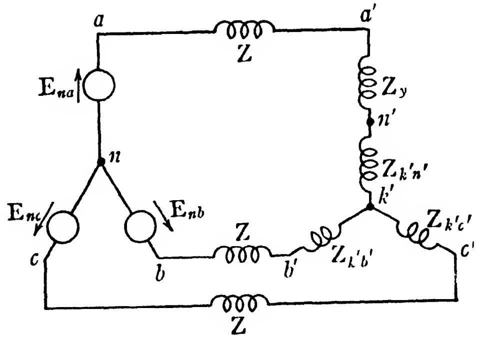


FIG. 48. Transformation of circuit of Fig 47 into three-wire unbalanced Y-connected system.

The resulting circuit (Fig. 48) is electrically unsymmetrical. However, the circuit may readily be solved by considering each line conductor in turn, as follows, using E_{na} as reference:

$$I_{aa'} = \frac{E_{na}}{Z + Z_y} = \frac{2{,}580}{5.1 + j2.6 + 86 + j53.3} = \frac{2{,}580}{107\angle 31.5^\circ} = 24.1\angle -31.5^\circ \text{ amp,} \quad [154]$$

$$\mathbf{V}_{k'n'} = -\mathbf{V}_{n'k'} = -\mathbf{I}_{aa'}(\mathbf{Z}_{k'n'}) = (24.1\underline{/148.5})(23.6\underline{/34.9}) = 569\underline{/183.4} \text{ v,} \qquad [155]$$

$$\left.\begin{aligned} \mathbf{I}_{bb'} &= \frac{\mathbf{E}_{nb} - \mathbf{V}_{k'n'}}{\mathbf{Z} + \mathbf{Z}_{k'b'}} = \frac{-1{,}290 - j2{,}240 + 569 + j34}{5.1 + j2.6 + 47.0 + j26.1} = \frac{-721 - j2{,}206}{52.1 + j28.7} \\ &= \frac{2{,}320\underline{/-108.1^\circ}}{59.6\underline{/28.8^\circ}} = 38.9\underline{/-136.9^\circ} = -28.4 - j26.6 \text{ amp,} \end{aligned}\right\} \qquad [156]$$

$$\left.\begin{aligned} \mathbf{I}_{cc'} &= \frac{\mathbf{E}_{nc} - \mathbf{V}_{k'n'}}{\mathbf{Z} + \mathbf{Z}_{k'c'}} = \frac{-1{,}290 + j2{,}240 + 569 + j34}{59.6\underline{/28.8^\circ}} \\ &= \frac{-721 + j2{,}274}{59.6\underline{/28.8^\circ}} = \frac{2{,}380\underline{/107.6^\circ}}{59.6\underline{/28.8^\circ}} = 39.9\underline{/78.8^\circ} = 7.75 + j39.2 \text{ amp.} \end{aligned}\right\} \qquad [157]$$

As a check on the numerical work, Eqs. 154, 156, and 157 are added, giving

$$\mathbf{I}_{aa'} = 20.5 - j12.6, \qquad [154]$$

$$\mathbf{I}_{bb'} = -28.4 - j26.6, \qquad [156]$$

$$\underline{\mathbf{I}_{cc'} = 7.75 + j39.2,} \qquad [157]$$

$$\Sigma \mathbf{I} = -0.15 + j0.00, \qquad [158]$$

a satisfactory check. By use of the line currents, the line-to-neutral voltages at the point P are now determined:

$$\mathbf{V}_{a'n'} = \mathbf{I}_{aa'}\mathbf{Z}_{a'n'} = (24.1\underline{/-31.5^\circ})(101\underline{/31.8^\circ}) = 2{,}440\underline{/0.3^\circ} \text{ v,} \qquad [159]$$

$$\left.\begin{aligned} \mathbf{V}_{b'n'} &= \mathbf{I}_{bb'}\mathbf{Z}_{b'k'} + \mathbf{V}_{k'n'} = (38.9\underline{/-136.9^\circ})(53.8\underline{/29.1^\circ}) + 569\underline{/183.4^\circ} \\ &= 2{,}090\underline{/-107.8^\circ} + 569\underline{/183.4^\circ} = -647 - j2{,}000 - 569 - j34 \\ &= -1{,}216 - j2{,}034 = 2{,}370\underline{/-120.9^\circ} \text{ v,} \end{aligned}\right\} \qquad [160]$$

$$\left.\begin{aligned} \mathbf{V}_{c'n'} &= \mathbf{I}_{cc'}\mathbf{Z}_{c'k'} + \mathbf{V}_{k'n'} = (39.9\underline{/78.8^\circ})(53.8\underline{/29.1^\circ}) + 569\underline{/183.4^\circ} \\ &= 2{,}140\underline{/107.9^\circ} + 569\underline{/183.4^\circ} = -657 + j2{,}050 - 569 - j34 \\ &= -1{,}226 + j2{,}016 = 2{,}360\underline{/121.3^\circ} \text{ v.} \end{aligned}\right\} \qquad [161]$$

(c) *Discussion of Physical Situation.* The line-to-line neutral voltages at P are shown vectorially in Fig. 49. The voltages at P remain almost symmetrical, except for a 5.6 per cent increase in magnitude of the voltage $V_{a'n'}$ from phase a to neutral. The voltage between conductors b and c becomes

$$\mathbf{V}_{b'c'} = \mathbf{V}_{b'n'} - \mathbf{V}_{c'n'} = -1{,}216 - j2{,}034 + 1{,}226 - j2{,}016 = -j4{,}050 \text{ v,} \qquad [162]$$

which is practically normal. In this particular situation, even though the line currents are unbalanced, no current is present in the neutral, whereas in general there is neutral current in a Y-connected system when conditions are unbalanced. It is of interest to determine the voltage $V_{m'n'}$ (Fig. 46), which is the approximate voltage to ground at the end of the broken conductor on the W side of the break at T.

$$\mathbf{V}_{m'n'} = \tfrac{1}{2}\mathbf{V}_{c'b'} + \mathbf{V}_{b'n'} = j2{,}025 - 1{,}216 - j2{,}034 = -1{,}216 - j9, \qquad [163]$$

$$V_{m'n'} = 1{,}216 \text{ v.} \qquad [163a]$$

The voltage which is present at this broken conductor is dangerous to life, even though it is on the side of the break which is away from the source of supply to phase a, which source is de-energized by the blowing of the fuse at P. Operating utility men call this

condition a *feedback* because the conductor beyond the break is energized by a feed through the transformers connected between it and the energized conductors.

The voltages from conductors b and c to the former c phase conductor (now called m') are readily calculated:

$$\mathbf{V}_{b'm'} = \mathbf{V}_{b'n'} - \mathbf{V}_{m'n'} = -1{,}216 - j2{,}034 + 1{,}216 + j9 = -j2{,}025 \text{ v.} \qquad [164]$$

$$\mathbf{V}_{c'm'} = \mathbf{V}_{c'n'} - \mathbf{V}_{m'n'} = -1{,}226 + j2{,}016 + 1{,}216 + j9 = j2{,}035 \text{ v.} \qquad [165]$$

Since the normal voltage between any two conductors between P and W is 4,000 volts, the customers served from transformers connected to the a-phase conductor receive

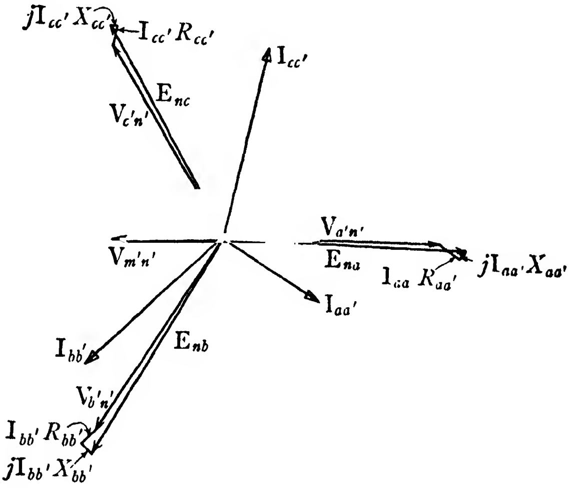


FIG. 49. Vector diagram of three phase rural distribution line operating with broken conductor in phase a

about 50 per cent of normal voltage, based on the assumptions made in (b) concerning the reduction of connected load.

(d) *Vector Diagram.* As a check on the precision of the calculations, a vector diagram indicating all the circuit conditions is plotted in Fig. 49. The voltage drops in the line conductors from S to P are first calculated:

IR drops:

$$\text{phase } a\text{: } \mathbf{I}_{aa'}R_{aa'} = (20.5 - j12.6)(5.1) = 104.6 - j64.2 \text{ v,} \qquad [166]$$

$$\text{phase } b\text{: } \mathbf{I}_{bb'}R_{bb'} = (-28.4 - j26.6)(5.1) = -145 - j136 \text{ v,} \qquad [167]$$

$$\text{phase } c\text{: } \mathbf{I}_{cc'}R_{cc'} = (7.75 + j39.2)(5.1) = 39.5 + j200 \text{ v;} \qquad [168]$$

IX drops:

$$\text{phase } a\text{: } \mathbf{I}_{aa'}jX_{aa'} = (20.5 - j12.6)(j2.6) = 32.8 + j53.4 \text{ v,} \qquad [169]$$

$$\text{phase } b\text{: } \mathbf{I}_{bb'}jX_{bb'} = (-28.4 - j26.6)(j2.6) = 69.1 - j73.9 \text{ v,} \qquad [170]$$

$$\text{phase } c\text{: } \mathbf{I}_{cc'}jX_{cc'} = (7.75 + j39.2)(j2.6) = -101.8 + j20.1 \text{ v.} \qquad [171]$$

The work can now be checked by calculation of the source voltages:

$$\mathbf{E}_{na} = \mathbf{V}_{a'n'} + \mathbf{I}_{aa'}\mathbf{Z} = 2{,}440 + j13 + (104.6 - j64.2) + (32.8 + j53.4)$$
$$= 2{,}577 + j2.2 = 2{,}577\underline{/0^\circ} \text{ v}, \qquad [172]$$

$$\mathbf{E}_{nb} = \mathbf{V}_{b'n'} + \mathbf{I}_{bb'}\mathbf{Z} = -1{,}216 - j2{,}034 + (-145 - j136) + (69.1 - j73.9)$$
$$= -1{,}292 - j2{,}244 = 2{,}585\underline{/-120.1^\circ} \text{ v}, \qquad [173]$$

$$\mathbf{E}_{nc} = \mathbf{V}_{c'n'} + \mathbf{I}_{cc'}\mathbf{Z} = -1{,}226 + j2{,}016 + (39.5 + j200) + (-101.8 + j20.1)$$
$$= -1{,}290 + j2{,}236 = 2{,}580\underline{/+120^\circ} \text{ v}. \qquad [174]$$

These values agree very well with the given source voltages.

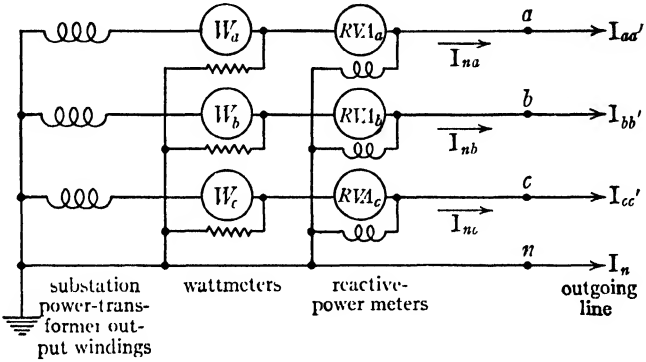


FIG. 50. Connections of substation instruments indicating active and reactive power delivered to three-phase rural distribution line.

(e) *Calculation of Source Power, Load Unbalanced.* The connections are shown in Fig. 50, except that the necessary instrument transformers are omitted to simplify the diagram.

Wattmeter readings:

$$P_a = E_{na}I_{na} \cos \measuredangle_{\mathbf{E}_{na}}^{\mathbf{I}_{na}}$$

$$= (2{,}580)(24.1) \cos(-31.5^\circ) = 53.1 \text{ kw}, \qquad [175]$$

$$P_b = E_{nb}I_{nb} \cos \measuredangle_{\mathbf{E}_{nb}}^{\mathbf{I}_{nb}}$$

$$= (2{,}580)(38.9) \cos(-16.9^\circ) = 96.3 \text{ kw}, \qquad [176]$$

$$P_c = E_{nc}I_{nc} \cos \measuredangle_{\mathbf{E}_{nc}}^{\mathbf{I}_{nc}}$$

$$= (2{,}580)(39.9) \cos(-41.2^\circ) = \underline{77.6 \text{ kw}}, \qquad [177]$$

$$\text{Total active power } P = 227.0 \text{ kw}. \qquad [178]$$

Reactive-power meter readings:

$$Q_a = E_{na}I_{na} \sin \measuredangle_{E_{na}}^{I_{na}}$$

$$= (2{,}580)(24.1) \sin(-31.5°) = -32.4 \text{ kvar}, \quad [179]$$

$$Q_b = E_{nb}I_{nb} \sin \measuredangle_{E_{nb}}^{I_{nb}}$$

$$= (2{,}580)(38.9) \sin(-16.9°) = -29.2 \text{ kvar}, \quad [180]$$

$$Q_c = E_{nc}I_{nc} \sin \measuredangle_{E_{nc}}^{I_{nc}}$$

$$= (2{,}580)(39.9) \sin(-41.2°) = -67.7 \text{ kvar}, \quad [181]$$

$$\text{Total reactive power } Q = -129.3 \text{ kvar}. \quad [182]$$

$$\text{Vector power} = P + jQ = 227 \text{ kw} - j129 \text{ kvar}. \quad [183]$$

$$\text{Power factor} = \frac{P}{\sqrt{P^2 + Q^2}} = \frac{227}{\sqrt{(227)^2 + (129)^2}} = \frac{227}{261} = 0.870. \quad [184]$$

The power factors of the individual phases, however are

$$\text{phase } a\text{:} \quad \frac{53.1}{\sqrt{(53.1)^2 + (32.4)^2}} = \frac{53.1}{62.3} = 0.853, \quad [185]$$

$$\text{phase } b\text{:} \quad \frac{96.3}{\sqrt{(96.3)^2 + (29.2)^2}} = \frac{96}{100.1} = 0.956, \quad [186]$$

$$\text{phase } c\text{:} \quad \frac{77.6}{\sqrt{(77.6)^2 + (67.7)^2}} = \frac{77.6}{103.2} = 0.752. \quad [187]$$

PROBLEMS

1. A three-wire single-phase secondary circuit delivers energy from a transformer to a group of residences having two-wire wiring as shown in Fig. 51. Each square represents a residence whose load is taken to be 500 w at 115 v and has unity power factor

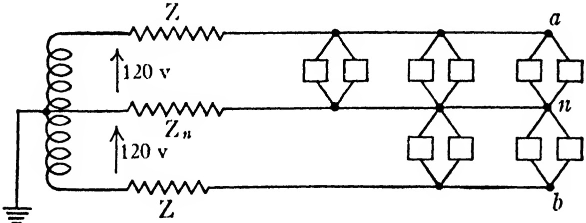


FIG. 51. Three-wire single-phase distribution for residences, Prob. 1.

The line impedances Z and Z_n are each $0.10 + j0.05$ ohm. What are the voltages V_{an} and V_{nb}? What changes should be made in the connections, and what are the voltages V_{an} and V_{nb} after the change?

2. Three equal impedance units, each of which has an equivalent resistance of 2 ohms and an inductive reactance of 1.25 ohms at source frequency, are Y connected to a balanced three-phase circuit whose line-to-line voltage is 220 v. What current does each unit take? What is the total power supplied to the impedances?

3. Three equal impedance units, each of which has an equivalent resistance of 2 ohms in series with a capacitive reactance of 1 ohm, are Y connected across a balanced three-phase circuit. The current in each impedance is 80 amp. Each line conductor between the source and the impedance units has an impedance of $0.5 + j0.10$ ohm. What is the magnitude of the line-to-line source voltage? A vector diagram is to be drawn illustrating the current and voltage of each circuit element.

4. A balanced three-phase Y-connected load takes 3,830 w at 0.80 power factor, lagging current. It is supplied through a three-phase line having an impedance of $0.80 + j0.60$ ohm/wire. The supply voltage is balanced, 230 v between wires.

(a) What is the line-to-neutral voltage at the load?
(b) What is the voltage between lines at the load?
(c) What is the line current?
(d) What is the efficiency of transmission?

5. A certain three-conductor cable can, for the purpose of calculating charging current, be represented approximately by the symmetrical network of capacitances

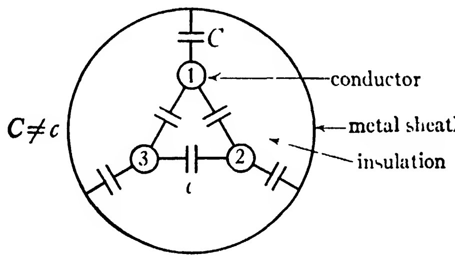


FIG. 52. Diagrammatic representation of three-conductor cable, Prob. 5.

of Fig. 52. This network does not take into account the resistance of the insulation nor the resistances and inductances of the conductors, which for this problem may be ignored. With all the conductors directly connected together, the capacitance measured from conductors to sheath is

$$C_{s(123)} = 0.201\ \mu f \qquad [188]$$

and, with the above connections removed and conductors 2 and 3 directly connected to the sheath, the capacitance measured from conductor 1 to the sheath is

$$C_{1(s23)} = 0.0738\ \mu f. \qquad [189]$$

With the above connections removed, sinusoidal potentials are impressed on the three conductors. The complex expressions representing the potential of each conductor with respect to the sheath are

$$V_{s1} = 133\underline{/0°}\ v, \qquad [190]$$

$$V_{s2} = 133\underline{/-120°}\ v, \qquad [191]$$

$$V_{s3} = 133\underline{/120°}\ v. \qquad [192]$$

The frequency is 60 ~. What is the charging current entering each conductor?

6. In the circuit of Fig. 53 the impedances Z_{na}, Z_{nb}, and Z_{nc} are each $100 + j173$ ohms. The impedance Z_{a3b} is $173 - j100$ ohms. The phase order is *abc*. What does a voltmeter of negligible admittance read if connected between (a) terminals 1–2, (b) terminals 2–3, (c) terminals 1–3? A vector diagram is to be drawn to scales of 1 in to 100 v and 1 in to 0.5 amp.

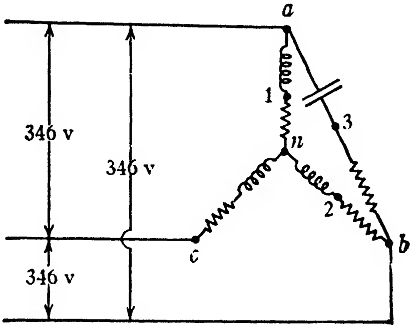


FIG. 53. Unbalanced load, Prob. 6.

7. Impedances are connected as shown by the designating subscripts from points *p* and *q* to the three-line conductors *a*, *b*, and *c* of a three-phase system having balanced line-to-line voltages of 240 v. The phase order is *abc*. The values of the impedances are

$$Z_{ap} = 8.66 + j5.00 \text{ ohms,} \qquad [193]$$

$$Z_{pc} = 0.00 - j10.0 \text{ ohms,} \qquad [194]$$

$$Z_{aq} = 8.66 - j5.00 \text{ ohms,} \qquad [195]$$

$$Z_{qb} = 0.00 + j10.0 \text{ ohms.} \qquad [196]$$

(a) What is the line current entering the terminal *a*?
(b) What is the voltage V_{pq} if V_{ab} is used as the axis of reference?

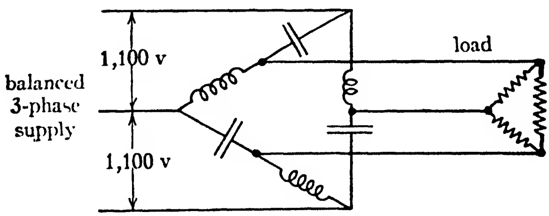


FIG. 54 Constant current three phase network, Prob. 8.

8. A constant-current three-phase network is made up as shown in Fig. 54. The inductive reactors have an impedance $0 + j100$ ohms and the capacitors have an impedance $0 - j100$ ohms.

What is the current in the balanced load resistors when they have the values (a) 10 ohms, (b) 100 ohms, (c) 1,000 ohms? A vector diagram is to be drawn for each case.

9. A three-phase power circuit using three 2 AWG annealed copper wires delivers energy to a balanced *Y*-connected load of 60 amp at 90% power factor located 500 ft

from the source. What is the efficiency of the circuit when the line-to-line load voltage is 220 v? What is the efficiency when the line-to-line load voltage is 550 v, the current remaining the same? The temperature of the line conductors is 20 C.

10. A balanced Y-connected load and a balanced Δ-connected load are connected in parallel at the end of a three-phase transmission line which has a resistance of 2.0 ohms and an inductive reactance of 1.0 ohm per conductor. The impedances of each phase of the Y- and Δ-connected loads are, respectively,

$$Z_Y = 25 + j15, \tag{197}$$

$$Z_\Delta = 45 + j15. \tag{198}$$

What line potential is necessary at the sending end of the line in order to maintain 2,300 v between lines at the load? A vector diagram is to be drawn.

11. What value of capacitive reactance should be connected in Y across the load terminals of Prob. 10 to correct the power factor to 100% at the load?

If the generator voltage remains as in Prob. 10 what is the load voltage after the addition of the condensers? A vector diagram is to be drawn.

12. In Prob. 10 what value of capacitive reactance connected in series in each line corrects the power factor to 100% at the generator? If the line voltage at the load is to be held at 2,300 v, what is the required generator voltage? A vector diagram is to be drawn.

13. A balanced three-phase load is connected to balanced three-phase voltages of 230 v between lines and is instrumented for power by the two-wattmeter method. The phase order is abc. The wattmeter in line a reads +2,000 w; the wattmeter in line b reads +1,000 w.

(a) What is the line current?
(b) What is the power factor of the load?
(c) What are the load impedances, assuming a Y connection? assuming a Δ connection?

14. Three impedances of $200 + j0$, $100 - j100$, and $100 + j100$ ohms are connected between lines ab, bc, and ca, respectively, of a three-phase three-wire system, the corresponding line-to-line voltages of which are 200, 141, and 141 v. The phase order is abc.

(a) What is the power input to each impedance?
(b) What is the total power?
(c) What are the individual wattmeter readings by the two-wattmeter method for each possible combination of proper instrument connections?
(d) A vector diagram is to be drawn showing each line voltage, line current, and phase current.

15. In Fig. 55, the load between a and c is 10 kw at unity power factor, the load between a and b is 10 kw at 0.80 power factor lagging, and the load between b and c is 15 kw at 0.70 power factor lagging. The phase order is abc. What does each wattmeter indicate?

16. In Fig. 56, load A represents a three-phase 10-hp motor operating at full load with an 80% power factor lagging, and at 90% efficiency; load B represents a single-phase 5-hp motor operating at full load with an 85% power factor lagging and at 83% efficiency. The phase order is abc.

(a) What does each wattmeter read?
(b) What line current is in each line wire?
(c) What is the total reactive power?

17. A balanced three-phase load takes 25 kw at 0.85 power factor from a balanced three-phase line. What does a wattmeter read if it is connected as shown in Fig. 57?

18. Balanced line voltages of 200 v and phase order *abc* are impressed on the load of Fig. 58.

(a) What are the magnitudes of the line currents?
(b) What is the wattmeter indication?
(c) What are the indications of two wattmeters connected in lines *a* and *b* so that the algebraic sum of the two indications gives the total power?

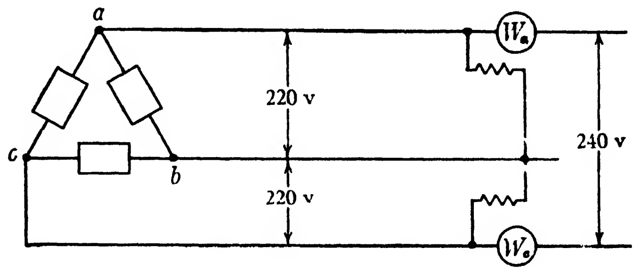


FIG. 55. Circuit for wattmeter reading computation, Prob. 15.

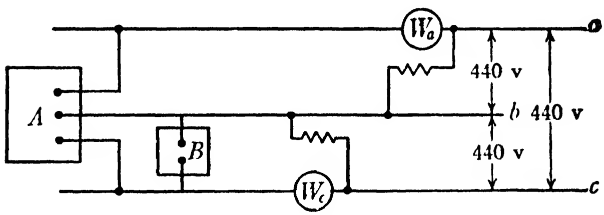


FIG. 56. Circuit for active and reactive-power computation, Prob. 16.

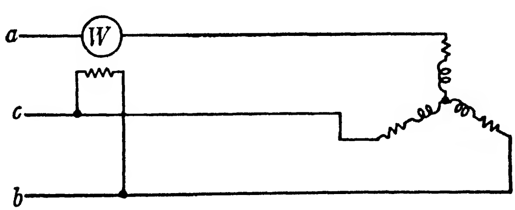


FIG. 57. Measurement of reactive power in balanced load, Prob. 17.

19. The following questions relate to Fig. 59:

(a) What are the readings of the two wattmeters?
(b) What are the readings of the two wattmeters if line *b* breaks at point *p*?
(c) What are the readings of the two wattmeters if line *b* is closed at *p* but breaks at point *p′*?
(d) In which, if either, of cases (b) or (c) does the sum of the readings of the two wattmeters equal the total power delivered to the load?

20. In Fig. 60 balanced line voltages of 100 v are impressed with phase order *abc*, and V_{ab} is $100 + j0$ v.

(a) What are the three line currents?
(b) What are the readings of the wattmeters W_a and W_b?
(c) What is the total power dissipated?

A vector diagram is to be drawn.

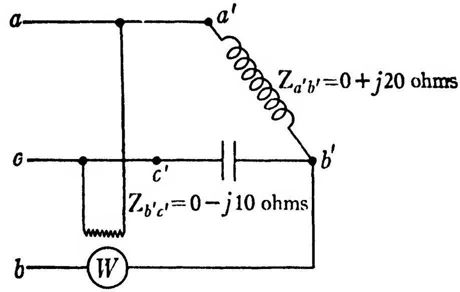


FIG. 58. Circuit for Prob. 18.

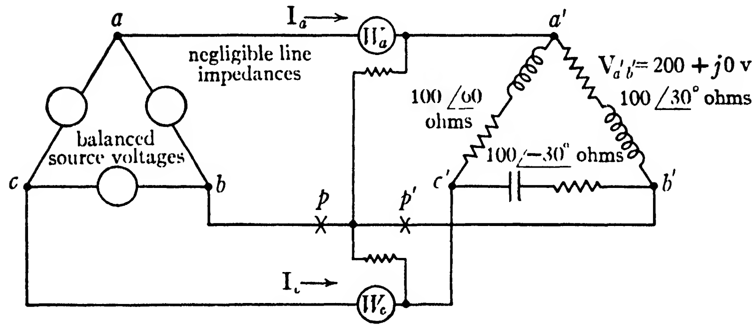


FIG. 59. Circuit for Prob. 19.

21. (a) Figure 61a represents a two-phase three-wire system, having resistance R' of 0.50 ohm per wire, used in a factory for delivering power to a bank of six resistors R, each of 45 ohms resistance, used in an electric furnace for drying enamel. Each source voltage is 240 v in magnitude. An automatic control system operates the contactors K in such a manner that a suitable temperature is maintained in the furnace. During a representative month, power is used 200 hr, at a steady rate. Power and energy are measured at the contactors K under the following monthly rate:

Demand charge: \$1.00/kw
Energy charge: 0.02/kwhr.

The meters can be read to the nearest 0.1 kw of demand and the nearest 10 kwhr of usage. What is the cost of electric service for one month?

(b) The system is to be changed to a balanced three-phase system by changing the source to a Δ-connected balanced source of the same voltage per phase, and by using the same load resistors, grouped with two resistors in parallel in each branch of the

Δ connection, as shown in Fig. 61b. How many hours a month does power flow, using the three-phase system? What is the cost of electric service for one month, using the three-phase system?

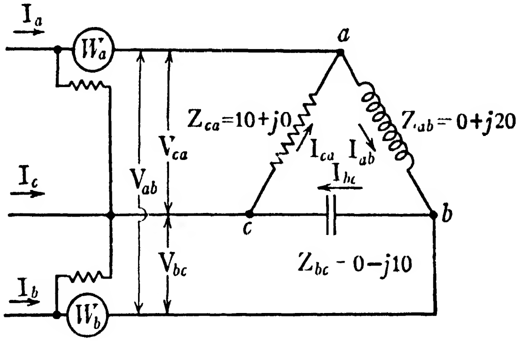
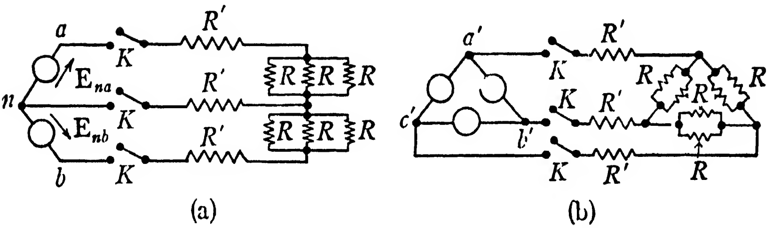


FIG. 60. Circuit for Prob. 20.



$$E_{na} = E_{nb} = E_{a'b'} = E_{b'c'} = E_{c'a'} = 240 \text{ v}$$
$$\mathbf{E}_{na} = j\mathbf{E}_{nb}$$

FIG. 61. Power supply for an electric furnace, Prob. 21.

22. The output of a three-phase, 2,300-v power plant is measured by a polyphase wattmeter with current coils in lines a and b. Three ammeters give the line currents. A single voltmeter is arranged so that it may be connected between either lines a and b, b and c, or c and a. During a test of this plant the circuit of the potential coil of the wattmeter which was connected to line b opened. Under this condition the polyphase wattmeter reads 260 kw. The ammeters in lines a, b, and c read, respectively, 125, 101, and 115 amp. The potentials between the lines a and b, b and c, and c and a were each 2,300 v. What was the output of the plant? Of the two possible answers the one corresponding to the higher power factor is desired.

CHAPTER XI

Elementary Theory of Symmetrical Components

1. Addition of three symmetrical sets of voltages having different phase orders

As indicated in Ch. X, certain problems arising from the operation of unsymmetrical polyphase systems are solved most readily by an analytical procedure known as the *method of symmetrical components*. This chapter presents the underlying theory of that method, which for the present may be considered simply an alternative means for analyzing the behavior of unbalanced polyphase systems. Actually, in many problems involving rotating machinery, transmission-line faults, and telephone interference caused by power transmission lines, the method of symmetrical components constitutes the only convenient, rigorous, analytical approach. Applications to such problems are brought out where needed in other volumes of this series.

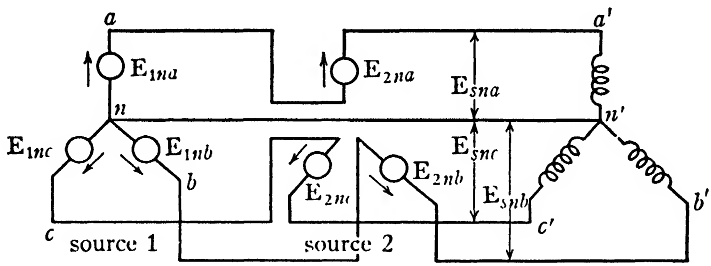


Fig. 1. Two symmetrical three-phase sources connected in series.

The ideas underlying the analysis of unbalanced three-phase circuits by the method of symmetrical components may be introduced by consideration of the effect of connecting two identical-frequency symmetrical three-phase sources in series, as shown in Fig. 1. In order to make such a connection, it is necessary to open the neutral point of one of the sources. The two sets of electromotive forces under consideration have different phase angles and phase-voltage magnitudes, and have positive- and negative-sequence phase orders, respectively. The vector diagrams of sources 1 and 2 are presented in Figs. 2a and 2b, respectively. These two symmetrical sets of source voltage are added vectorially in Fig. 2c, where it is clearly evident that the resulting voltages E_{sna}, E_{snb}, and E_{snc} compose a three-phase voltage source that is decidedly unsym-

metrical. If three zero-sequence voltages, E_{0na}, E_{0nb}, and E_{0nc}, are now added in series in phases a, b, and c, respectively, of the circuits of Fig. 1, as shown by the vector diagrams of Fig. 3, there results the unbalanced

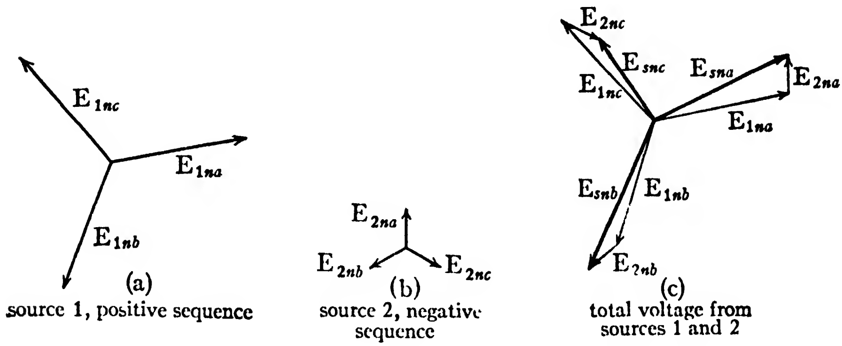


FIG. 2. Positive-sequence and negative-sequence symmetrical voltages added to produce an unsymmetrical three phase set.

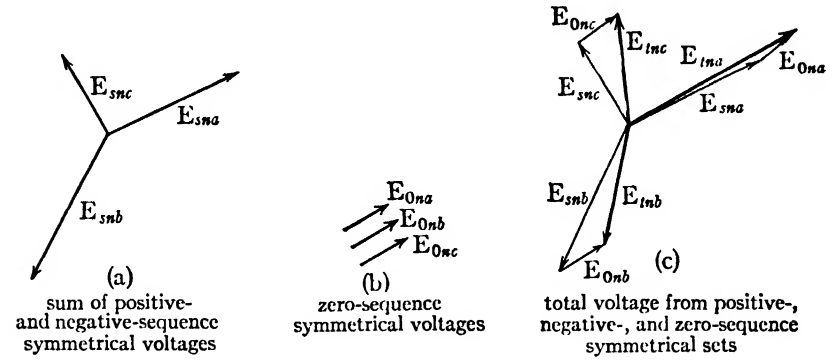


FIG. 3. Zero-sequence symmetrical set added to sum of positive- and negative-sequence symmetrical sets.

set of three voltages, E_{tna}, E_{tnb}, and E_{tnc} of Fig. 3c. The zero-sequence voltages could be represented, of course, by a third source connected in series in each phase of Fig. 1.

2. A METHOD OF VISUALIZING THE RESOLUTION OF THREE VECTORS INTO SYMMETRICAL COMPONENTS

Since three symmetrical sets of three plane vectors having, respectively, zero-, positive-, and negative-sequence phase orders can be added to form an unsymmetrical set of three vectors, it is natural to expect that the converse relation is also true. Actually any unsymmetrical set of

three plane vectors, such as those used in the analysis of three-phase systems, can always be resolved into symmetrical-component sets of vectors. In the most general case, three such sets — one zero-sequence, one positive-sequence, and one negative-sequence symmetrical-component set can be uniquely determined. As is shown later, the zero-sequence set often does not exist. Before the analytical method of determining the values of the three symmetrical-component sets is undertaken, consideration of a helpful method of visualizing the scheme of resolution is in order.

In Fig. 4a are shown three plane vectors V_a, V_b, and V_c which for comparison are made identical in value, respectively, to the vectors E_{tna}, E_{tnb}, and E_{tnc} of Fig. 3c. The letter V, as used in Fig. 4 and the following discussion, stands for vector rather than for voltage, so that the system can be analyzed in general terms applicable to any situation which three plane vectors can represent. In Art. 1, three symmetrical-component sets having respectively positive-, negative-, and zero-sequence phase order are added to give an unbalanced set of three vectors. If the individual vectors of the positive-sequence and negative-sequence sets are added, the resultant vector is zero, and only the zero-sequence vectors add up to a resultant. Hence to start the analysis, the resultant of any three unsymmetrical vectors is assumed to be the sum of the zero-sequence component vectors, and, likewise, if the sum of three vectors is zero, they are assumed to have no zero-sequence components. In Fig. 4a, the three vectors V_a, V_b, and V_c are added vectorially, the resultant vector V_{t0} representing the sum of the individual zero-sequence component vectors. If V_{t0} is divided into three equal parts, as shown, each part represents an individual zero-sequence component vector and is identical in value to the vectors of Fig. 3b. Thus from this analysis it is evident that

$$V_0 = \tfrac{1}{3}(V_a + V_b + V_c), \qquad \blacktriangleright[1]$$

in which V_0 is the value of each individual zero-sequence component vector.

The next step is to determine a method of visualizing the significance of the positive- and negative-sequence symmetrical-component sets. Assuming that such sets exist, the effect of rotation operations is considered. If V_b is rotated 120 degrees and V_c is rotated -120 degrees, the six individual vectors of their zero- and negative-sequence component sets add to zero, as shown in Figs. 4b and 4d. However, the individual vectors of the positive-sequence components of V_b and V_c are rotated into line with the positive-sequence component of V_a as shown in Fig. 4c, and the sum of these three positive-sequence components

becomes $3V_{1a}$. The result of this operation is then

$$3V_{1a} = V_a + V_b\angle 120^\circ + V_c\angle -120^\circ, \qquad [2]$$

in which V_{1a} is the value of the a vector of the positive-sequence symmetrical-component set. If V_b is rotated -120 degrees instead of $+120$

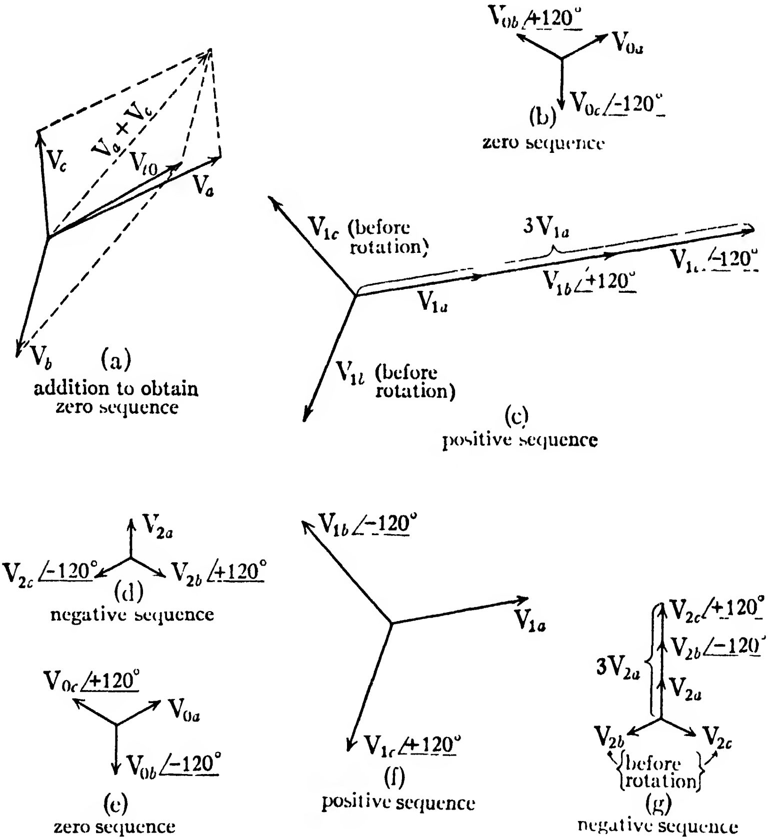


FIG. 4. A diagrammatic method of visualizing the meaning of symmetrical-component sets. In (a) three vectors V_a, V_b, V_c, are added to give V_{t0} equal to 3 times an individual zero-sequence component vector. In (b), (c), and (d) is shown the effect on the symmetrical-component sets of rotating V_b through $+120^\circ$ and V_c through -120°. In (e), (f), and (g) is shown the effect on the symmetrical-component sets of rotating V_b through -120° and V_c through $+120^\circ$.

degrees, and V_c is rotated $+120$ degrees instead of -120 degrees, the six individual vectors of their zero- and positive-sequence symmetrical-component sets add to zero, as shown in Figs. 4e and 4f. However, the

individual vectors of the negative-sequence components of V_b and V_c are rotated into line with the negative-sequence component of V_a as shown in Fig. 4g, and the sum of these three negative-sequence components becomes $3V_{2a}$. The result of this operation is then

$$3V_{2a} = V_a + V_b\underline{/-120°} + V_c\underline{/120°}, \qquad \blacktriangleright[3]$$

in which V_{2a} is the value of the a vector of the negative-sequence symmetrical-component set.

3. Analytical determination of symmetrical components

An analytical derivation of formulas corresponding to Eqs. 1 to 3 can now be presented. The vector V_a of Fig. 4a is resolved into three components designated V_{0a}, V_{1a}, and V_{2a}, and V_b and V_c are similarly resolved, giving the three relations

$$V_a = V_{0a} + V_{1a} + V_{2a}, \qquad [4]$$

$$V_b = V_{0b} + V_{1b} + V_{2b}, \qquad [5]$$

$$V_c = V_{0c} + V_{1c} + V_{2c}. \qquad [6]$$

As yet the nine components may be any vectors. However, the arbitrary restriction can be imposed that V_{1a}, V_{1b}, and V_{1c} shall form a symmetrical positive-sequence set.

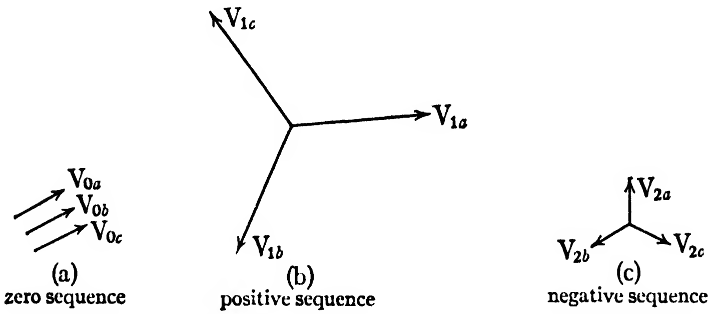


FIG. 5. Three symmetrical-component sets representing the vectors V_a, V_b, and V_c of Fig. 4a.

In other words, V_{1a}, V_{1b}, and V_{1c} must be equal in length, must be 120 degrees apart, and must have the phase order abc. The following relations thus apply:

$$V_{1b} = V_{1a}\underline{/-120°} \qquad [7]$$

and

$$V_{1c} = V_{1a}/120°, \qquad [8]$$

in which the magnitude and angle of V_{1a} are as yet unknown. Such a set is shown in Fig. 5b.

In a similar way V_{2a}, V_{2b}, and V_{2c} are arbitrarily required to form a symmetrical set, but to have a negative phase sequence, that is, to have the phase order *acb*. Thus the following restrictions are imposed:

$$V_{2b} = V_{2a}\underline{/120^\circ} \qquad [9]$$

and

$$V_{2c} = V_{2a}\underline{/-120^\circ}, \qquad [10]$$

in which the magnitude and angle of V_{2a} are as yet unknown. Such a set is shown in Fig. 5c.

The third set, V_{0a}, V_{0b}, and V_{0c}, is also arbitrarily made a symmetrical set, and of zero sequence; that is,

$$V_{0a} = V_{0b} = V_{0c}, \qquad [11]$$

in which a magnitude and angle of V_{0a} are as yet unknown. Such a set is shown in Fig. 5a.

While the foregoing restrictions seem quite arbitrary, they are justified if V_{0a}, V_{1a}, and V_{2a} can be uniquely determined from the original vectors V_a, V_b, and V_c, and if the resulting symmetrical vector sets are useful. The steps necessary to determine the uniqueness of the resolution expressed in Eqs. 7 to 11 are now carried out. If Eqs. 7 to 11 are substituted into Eqs. 4 to 6, these relations result:

$$V_a = V_{0a} + V_{1a} + V_{2a}, \qquad [4a]$$

$$V_b = V_{0a} + V_{1a}\underline{/-120^\circ} + V_{2a}\underline{/120^\circ}, \qquad [5a]$$

$$V_c = V_{0a} + V_{1a}\underline{/120^\circ} + V_{2a}\underline{/-120^\circ}. \qquad [6a]$$

Equations 4a to 6a are easily solved for V_{0a} by adding them, with the result, identical with Eq. 1:

$$V_{0a} = \tfrac{1}{3}(V_a + V_b + V_c). \qquad ▶[1]$$

If the terms of Eq. 5a are rotated +120 degrees and the terms of Eq. 6a are rotated −120 degrees, and these results are added to Eq. 4a, the solution for V_{1a} is obtained as

$$V_{1a} = \tfrac{1}{3}(V_a + V_b\underline{/120^\circ} + V_c\underline{/-120^\circ}). \qquad ▶[2]$$

If the terms of Eq. 5a are rotated −120 degrees and the terms of Eq. 6a are rotated +120 degrees, and these results are added to Eq. 4a, the solution for V_{2a} is obtained as

$$V_{2a} = \tfrac{1}{3}(V_a + V_b\underline{/-120^\circ} + V_c\underline{/120^\circ}). \qquad ▶[3]$$

For convenience, the vectors V_{0a}, V_{1a}, and V_{2a} are called the *key vectors* of the zero-, positive-, and negative-sequence symmetrical-component sets, respectively.

▶If V_a, V_b, and V_c have the phase order *abc*, the key vector of the positive-sequence set is equal to one-third the vector sum of V_a, V_b rotated $+120$ degrees, and V_c rotated -120 degrees; and the key vector of the negative-sequence set is equal to one-third the vector sum of V_a, V_b rotated -120 degrees, and V_c rotated $+120$ degrees.◀

The proof for the negative-sequence set is left for the student.

The three sets of components of the original vectors in Fig. 4a as determined in Eqs. 1, 2, and 3 are shown in Fig. 5. Equations 1 to 3 and 4a to 6a show that a unique solution of the set represented by Eqs. 4a to 6a is, in general, possible. That this is reasonable can be seen through consideration of the number of degrees of freedom in the original set of three vectors V_a, V_b, and V_c, and in the symmetrical-component sets. Six numbers are required to describe the original vectors uniquely — three magnitudes and three angles, or, alternatively, three real and three imaginary numbers. These original vectors are therefore said to have six degrees of freedom. A unique description of the three symmetrical-component sets also requires six numbers, two for each set. For a given set, one number describes the common length of the vectors; another number, the angular position of one vector with respect to a reference axis. Thus the three symmetrical systems together also have six degrees of freedom. A general property of physical systems is that the number of degrees of freedom must be the same in any mathematical description of the system. The symmetrical components are seen to have the required number of degrees of freedom.

4. Zero-Sequence Balanced Systems of Voltages and Currents

The resolution of unsymmetrical systems of three vectors into their three component sets is of great value in the analysis of certain types of three-phase electrical systems. Since, as is shown in Ch. X, systems of three vectors are used to represent steady-state line or phase currents, line or phase voltages, or impedances in three-phase systems, it is evident that unbalanced voltages, currents, and impedances may be represented by component sets which are symmetrical and, in some cases, easier to manipulate mathematically than are unsymmetrical sets of vectors.

Before the method of symmetrical components is applied to polyphase circuits, the properties of zero-sequence systems are demonstrated. Figure 6 shows a three-phase, Y-connected generator, connected by a three-phase transmission line with neutral conductor to a Y-connected

load having equal impedances Z' in the three phases. The impedances in the line wires are each Z, and in the neutral the impedance is Z_n. The vectors E_{0na}, E_{0nb}, and E_{0nc} are the phase electromotive forces of the

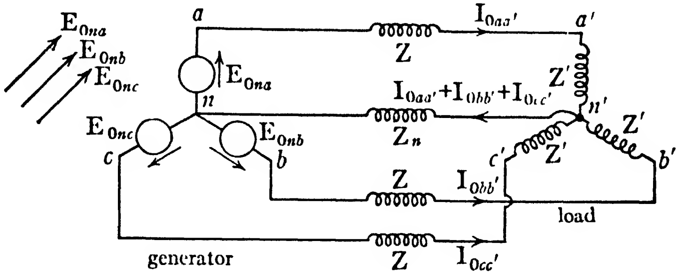


FIG. 6. Vector diagram and circuit diagram of zero-sequence components of voltage and current in a Y-connected three-phase system.

generator; $V_{0a'n'}$, $V_{0b'n'}$, and $V_{0c'n'}$, the phase voltage drops of the load; and $I_{0aa'}$, $I_{0bb'}$, and $I_{0cc'}$, the phase currents. Since from the definition of zero-sequence currents and voltages

$$I_{0aa'} = I_{0bb'} = I_{0cc'}, \qquad [11a]$$

$$E_{0na} = E_{0nb} = E_{0nc}, \qquad [11b]$$

and

$$V_{0a'n'} = V_{0b'n'} = V_{0c'n'}, \qquad [11c]$$

it is convenient for visualization to change the diagram of Fig. 6 to the form shown in Fig. 7, in which each source phase-voltage rise is called E_0

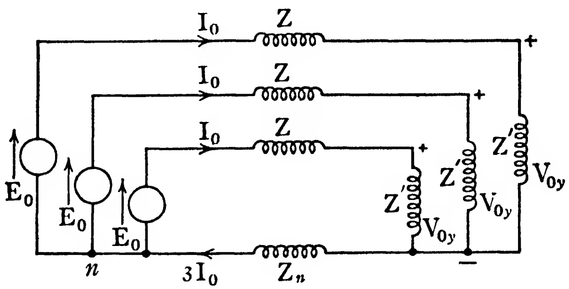


FIG. 7. Simplified circuit diagram showing zero-sequence components of voltage and current in a Y-connected three-phase system.

and each load phase-voltage drop is called V_{0y}. From this diagram, it is clear that, if the neutral is omitted, the circuit is open and there can be no current. The neutral conductor of a transmission system is definitely involved when zero-sequence currents are present.

The equation for any one phase of the Y-connected circuit for zero-sequence quantities is, from Fig. 7,

$$E_0 - I_0Z - I_0Z' - 3I_0Z_n = 0, \tag{12}$$

or

$$E_0 = I_0(Z + Z' + 3Z_n), \tag{12a}$$

from which

$$I_0 = \frac{E_0}{Z + Z' + 3Z_n}. \tag{12b}$$

If the generator and loads are connected in Δ, zero-sequence currents are not possible in the transmission lines since there can be no neutral for a return path and the sum of the line currents hence is zero. Zero-sequence currents exist, however, in the Δ connections themselves if the vector sum of the Δ currents is not zero.

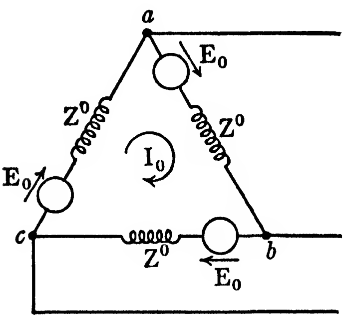


FIG. 8. Circuit diagram of zero-sequence components of voltage and current in Δ-connected generator windings.

For the Y connection, the line-to-line zero-sequence voltages at both generator and load are zero since the line-to-line voltages E_{ab}, E_{bc}, and E_{ca} are each the difference between two of the equal-phase voltages E_0. In a symmetrical Δ-connected generator, the terminal zero-sequence voltage is also zero even though zero-sequence induced voltages exist in the phases. In Fig. 8, E_0 is the zero-sequence electromotive force generated in each phase and the impedance to zero-sequence currents is Z^0 per phase* in the generator windings. The voltage equation around the Δ is

$$E_0 - I_0Z^0 + E_0 - I_0Z^0 + E_0 - I_0Z^0 = 0, \tag{13}$$

or

$$3E_0 - 3I_0Z^0 = 0, \tag{13a}$$

or

$$E_0 - I_0Z^0 = 0. \tag{13b}$$

The left-hand member of Eq. 13b is the zero-sequence terminal voltage of any phase of the generator, which is thus shown to be zero. The im-

* The impedances which machines offer to the various sequence components are derived in this series in the volume on rotating electric machinery.

pedance Z^0 to zero-sequence currents is usually very small, and currents given by

$$I_0 = \frac{E_0}{Z^0} \tag{13c}$$

are circulating in the Δ. For this reason the Δ connection is to be avoided whenever it is necessary to suppress zero-sequence currents in generator or transformer windings. It should be noticed that, although I_0 is indicated as an assumed loop current in Fig. 8, I_0 is also the actual zero-sequence component of current in each phase winding, because no zero-sequence current is present in the line conductors.

5. Circuits with Balanced Impedances and Unbalanced Applied Voltages

The solution of circuit problems involving *balanced impedances* to which *unbalanced voltages* are applied is easily handled by the method developed in Art. 3 for resolving an unbalanced set of vectors into three balanced sets. The general method is to resolve the voltages into their symmetrical

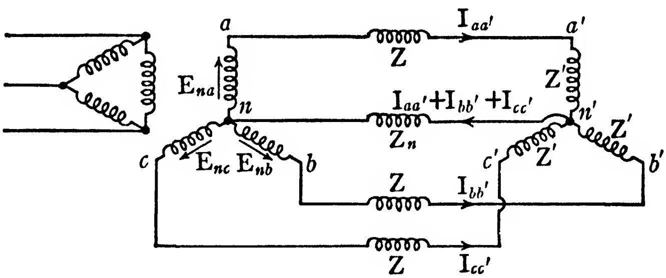


FIG. 9. Circuit diagram of a Y-connected three-phase system having unbalanced source voltages and balanced impedances. (In Art. 6, the impedances are treated as unbalanced.)

components; to find the corresponding balanced or symmetrical-component sets of currents by balanced-circuit methods; and then, if desirable, to add the components of current in each phase to obtain the actual phase currents. Often, however, the results are most useful in their symmetrical-component form. As an example of this method, the symmetrical components of current in the circuit of Fig. 9 are now determined.

Since all symmetrical-component quantities relate to the key vectors of phase a, it is unnecessary to retain the subscript showing the phase. Unless otherwise stated, the symmetrical-component quantities are understood to be for phase a from this point on.

The three unbalanced voltages E_{na}, E_{nb}, and E_{nc}, having the phase

order *abc*, are first resolved into their symmetrical components by application of Eqs. 1, 2, and 3, as follows:

$$\mathbf{E}_0 = \tfrac{1}{3}(\mathbf{E}_{na} + \mathbf{E}_{nb} + \mathbf{E}_{nc}), \tag{1a}$$

$$\mathbf{E}_1 = \tfrac{1}{3}(\mathbf{E}_{na} + \mathbf{E}_{nb}\underline{/120°} + \mathbf{E}_{nc}\underline{/-120°}), \tag{2a}$$

$$\mathbf{E}_2 = \tfrac{1}{3}(\mathbf{E}_{na} + \mathbf{E}_{nb}\underline{/-120°} + \mathbf{E}_{nc}\underline{/120°}). \tag{3a}$$

In each of these equations, the left-hand member refers to the *a*-phase symmetrical-component vector, or key vector, of the system. The zero-sequence line current is next determined. Equation 12b gives

$$\mathbf{I}_0 = \frac{\mathbf{E}_0}{\mathbf{Z} + \mathbf{Z}' + 3\mathbf{Z}_n}. \tag{12b}$$

Next, the positive-sequence current in line *a* is determined from Eq. 12, p. 532,

$$\mathbf{I}_1 = \frac{\mathbf{E}_1}{\mathbf{Z} + \mathbf{Z}'}, \tag{14}$$

by the usual procedure for balanced circuits. The negative-sequence current in line *a* is obtained similarly as

$$\mathbf{I}_2 = \frac{\mathbf{E}_2}{\mathbf{Z} + \mathbf{Z}'}. \tag{15}$$

Since each of the currents results from a balanced set of voltages, one phase only need be considered.

The zero-sequence currents in lines *b* and *c* are by definition just the same as that in line *a*. From Eqs. 7 and 8 the positive-sequence current in line *b* is seen to have the same magnitude as that in line *a* but lags it by 120 degrees, and the positive-sequence current in line *c* is seen to have the same magnitude as that in line *a* but leads it by 120 degrees. Similarly, from Eqs. 9 and 10 the negative-sequence currents in lines *b* and *c* are seen to have the same magnitude as that in line *a* but have the phase order opposite to that of the positive-sequence set.

This symmetrical-component solution of the circuit of Fig. 9 is distinguished from solution by the usual Kirchhoff equations in that the *three components* of current in *one* phase are determined from three independent equations involving this *one* phase, whereas in the usual method the *three actual phase currents* are determined by simultaneous solution of three independent equations involving the *three* phases.

It has already been stated that the components themselves are sometimes desired, but in many cases the total currents are required. The total currents are determined by addition of the components as follows:

$$\mathbf{I}_{aa'} = \mathbf{I}_0 + \mathbf{I}_1 + \mathbf{I}_2, \tag{16}$$

or

$$\mathbf{I}_{aa'} = \frac{\mathbf{E}_0}{\mathbf{Z} + \mathbf{Z}' + 3\mathbf{Z}_n} + \frac{\mathbf{E}_1}{\mathbf{Z} + \mathbf{Z}'} + \frac{\mathbf{E}_2}{\mathbf{Z} + \mathbf{Z}'}. \qquad [16a]$$

The other two currents are

$$\mathbf{I}_{bb'} = \mathbf{I}_0 + \mathbf{I}_1\underline{/-120^\circ} + \mathbf{I}_2\underline{/120^\circ}, \qquad [17]$$

$$\mathbf{I}_{cc'} = \mathbf{I}_0 + \mathbf{I}_1\underline{/120^\circ} + \mathbf{I}_2\underline{/-120^\circ}. \qquad [18]$$

Thus far there is no obvious advantage in using the symmetrical-component method, but it should be pointed out that, if the voltage source and the load are connected by a transmission line, the impedance in the denominator of Eq. 12b, called the impedance to zero-sequence currents, is not related to the impedances to positive- or negative-sequence currents $Z + Z'$ in Eqs. 14 and 15 in so simple a manner, because of the mutual effects between phases. Also, if the load contains rotating machines, these machines offer quite different impedances to currents of the three sequences. These impedances to the various symmetrical-component currents are readily calculated or found from test. The appreciation of these facts should make it evident that the component method, in usual power-system applications, is much simpler than the one involving complicated mutual effects. A few elementary examples of the usefulness of the component method in analyzing power-system problems are given in Art. 11.

6. Unbalanced Y-connected impedances with neutral

When the load impedances or the line impedances of a Y-connected system, or both, are unbalanced, there are two possible methods of solution, following the resolution of the applied voltages into symmetrical components.

In the first method the procedure is as follows: First, the currents resulting from the zero-sequence voltages are calculated. These currents are different in each phase because of the impedance unbalance and therefore have to be determined for each phase. They thus constitute an unsymmetrical set which is one of three sets of components of the actual line currents. A similar procedure using the positive-sequence applied voltages yields the second unsymmetrical set of currents. The third set results from the negative-sequence set of voltages. The sum of these three unsymmetrical sets of currents then gives the total line currents. This solution does not adhere to the principle of working entirely in *one* phase and this method probably never has any advantage over the methods described in Ch. X.

In the second and better method, the solution is confined to phase *a*

by the following scheme: In the circuit of Fig. 9 the three Y-connected unbalanced impedances $Z_{an'}$, $Z_{bn'}$, and $Z_{cn'}$ are the impedances of the series combination of the line and load impedances. The line-to-neutral voltage drops caused by the currents $I_{aa'}$, $I_{bb'}$, and $I_{cc'}$ are

$$V_{an'} = Z_{an'}I_{aa'}, \tag{19}$$

$$V_{bn'} = Z_{bn'}I_{bb'}, \tag{20}$$

$$V_{cn'} = Z_{cn'}I_{cc'}. \tag{21}$$

If all the voltages from generator terminals to load neutral and all the line currents are expressed in terms of symmetrical components in phase a, and if for simplicity the subscripts n' are dropped, the expressions for the symmetrical components become

$$\left.\begin{aligned} V_0 &= \tfrac{1}{3}(V_a + V_b + V_c) \\ &= \tfrac{1}{3}(Z_aI_0 + Z_aI_1 + Z_aI_2 + Z_bI_0 + Z_bI_1\underline{/-120^\circ} + Z_bI_2\underline{/120^\circ} \\ &\qquad + Z_cI_0 + Z_cI_1\underline{/120^\circ} + Z_cI_2\underline{/-120^\circ}), \\ V_0 &= \tfrac{1}{3}(Z_a + Z_b + Z_c)I_0 + \tfrac{1}{3}(Z_a + Z_b\underline{/-120^\circ} + Z_c\underline{/120^\circ})I_1 \\ &\qquad + \tfrac{1}{3}(Z_a + Z_b\underline{/120^\circ} + Z_c\underline{/-120^\circ})I_2; \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} V_1 &= \tfrac{1}{3}(V_a + V_b\underline{/120^\circ} + V_c\underline{/-120^\circ}) \\ &= \tfrac{1}{3}(Z_aI_0 + Z_aI_1 + Z_aI_2 + Z_bI_0\underline{/120^\circ} + Z_bI_1 + Z_bI_2\underline{/-120^\circ} \\ &\qquad + Z_cI_0\underline{/-120^\circ} + Z_cI_1 + Z_cI_2\underline{/120^\circ}), \\ V_1 &= \tfrac{1}{3}(Z_a + Z_b\underline{/120^\circ} + Z_c\underline{/-120^\circ})I_0 + \tfrac{1}{3}(Z_a + Z_b + Z_c)I_1 \\ &\qquad + \tfrac{1}{3}(Z_a + Z_b\underline{/-120^\circ} + Z_c\underline{/120^\circ})I_2; \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} V_2 &= \tfrac{1}{3}(V_a + V_b\underline{/-120^\circ} + V_c\underline{/120^\circ}) \\ &= \tfrac{1}{3}(Z_aI_0 + Z_aI_1 + Z_aI_2 + Z_bI_0\underline{/-120^\circ} + Z_bI_1\underline{/120^\circ} + Z_bI_2 \\ &\qquad + Z_cI_0\underline{/120^\circ} + Z_cI_1\underline{/-120^\circ} + Z_cI_2), \\ V_2 &= \tfrac{1}{3}(Z_a + Z_b\underline{/-120^\circ} + Z_c\underline{/120^\circ})I_0 + \tfrac{1}{3}(Z_a + Z_b\underline{/120^\circ} \\ &\qquad + Z_c\underline{/-120^\circ})I_1 + \tfrac{1}{3}(Z_a + Z_b + Z_c)I_2. \end{aligned}\right\} \tag{24}$$

In these equations, only three combinations of the impedances Z_a, Z_b, and Z_c appear as coefficients of the symmetrical components of current, and these have exactly the same form as Eqs. 1, 2, and 3. They are, therefore, designated as the *symmetrical components of unbalanced impedances* as follows:

$$Z_0 = \tfrac{1}{3}(Z_a + Z_b + Z_c), \tag{▶25}$$

$$Z_1 = \tfrac{1}{3}(Z_a + Z_b\underline{/120^\circ} + Z_c\underline{/-120^\circ}), \tag{▶26}$$

$$Z_2 = \tfrac{1}{3}(Z_a + Z_b/-120^\circ + Z_c/120^\circ). \tag{▶27}$$

These are the phase-a components. In terms of these symmetrical components of impedance, Eqs. 22, 23, and 24 can be written

$$V_0 = Z_0I_0 + Z_2I_1 + Z_1I_2, \qquad \blacktriangleright[22a]$$

$$V_1 = Z_1I_0 + Z_0I_1 + Z_2I_2, \qquad \blacktriangleright[23a]$$

$$V_2 = Z_2I_0 + Z_1I_1 + Z_0I_2. \qquad \blacktriangleright[24a]$$

These equations make evident the important facts that in portions of a three-phase circuit where the impedances are balanced, zero-sequence voltage drops are associated with zero-sequence currents only; positive-sequence voltage drops, with positive-sequence currents only; and negative-sequence voltage drops, with negative-sequence currents only.

The distinction between these symmetrical components of unbalanced impedances and balanced impedances to currents of the different sequences is important. Also it should be noted that balanced currents and voltages of a single frequency are of positive sequence only or of negative sequence only, whereas balanced impedances are of zero sequence. A three-phase induction motor while running at normal speed offers certain balanced impedances to negative-sequence currents.* These might be called zero-sequence or balanced impedances to negative-sequence currents. In the literature dealing with the symmetrical-component method, the names "positive-sequence impedance," "negative-sequence impedance," and "zero-sequence impedance" are used for both concepts. The conditions of any particular problem reveal whether these names denote components of unbalanced impedances or balanced impedances to currents of the various sequences.

For a specific problem, the circuit of Fig. 9 is modified so that the line impedances are balanced and each is equal to Z, but the load impedances are unbalanced and equal to Z_a, Z_b, and Z_c, respectively, in the three phases. These unbalanced load impedances are first resolved into symmetrical components Z_0, Z_1, and Z_2 by equations 25, 26, and 27. The voltages of Eqs. 22a, 23a, and 24a are then seen to be the components of voltage on phase a of the load. From the circuit of Fig. 9 the following equations are written:

$$E_0 - (Z + 3Z_n)I_0 = V_{0a'n'} = Z_0I_0 + Z_2I_1 + Z_1I_2, \qquad [28]$$

$$E_1 - ZI_1 = V_{1a'n'} = Z_1I_0 + Z_0I_1 + Z_2I_2, \qquad [29]$$

$$E_2 - ZI_2 = V_{2a'n'} = Z_2I_0 + Z_1I_1 + Z_0I_2. \qquad [30]$$

* This is demonstrated in this series in the volume on rotating electric machinery.

Equations 28, 29, and 30 can be rearranged into the more convenient form

$$(Z + 3Z_n + Z_0)I_0 + \qquad Z_2I_1 + \qquad Z_1I_2 = E_0, \qquad [28a]$$

$$Z_1I_0 + (Z + Z_0)I_1 + \qquad Z_2I_2 = E_1, \qquad [29a]$$

$$Z_2I_0 + \qquad Z_1I_1 + (Z + Z_0)I_2 = E_2, \qquad [30a]$$

which can be solved for I_0, I_1, and I_2.

7. Unbalanced Y-connected impedances without neutral

When unbalanced voltages are applied to an unbalanced Y-connected load without neutral, there can be no zero-sequence current, since the sum of the three line currents is zero. The equations in terms of phase-a quantities are:

$$E_1 - I_1Z = V_{1y} = Z_0I_1 + Z_2I_2, \qquad [31]$$

$$E_2 - I_2Z = V_{2y} = Z_1I_1 + Z_0I_2. \qquad [32]$$

After the two components of current have been obtained from Eqs. 31 and 32, the zero-sequence voltage drop V_{0y} at the load can be found by substitution in Eq. 22a, which gives

$$V_{0y} = Z_2I_1 + Z_1I_2. \qquad [33]$$

An interesting and useful relation can be developed by connecting an unbalanced Y-connected load in parallel with a balanced Y-connected

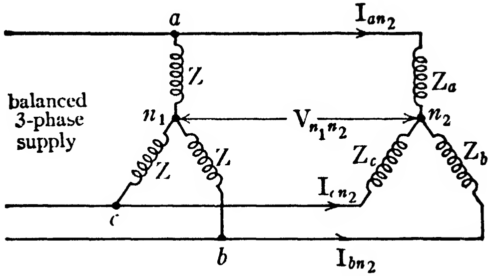


FIG. 10. Balanced load in parallel with unbalanced load, both connected to a balanced supply line.

load, both supplied from a symmetrical three-phase system. Such a situation is represented in Fig. 10. Since the impedances Z are equal, and the supply voltages are balanced,

$$V_{an_1} = V, \qquad [34]$$

$$V_{bn_1} = V\angle{-120°}, \qquad [35]$$

and

$$V_{cn_1} = V\angle{+120°}, \qquad [36]$$

assuming the supply phase order to be abc. The addition of voltage drops around the three loops an_2n_1a, bn_2n_1b, and cn_2n_1c gives

$$\mathbf{V} + \mathbf{V}_{n_1n_2} - \mathbf{I}_{an_2}\mathbf{Z}_a = 0, \tag{37}$$

$$\mathbf{V}\underline{/-120^\circ} + \mathbf{V}_{n_1n_2} - \mathbf{I}_{bn_2}\mathbf{Z}_b = 0, \tag{38}$$

$$\mathbf{V}\underline{/+120^\circ} + \mathbf{V}_{n_1n_2} - \mathbf{I}_{cn_2}\mathbf{Z}_c = 0; \tag{39}$$

whence

$$3\mathbf{V}_{n_1n_2} = \mathbf{I}_{an_2}\mathbf{Z}_a + \mathbf{I}_{bn_2}\mathbf{Z}_b + \mathbf{I}_{cn_2}\mathbf{Z}_c, \tag{40}$$

$$\mathbf{V}_{n_1n_2} = \tfrac{1}{3}(\mathbf{V}_{an_2} + \mathbf{V}_{bn_2} + \mathbf{V}_{cn_2}), \tag{41}$$

$$\mathbf{V}_{n_1n_2} = \mathbf{V}_0, \tag{42}$$

where $\mathbf{V}_0$ is the zero-sequence component of line to neutral voltages at the unbalanced load.

8. Unbalanced Δ-Connected Impedances

When unbalanced line voltages are applied to a balanced line supplying an unbalanced Δ-connected load, the method of symmetrical components can be applied, as is shown for the circuit of Fig. 11.

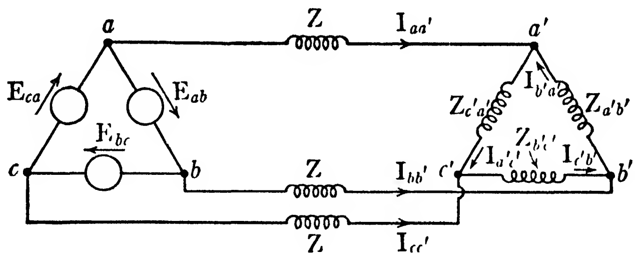


FIG. 11. Unbalanced voltages applied to balanced line supplying unbalanced load

Formulation of this problem in terms of symmetrical components requires special care, and is therefore instructive. As a first step Kirchhoff's voltage relations are written for loop $abb'a'a$ in terms of the line currents and the load voltage, the question of how the phase voltages are related to the phase currents and phase impedances in the load being omitted for the moment. This relation is

$$-\mathbf{E}_{ab} + Z\mathbf{I}_{bb'} + \mathbf{V}_{b'a'} - Z\mathbf{I}_{aa'} = 0, \tag{43}$$

which is true for actual currents and voltages, for their positive-sequence components, and for their negative-sequence components. That this is true can be seen by considering $-\mathbf{E}_{ab} + \mathbf{V}_{b'a'}$ as a single voltage applied

in association with the corresponding voltages of loops $bcc'b'b$ and $caa'c'c$ to the balanced set of impedances Z, which then appear to be connected in Y. The positive-sequence components of $\mathbf{E}_{ab}$ and $\mathbf{V}_{b'a'}$ result in positive-sequence currents only. Similarly, the negative-sequence voltage components of $\mathbf{E}_{ab}$ and $\mathbf{V}_{b'a'}$ result only in negative-sequence currents.

Taking Eq. 43 for positive-sequence quantities and employing the relations between line and phase currents in a balanced Δ-connected system give

$$\mathbf{E}_{1ab} - 3Z\mathbf{I}_{1b'a'} = \mathbf{V}_{1b'a'}. \qquad [43a]$$

Similarly, taking Eq. 43 for negative-sequence quantities gives

$$\mathbf{E}_{2ab} - 3Z\mathbf{I}_{2b'a'} = \mathbf{V}_{2b'a'}. \qquad [43b]$$

From the nature of the delta connection it is known that

$$\mathbf{V}_{0b'a'} = 0. \qquad [44]$$

The second step in the solution relating the phase load currents and voltages in terms of the load impedances - is most readily taken after the various sequence components of impedance have been calculated as shown in Art. 6:

$$Z_0 = \tfrac{1}{3}(Z_{a'b'} + Z_{b'c'} + Z_{c'a'}), \qquad [25a]$$

$$Z_1 = \tfrac{1}{3}(Z_{a'b'} + Z_{b'c'}\underline{/120^\circ} + Z_{c'a'}\underline{/-120^\circ}), \qquad [26a]$$

$$Z_2 = \tfrac{1}{3}(Z_{a'b'} + Z_{b'c'}\underline{/-120^\circ} + Z_{c'a'}\underline{/120^\circ}). \qquad [27a]$$

From Eqs. 22a to 24a the various components of $\mathbf{V}_{b'a'}$ can be expressed in terms of the various components of $\mathbf{I}_{b'a'}$ as follows, dropping the phase subscripts since phase $b'a'$ only is involved:

$$\mathbf{V}_1 = Z_1\mathbf{I}_0 + Z_0\mathbf{I}_1 + Z_2\mathbf{I}_2, \qquad [23b]$$

$$\mathbf{V}_2 = Z_2\mathbf{I}_0 + Z_1\mathbf{I}_1 + Z_0\mathbf{I}_2, \qquad [24b]$$

$$0 = \mathbf{V}_0 = Z_0\mathbf{I}_0 + Z_2\mathbf{I}_1 + Z_1\mathbf{I}_2. \qquad [22b]$$

Using Eqs. 23b, 24b, and 22b in Eqs. 43a, 43b, and 44 and rearranging the terms give

$$Z_1\mathbf{I}_0 + (Z_0 + 3Z)\mathbf{I}_1 + Z_2\mathbf{I}_2 = \mathbf{E}_{1ab}, \qquad [43c]$$

$$Z_2\mathbf{I}_0 + Z_1\mathbf{I}_1 + (Z_0 + 3Z)\mathbf{I}_2 = \mathbf{E}_{2ab}, \qquad [43d]$$

$$Z_0\mathbf{I}_0 + Z_2\mathbf{I}_1 + Z_1\mathbf{I}_2 = 0, \qquad [44a]$$

which can be solved for the components of current. If the line currents are desired, they can easily be found from the positive- and negative-

sequence phase currents. The zero-sequence current in the Δ does not appear in the line. Although the Δ has no zero-sequence induced voltages, the unbalanced impedances may give rise to zero-sequence components of drop even with positive-sequence currents in the Δ. The drops due to positive-sequence and to negative-sequence currents must be compensated by drops due to zero-sequence current as shown in Eq. 44a.

9. General observations

A summary of useful observations can now be made concerning symmetrical components:

(a) In a Y connection without neutral, the zero-sequence component of current is zero.

(b) The zero-sequence component of line voltages $\mathbf{E}_{ab}$, $\mathbf{E}_{bc}$, and $\mathbf{E}_{ca}$ or $\mathbf{V}_{a'b'}$, $\mathbf{V}_{b'c'}$, and $\mathbf{V}_{c'a'}$ is always zero.

(c) Zero-sequence currents may exist in a Δ connection but cannot pass into the line.

(d) Zero-sequence voltages may exist in the phases of a Y-connected load although they cannot appear in the line voltages.

(e) If the line voltages are balanced and of sequence *abc*, the negative-sequence component of the phase voltages of a Y connection is zero.

(f) If the line currents are balanced and of sequence *abc*, the negative-sequence component of the phase currents of a Δ connection is zero.

10. Power in terms of symmetrical components

As shown in Art. 10, Ch. X, the average power P delivered to a three-phase Y-connected circuit in which the line currents and phase-voltage drops to neutral are $\mathbf{I}_{an}$, $\mathbf{I}_{bn}$, $\mathbf{I}_{cn}$, $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$, respectively, is given by

$$P = V_{an}I_{an} \cos \angle_{\mathbf{V}_{an}}^{\mathbf{I}_{an}} + V_{bn}I_{bn} \cos \angle_{\mathbf{V}_{bn}}^{\mathbf{I}_{bn}} + V_{cn}I_{cn} \cos \angle_{\mathbf{V}_{cn}}^{\mathbf{I}_{cn}}. \qquad [45]$$

The wiring diagram representing load conditions in such a circuit is shown in Fig. 12. The symmetrical components of voltage and current are

$$\mathbf{V}_{an} = \mathbf{V}_0 + \mathbf{V}_1 + \mathbf{V}_2, \qquad [4b]$$

$$\mathbf{V}_{bn} = \mathbf{V}_0 + \mathbf{V}_1\underline{/-120^\circ} + \mathbf{V}_2\underline{/120^\circ}, \qquad [5b]$$

$$\mathbf{V}_{cn} = \mathbf{V}_0 + \mathbf{V}_1\underline{/120^\circ} + \mathbf{V}_2\underline{/-120^\circ}, \qquad [6b]$$

$$\mathbf{I}_{an} = \mathbf{I}_0 + \mathbf{I}_1 + \mathbf{I}_2, \qquad [4c]$$

$$\mathrm{I}_{bn} = \mathrm{I}_0 + \mathrm{I}_1\underline{/-120^\circ} + \mathrm{I}_2\underline{/120^\circ}, \qquad [5c]$$

$$\mathrm{I}_{cn} = \mathrm{I}_0 + \mathrm{I}_1\underline{/120^\circ} + \mathrm{I}_2\underline{/-120^\circ}. \qquad [6c]$$

The total power absorbed in each phase of the load in terms of symmetrical components is the sum of the powers resulting from each prod-

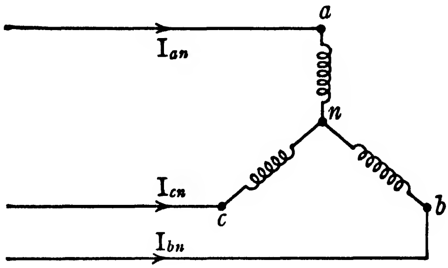


FIG. 12. Wiring diagram of three-phase Y-connected unbalanced load.

uct of V, I components, and the cosine of the angle between V and I for that phase. Thus, by multiplying the terms of Eqs. 4b and 4c, the power in phase a is found to be:

$$\left.\begin{aligned} P_a = {} & V_0I_0 \cos \measuredangle_{\mathrm{V}_{0a}}^{\mathrm{I}_{0a}} + V_0I_1 \cos \measuredangle_{\mathrm{V}_{0a}}^{\mathrm{I}_{1a}} + V_0I_2 \cos \measuredangle_{\mathrm{V}_{0a}}^{\mathrm{I}_{2a}} \\ & + V_1I_0 \cos \measuredangle_{\mathrm{V}_{1a}}^{\mathrm{I}_{0a}} + V_1I_1 \cos \measuredangle_{\mathrm{V}_{1a}}^{\mathrm{I}_{1a}} + V_1I_2 \cos \measuredangle_{\mathrm{V}_{1a}}^{\mathrm{I}_{2a}} \\ & + V_2I_0 \cos \measuredangle_{\mathrm{V}_{2a}}^{\mathrm{I}_{0a}} + V_2I_1 \cos \measuredangle_{\mathrm{V}_{2a}}^{\mathrm{I}_{1a}} + V_2I_2 \cos \measuredangle_{\mathrm{V}_{2a}}^{\mathrm{I}_{2a}} \end{aligned}\right\}. \qquad [46]$$

In phase b, the product term involving V_1 and I_2 is $V_1I_2 \cos \measuredangle_{\mathrm{V}_{1b}}^{\mathrm{I}_{2b}}$,

and in phase c it is $V_1I_2 \cos \measuredangle_{\mathrm{V}_{1c}}^{\mathrm{I}_{2c}}$.

Adding these two terms to the $V_1I_2 \cos \measuredangle_{\mathrm{V}_{1a}}^{\mathrm{I}_{2a}}$ term of phase a gives

$$\Sigma P(V_1I_2) = V_1I_2\left(\cos \measuredangle_{\mathrm{V}_{1a}}^{\mathrm{I}_{2a}} + \cos \measuredangle_{\mathrm{V}_{1b}}^{\mathrm{I}_{2b}} + \cos \measuredangle_{\mathrm{V}_{1c}}^{\mathrm{I}_{2c}}\right). \qquad [47]$$

Figure 13 shows how Eq. 47 can be changed to the following form:

$$\left.\begin{aligned} \Sigma P(V_1I_2) = V_1I_2\Big[& \cos \measuredangle_{\mathrm{V}_{1a}}^{\mathrm{I}_{2a}} + \cos\left(120^\circ + \measuredangle_{\mathrm{V}_{1a}}^{\mathrm{I}_{2a}}\right) \\ & + \cos\left(120^\circ - \measuredangle_{\mathrm{V}_{1a}}^{\mathrm{I}_{2a}}\right)\Big] \end{aligned}\right\}. \qquad [47a]$$

With $\measuredangle_{V_{1a}}^{I_{2a}}$ replaced by θ_{12}, Eq. 47a can be expanded as follows:

$$\left.\begin{aligned}\Sigma P(V_1 I_2) &= V_1 I_2[\cos\theta_{12} + \cos 120^\circ \cos\theta_{12} - \sin 120^\circ \sin\theta_{12} \\ &\qquad + \cos 120^\circ \cos\theta_{12} + \sin 120^\circ \sin\theta_{12}] \\ &= V_1 I_2[\cos\theta_{12} + 2(-\tfrac{1}{2})\cos\theta_{12}] = 0.\end{aligned}\right\} \quad [47b]$$

Similar manipulations show that the power from all the terms involving products of voltage and current components of unlike phase order is

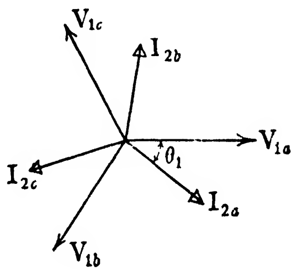


FIG. 13. Vector diagram showing positive-sequence symmetrical components of voltage and negative-sequence symmetrical components of current in load of Fig. 12.

zero. The terms involving products of components of like phase order are equal in all three phases, adding to give the total power

$$\left.\begin{aligned}P = P_a + P_b + P_c = 3\Big[V_0 I_0 \cos \measuredangle_{V_0}^{I_0} + V_1 I_1 \cos \measuredangle_{V_1}^{I_1} \\ + V_2 I_2 \cos \measuredangle_{V_2}^{I_2}\Big].\end{aligned}\right\} \quad \blacktriangleright[45a]$$

This does not mean that P_a, P_b, and P_c are each equal to one-third of P as given by Eq. 45a. These are given by the individual terms of Eq. 45, or in terms of symmetrical components by equations such as Eq. 46.

If the Y-phase currents and voltages are merely redefined as Δ-phase currents and voltages, the relation given by Eq. 45a can be seen to hold for Δ-connected circuits, since the steps between Eqs. 45 and 45a are the same in either situation.

The computation of I^2R losses in terms of symmetrical components is in general a complicated procedure which is simplified only when the resistances to currents of each sequence are balanced.[1]

[1] A derivation may be found, for example, in: O. G. C. Dahl, *Electric Circuits. Vol. I: Theory and Applications* (New York: McGraw-Hill Book Company, Inc., 1928), pp. 90–92; also in other references mentioned in the bibliography.

11. Study of faults on three-phase power circuits

When compared with the ordinary network methods of Ch. X, the method of symmetrical components offers in general no advantage for solving the simple problems of polyphase circuits. But when power-network circuits and problems involving rotating machines are encountered, it saves labor. Applications of the method to circuits containing rotating machines are presented in this series in the volume on rotating electric machinery. Certain examples of the use of the method in the study of three-phase power-system faults are presented here.

Figure 14 is a one-line diagram of one phase (phase a) of a three-phase power system consisting of a hydroelectric generating station G, a trans-

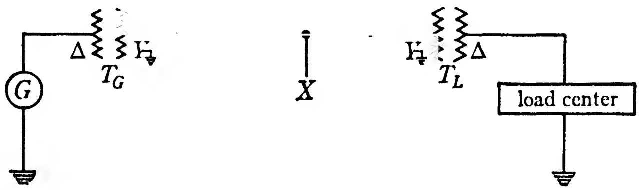


FIG. 14. One-line diagram of phase a of a three-phase transmission line connecting generating station G to a load center.

mission line with terminating transformers T_G and T_L, and a large urban load center having additional generating stations. In this one-line diagram the symbols indicate that transformer bank T_G is Δ connected on the left or input side, and Y connected with grounded neutral on the right-hand side. Transformer bank T_L is Y connected with grounded neutral on the left-hand side, and Δ connected on the right-hand or output side. Generating station G maintains under fault conditions balanced sinusoidal internal generated electromotive forces of $\mathbf{E}$ per phase. The generator impedance is Z_1^+ per phase for currents of positive sequence, Z_1^- for currents of negative sequence, and Z_1^0 for currents of zero sequence. The generators are Y connected with neutrals grounded. The generating capacity at the load center is so large that the voltages $\mathbf{E}_l$ per phase at this point can be considered balanced and independent of conditions on the transmission line. The transformers are represented by their approximate equivalent circuits which neglect the exciting currents, that is, by simple series impedances Z_{tg} and Z_{tl} for all sequences. The point of fault is marked X and the transmission-line impedances on the left and on the right of this point are called, respectively, Z_p and Z_q for currents of positive sequence and negative sequence, and Z_p^0 and Z_q^0 for currents of zero sequence.

The positive-sequence schematic diagram for phase a is shown in Fig. 15a. This diagram contains the balanced source voltages and all the

positive-sequence quantities. Part (b) of this figure shows the negative-sequence schematic diagram. There are no source voltages, since the internal generator voltages are balanced. The only place where the impedances are different from those in (a) is in the generator G. The zero-sequence diagram — part (c) — has the additional difference that the zero-sequence voltage at the Δ-connected low-tension transformer terminals is zero. The zero-sequence currents can circulate in the Δ but cannot exist in the line. These facts make it necessary, in the diagram, to place a short circuit at the low-tension transformer terminals but to leave the circuit open at the left of Z_{tg} and at the right of Z_{tl}. In this situation

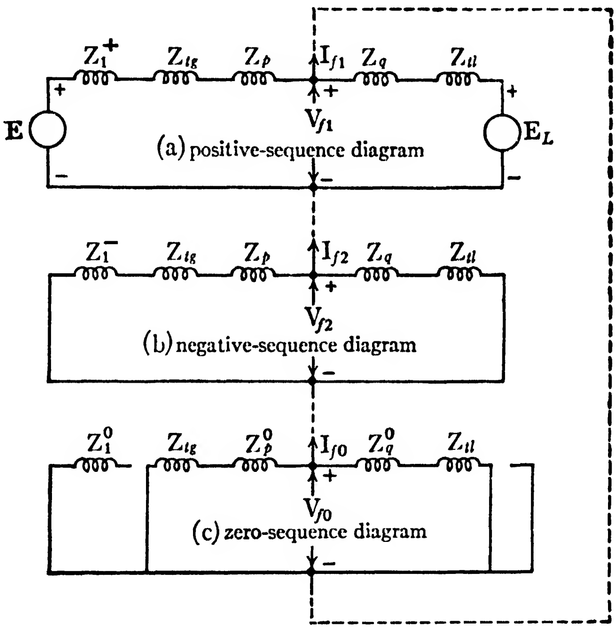


FIG. 15. Positive-, negative- and zero-sequence circuit schematic diagrams and interconnection for single line-to-ground short circuit on phase *a*.

there are no zero-sequence currents in generator G whether the neutral is grounded or not. When zero-sequence currents are to be allowed to circulate through the grounded generators, the transformers are connected Y–Y with both the high-tension and low-tension neutrals grounded, the circulating third harmonic currents of the transformers being carried by a third set of windings, called tertiaries, connected in Δ.*

The schematic diagrams of the three sequences are independent of the type of fault at X. The remainder of the problem comprises the inter-

* This is discussed in this series in the volume on magnetic circuits and transformers.

connection of these three networks corresponding to a particular kind of fault, and the solution of the complete network formed by this interconnection.

First the fault is considered to be a solid short circuit from conductor *a* to ground. Three conditions are always required to describe any kind

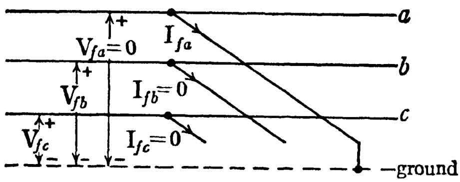


FIG. 16. Current and voltage conditions at a line-to-ground fault in phase *a*.

of fault at a single point along the line; the three which describe this particular fault are shown in Fig. 16:

$$I_{fb} = 0, \quad [48]$$

$$I_{fc} = 0, \quad [49]$$

$$V_{fa} = 0. \quad [50]$$

The subscript f means fault, and a, b, and c denote the phase involved. These three conditions in the three phases must now be transformed into three conditions in phase a. In terms of symmetrical components these are:

$$I_{f0} + I_{f1}\underline{/-120°} + I_{f2}\underline{/120°} = 0, \quad [48a]$$

$$I_{f0} + I_{f1}\underline{/120°} + I_{f2}\underline{/-120°} = 0, \quad [49a]$$

$$V_{f0} + V_{f1} + V_{f2} = 0. \quad [50a]$$

The first two of these, Eqs. 48a and 49a, can be simplified by subtraction, which gives

$$I_{f1} = I_{f2}; \quad [51]$$

substituting this into either of the equations gives the result:

$$I_{f0} = I_{f1} = I_{f2}. \quad [52]$$

These two conditions, together with Eq. 50a, describe the line-to-neutral fault on phase a in terms of components, as Eqs. 48 to 50 describe it in terms of total voltages and currents.

In Fig. 15, the voltage components in phase a at the fault are denoted by V_{f1}, V_{f2}, and V_{f0}, and the components of fault current are denoted by I_{f1}, I_{f2}, and I_{f0}. The three conditions of Eqs. 48a to 50a are fulfilled

if the three component circuits are connected in series as shown by the dotted line.

The resulting network can now be solved for currents and voltages at any desired points by the usual network methods. Since there are no source voltages in the negative- and zero-sequence diagrams, these are easily reduced to single impedances as follows:

$$Z^- = \frac{(Z_1^- + Z_{tg} + Z_p)(Z_q + Z_{tl})}{Z_1^- + Z_{tg} + Z_p + Z_q + Z_{tl}}, \tag{53}$$

$$Z^0 = \frac{(Z_{tg} + Z_p^0)(Z_q^0 + Z_{tl})}{Z_{tg} + Z_p^0 + Z_q^0 + Z_{tl}}, \tag{54}$$

and then Fig. 15 reduces to Fig. 17, which can be solved as a simple two-loop circuit by either the two-loop or the single-node method.

After currents are found from Fig. 17 and properly distributed through the branches of Fig. 15, one may find the total currents at any point in the system by combining the components at that point, being sure that

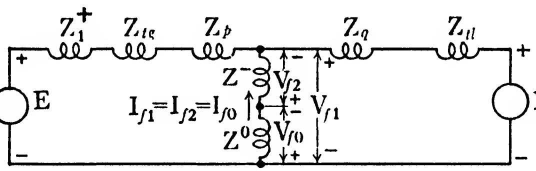


FIG. 17. Two-loop circuit representing impedances and currents and voltage components resulting from line-to ground fault in phase a.

the current directions in the circuit are the same for all components. Voltages at different points can also be found through combination of the component voltages as calculated from Fig. 17. Certain kinds of switching relays operate on one or another of the components of current or voltage. Therefore finding the total quantities is unnecessary, because the components themselves control the operation of the relays.

The interconnection just described for a single line-to-neutral short circuit on phase a is shown in Fig. 18 in a simplified form of Fig. 15. Faults other than the single line-to-neutral short circuit may occur. One of these is the double line-to-neutral short circuit in which line wires b and c are connected to ground while there is no fault on wire a. This condition is shown in Fig. 19 and is described by the three equations

$$V_{fb} = 0, \tag{55}$$

$$V_{fc} = 0, \tag{56}$$

$$I_{fa} = 0. \tag{57}$$

These are duals of Eqs. 48 to 50, obtained by interchanging voltage and current. The transformation of these three conditions in the three phases into three component conditions in phase a is the dual of the previous one, and the results are

$$V_{f_1} = V_{f_2} = V_{f_0}, \qquad [58]$$

$$I_{f_1} + I_{f_2} + I_{f_0} = 0. \qquad [59]$$

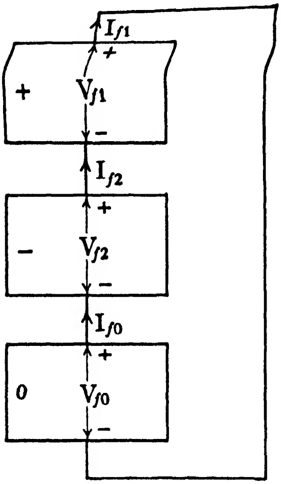


FIG. 18. Interconnection of sequence networks for single line-to-neutral short circuit on phase a.

These equations require a parallel interconnection of the three component networks of Fig. 15 as shown in Fig. 20.

The third common type of fault is the line-to-line short circuit. This is shown in Fig. 21 and is described analytically by the equations

$$I_{fa} = 0, \qquad [60]$$

$$I_{fb} = -I_{fc}, \qquad [61]$$

$$V_{fb} = V_{fc}. \qquad [62]$$

When transformed into components of phase a, these give

$$I_{f_0} = 0, \qquad [63]$$

$$I_{f_1} = -I_{f_2}, \qquad [64]$$

$$V_{f_1} = V_{f_2}. \qquad [65]$$

These equations require the parallel interconnections of the networks as shown in Fig. 22.

The simple interconnections discussed here are only selections from a large group of interconnections arising from all the various types of possible faults. Some of the problems are very complicated, especially if faults exist

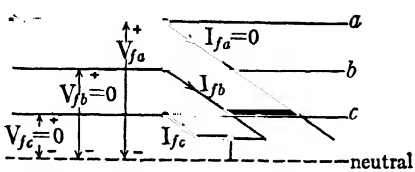


FIG. 19. Double line-to-neutral short circuit, phases b and c grounded

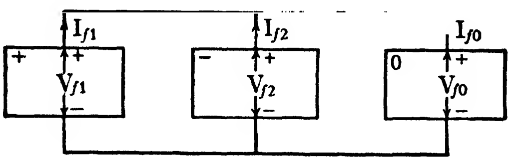


FIG. 20. Interconnection of networks for double line-to-neutral short circuit on phases b and c.

simultaneously at more than one point in the system. This kind of analysis, in conjunction with the use of a network analyzer, such as those

at the plants of the large manufacturing companies and some of the utility companies, and at the Massachusetts Institute of Technology, makes practicable the solutions of power problems which were formerly

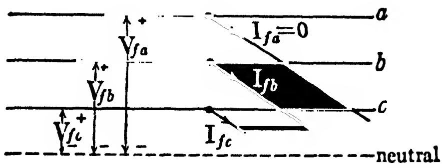


FIG. 21. Line-to-line short circuit from phase *b* to phase *c*.

so complicated and tedious as to be very costly, to say the least.* When these analyzers are used, the networks for the different sequences are set up and interconnected. Results are obtained chiefly by reading instruments rather than by calculation.

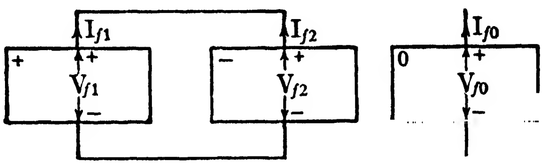


FIG. 22. Interconnection of networks for line-to-line short circuit from phase *b* to phase *c*.

12. Application of the Method of Symmetrical Components to a System of *n* Phases

The symmetrical-component method can be extended to the n-phase system by using n components in phase a rather than n total quantities in n phases. Then there are n sequences, numbered from zero to $n - 1$. If the set of n unsymmetrical vectors representing some circuit condition has a resultant other than zero, there is a zero-sequence component set of n equal vectors, each of which is equal to $1/n$ of the resultant. The next symmetrical-component set, called the first-order or positive-sequence set, consists of n symmetrical vectors, spaced at angles of $360/n$ degrees. The second-order set likewise has n symmetrical component vectors, spaced at angles of $(2 \times 360)/n$ degrees. Thus, the entire system of symmetrical-component sets needed to represent n original phases can be listed as follows:

* Chapter VIII, plates on pp. 443–446.

Sequence	*Order*	*Angle between vectors having adjacent phase letters*
0	0	0
Positive	1	$\dfrac{360°}{n}$
Not named	2	$\dfrac{2 \times 360°}{n}$
Not named	3	$\dfrac{3 \times 360°}{n}$
.	.	.
.	.	.
.	.	.
.	.	.
Negative	$n - 1$	$\dfrac{(n - 1) \times 360°}{n}$

The method of calculating the value of the key vector of each of the symmetrical-component sets having orders from one to $n - 1$ parallels the method presented for the three-vector problem. Since the work involved is obviously much greater, it is fortunate that there are few practical problems requiring the analysis of unbalanced systems of more than three phases. Although five-phase systems are not at present used

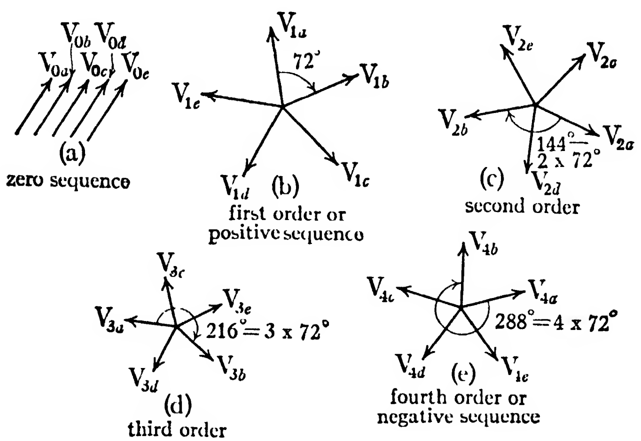


FIG. 23. Symmetrical-component sets representing a five-phase system.

in practice, they serve very nicely to indicate the type of symmetrical sets which result from the resolution of a system having more than three phases. Figures 23 and 24 illustrate the symmetrical-component sets of a five-phase and of a six-phase system, respectively. When the number of vectors or phases of a system are prime, as in the three- and five-phase

cases, each component set of sequence higher than zero has n individual vectors; whereas, when the number of phases is not prime, as in the six-

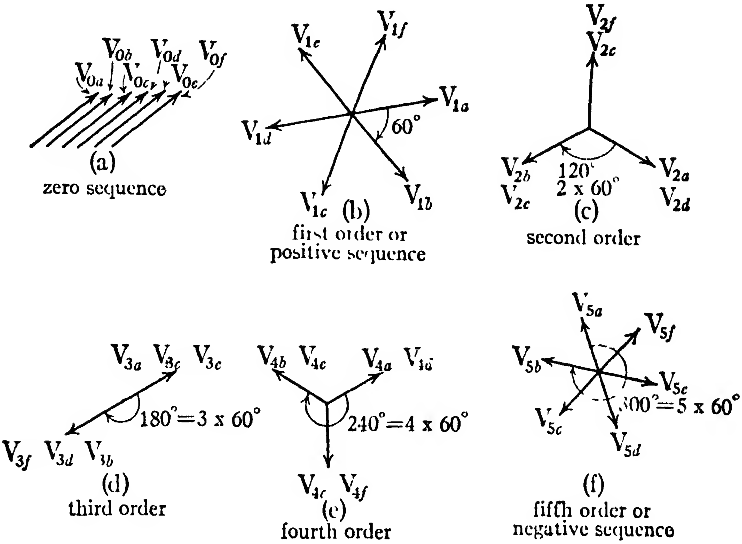


FIG. 24. Symmetrical-component sets representing a six phase system.

phase system, some of the component sets contain superimposed vectors and thus appear not to have n symmetrical-component vectors.

PROBLEMS

1. Equal impedances Z have the three-phase alternating voltages applied as shown in Fig. 25.

(a) What are the symmetrical components of the line voltages?
(b) What are the symmetrical components of the line currents?
(c) What are the positive sequence, negative sequence, and total power?

2. Three unequal single-phase inductive loads are connected between lines a-b, b-c, and c-a of a three-phase system. The line potentials are each 230 v with V_{ab} leading V_{bc}. The first load takes 100 amp at 0.707 power factor, the second takes 150 amp at 0.80 power factor, and the third takes 28.4 kw at 0.50 power factor.

(a) What are the line currents?
(b) What do two wattmeters connected for the two-wattmeter method of measuring power read if their current coils are in lines a and b?
(c) What are the symmetrical components of the phase currents?
(d) What are the symmetrical components of the line currents?

3. In Fig. 26, the phase order is abc and the line-to-line voltages are each 200 v.

(a) What are the symmetrical components of the line currents?
(b) What power is dissipated in each phase?

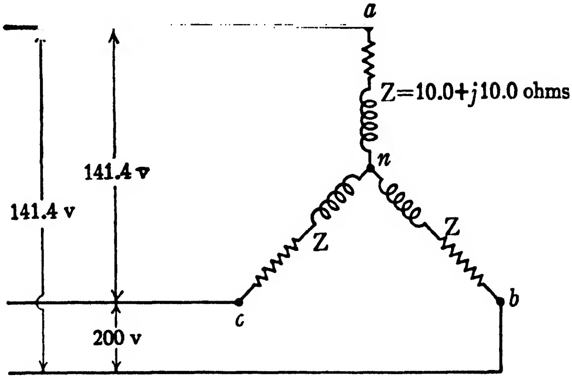


FIG. 25. Circuit for Prob. 1.

(c) What is the voltage between the load neutral and the true neutral of the system?
(d) If wattmeter current coils are inserted in lines a and b, and their voltage coils connected between c–a and c–b, respectively, what does each wattmeter read?
(e) A vector diagram is to be drawn.

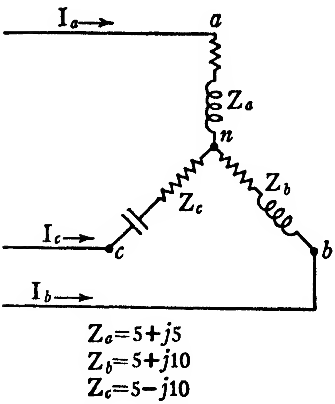


FIG. 26. Circuit for Prob. 3.

4. The following questions refer to the circuit of Fig. 27.

(a) What are the symmetrical components of phase voltages and line-to-line voltages at the load, the symmetrical components of line currents, and the total line currents with K open?
(b) What are the answers to (a) with K closed?

5. In the circuit of Fig. 28 the method of symmetrical components is to be used to find:

(a) the symmetrical components of $I_{aa'}$,
(b) the actual line currents $I_{aa'}$, $I_{bb'}$, $I_{cc'}$,
(c) the load voltages $V_{a'b'}$, $V_{b'c'}$, $V_{c'a'}$,
(d) the total power dissipated in the three load resistors R_2.

The answers should be checked by another method.

6. A symmetrical three-phase voltage source of phase order abc is connected through a circuit having balanced line-conductor impedances, to a balanced load, as shown in Fig. 29. Under these conditions, the line-to-neutral generator voltages are each 276 v, and

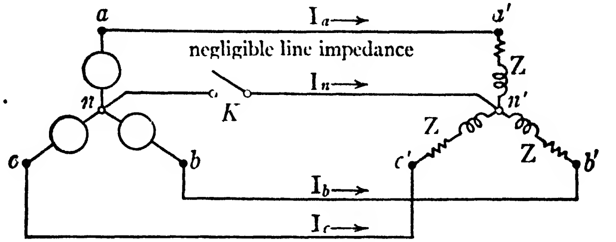
$$I_{aa'} = 36.8\underline{/-30°} \text{ amp,} \tag{66}$$

E_{na} being used as reference. Then the voltage E_{na} is changed in magnitude to 138 v, the phase angles of all source voltages and the magnitudes of E_{nb} and E_{nc} remaining unchanged. Under this new condition,

$$I_{aa'} = 17.01 - j10.60 \text{ amp} \tag{67}$$

referred to E_{na}. What is the vector value of the neutral impedance Z_n?

7. Three transformers having different internal impedance characteristics receive their energy from a balanced three-phase supply system which has constant voltage. The transformer output terminals are Δ connected, the no-load output voltage of each



$E_{na} = 120/0°$ v, $E_{nb} = 90/-120°$ v, $E_{nc} = 90/120°$ v; $Z = 100/30°$ ohms.

FIG. 27. Circuit of Prob. 4.

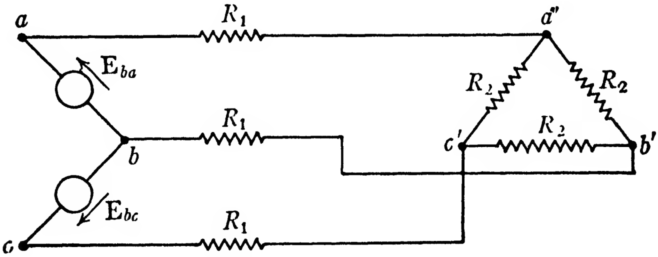
transformer being 240 v. Each transformer may be considered to be a single-phase generator having a series impedance as follows:

$$Z_{ab} = 0.50 + j0.866 \text{ ohm,} \tag{68}$$

$$Z_{bc} = 0.100 + j0.173 \text{ ohm,} \tag{69}$$

$$Z_{ca} = 0.50 + j0.866 \text{ ohm.} \tag{70}$$

What voltages are present at the transformer terminals when a Y-connected external load of $1.00 + j1.73$ ohms/phase is connected to the transformer bank? The magni-



$R_1 = 5.00$ ohms
$R_2 = 45.0$ ohms
$E_{ba} = E_{bc} = 120/0°$ v

FIG. 28. Circuit for Prob. 5.

tudes of internal impedances here are exaggerated in comparison with the impedances of practical transformers.

8. The following questions refer to the circuit of Fig. 30.

(a) What are the symmetrical components of the generator currents?
(b) What are the symmetrical components of the load currents?
(c) What are the load resistances?

9. A three-phase, Y-connected induction motor has an impedance/phase of $43.3 + j25.0$ ohms to currents of positive sequence and an impedance of $5.0 + j8.66$ ohms/phase to currents of negative sequence when supplying a particular mechanical

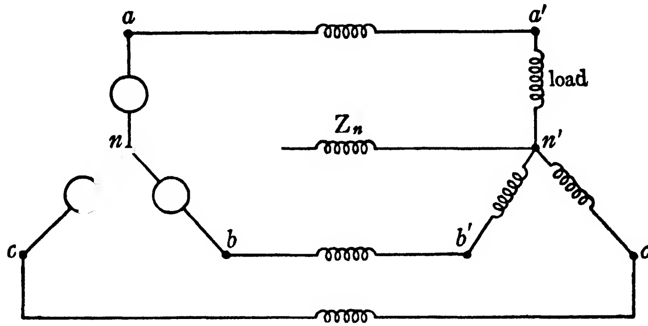


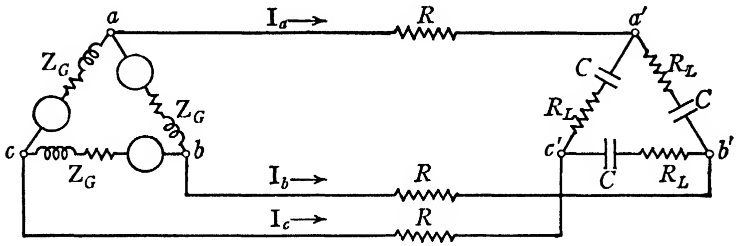
FIG. 29. Circuit for Prob. 6.

load. The motor receives power from a three-phase line having an impedance of $0.50 + j0.866$ ohm/conductor. There is no neutral connection to the motor. The line-to-line voltages at the source are

$$E_{ab} = 2{,}000 \text{ v}, \tag{71}$$

$$E_{bc} = 2{,}300 \text{ v}, \tag{72}$$

$$E_{ca} = 2{,}300 \text{ v}. \tag{73}$$



$E_{ab} = 200 + j0$ v, $E_{bc} = -117.3 - j203$ v, $E_{ca} = -82.7 + j143$ v
$\omega = 377$ radians/sec
$Z_G = 1.00 + j10.0$ ohms, $R = 3.00$ ohms; $C = 265\ \mu$f
$I_a = -14.13 + j7.16$ amp, $I_b = 15.87 + j10.16$ amp
$I_c = -1.74 - j17.32$ amp

FIG. 30. Circuit for Prob. 8.

(a) What are the positive- and negative-sequence currents, V_{ab} being taken as the axis of reference?

(b) What is the ratio of the magnitudes of positive- and negative-sequence voltages at the motor compared with the corresponding ratio for voltages at the source?

(c) What power is delivered to the motor?

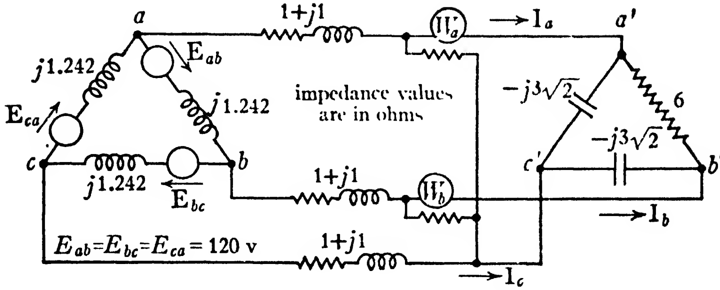


FIG. 31. Circuit for Prob 10.

10. Figure 31 shows a balanced Δ-connected source of phase order *abc*, transmitting power to an unbalanced Δ-connected load over a symmetrical three-phase conductor system.

(a) What are the line currents I_a, I_b, and I_c?
(b) What are the voltages across the phases of the load?
(c) What are the readings of the wattmeters W'_a and W'_b and the net power taken by the load?

CHAPTER XII

Electromechanically Coupled Systems

1. Introduction

Essentially every problem of generation, measurement, control, or utilization of electrical energy involves the conversion of that energy to or from some other form. The usefulness of electrical energy lies not merely in the ease with which it can be controlled while in that form, but also in the ease and convenience with which it can be converted to and from other forms. Although the conversion of electrical energy occurs in a practically infinite variety of devices, and engineering concern with these devices differs widely, a careful study of them makes clear certain fundamental relations between electric and other forms of energy which serve as a basis for the study of practically all forms. In particular, the equations describing the forces acting on electric charges in electric and magnetic fields, or relating the energy changes to the motion of the charges in the fields, if interpreted with all the ramifications of the modern physicist's conception of the constitution of matter. cover all situations. In engineering, however, it is more appropriate to classify energy conversion processes according to the nature of the energy available or delivered — for example, thermal, radiant, chemical, or mechanical. This chapter deals with the conversion processes to and from mechanical energy, although other forms of energy are of necessity considered. In any electrical device, for instance, conversion or diversion of some of the energy to thermal form is unavoidable; so a study of the various mechanisms of energy conversion that ignores this factor is incomplete.

In dealing with electromechanically coupled systems, it is very convenient at some times, and essential at others, to treat the combined system as an entity. This procedure has already been followed implicitly for the conversion of electric energy into heat. To consider a resistor as a device for converting energy into heat is entirely accurate, but to consider the resistance of conductors as a purely electrical attribute is certainly far more convenient. A similar simplification is obtained if mechanical devices coupled to an electrical system can be treated as equivalent electric-circuit elements, for in the preceding chapters the theory of electric networks has been developed to a point where it provides an extremely useful working tool for the analysis of the behavior of any electrical system.

The simplest approach to the problem of the electromechanically coupled system is, therefore, to apply the same methods of analysis first to the mechanical system and finally to the conversion mechanism itself.

2. Analogies and equivalents

The basic idea of an electromechanical analogy — comparison of inductance to mass, capacitance to compliance, voltage to force, and so on — is probably already familiar to the student. Such analogies are generally useful if they enable a mental picture of new phenomena to be formed in terms which have become familiar. Thus, when electrical phenomena are first considered, the electrical-mechanical analogy enables the relations in an electrical circuit to be formed in the familiar terms of mechanical phenomena. Now, having developed electric-circuit theory to a high degree, it becomes desirable to use this same analogy to make possible a more convenient treatment of mechanical systems. Finally, when combined electrical-mechanical systems are considered, such an analogy permits the substitution for the mechanical system of an electrical network which is an exact *equivalent*. The problem is thereby reduced entirely to electrical terms for solution.

The full significance of an analogy is not always appreciated. Customarily an analogy is introduced simply as an aid in visualizing the correspondence between two different systems. There is no objection to such a practice, but the validity of the analogy depends solely on the mathematical proof of the correspondence between the two systems. In particular, the existence of a rigorous analogy implies an exact similarity in the form of the mathematical equations which describe the behavior of the two systems under consideration.

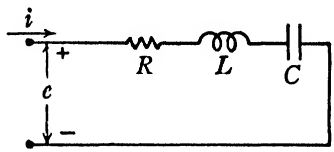


FIG. 1. Electric circuit having resistance, inductance, and capacitance in series.

As an example, the simple series electrical circuit of Fig. 1 and the simple vibrating mechanical system of Fig. 2a are considered. The relations in the electrical circuit are expressed by the equation:

$$e = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt. \qquad [1]$$

Equation 1 is of so familiar a form that the reasoning upon which it is based is likely to be overlooked. Actually it depends on the principle of adding the voltages across individual circuit elements to obtain the total voltage across all of them connected in series. In the mechanical system of Fig. 2a, each individual element can be treated as a free body. The individual elements are indicated in Fig. 2b. The equations for the individual elements are:

$$f_1 - L_M\frac{d\Omega}{dt} = 0, \qquad [2]$$

$$f_2 - R_M v = 0, \qquad [3]$$

$$f_3 - \frac{1}{C_M}\int v\,dt = 0. \qquad [4]$$

In these equations f represents force; v represents velocity; L_M represents mass, or mechanical inductance; R_M represents mechanical resistance; C_M represents compliance (deflection per unit force), or mechanical capacitance; and t represents time.

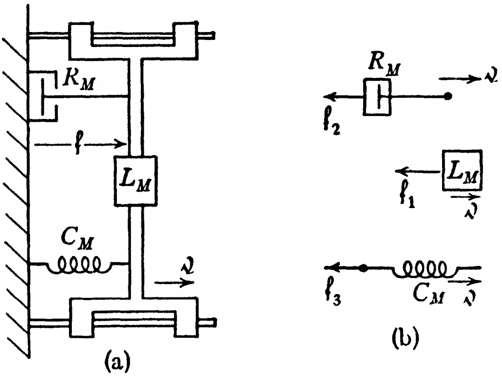


FIG. 2. Mechanical system having mechanical resistance (friction), mechanical inductance (mass), and mechanical capacitance (compliance) in parallel.

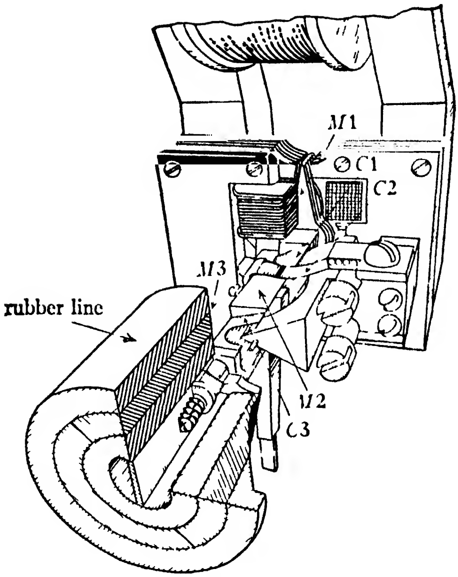
In Eqs. 2, 3, and 4, the positive direction of force is arbitrarily taken to be to the left in Fig. 2b. It is fully as essential to assign a positive reference direction for force in the analysis of any mechanical system as for voltage in an electrical system. Likewise, the positive direction of velocity v must be arbitrarily established in order that the relations between force and velocity may be consistently determined.

The total force f acting on the moving system is determined by addition of the reaction forces of Eqs. 2, 3, and 4, which results in the relation

$$f = L_M \frac{dv}{dt} + R_M v + \frac{1}{C_M}\int v\,dt. \qquad [5]$$

There is a rigorous analogy between the electrical and the mechanical systems, since Eqs. 1 and 5 describing their behavior are identical in form.

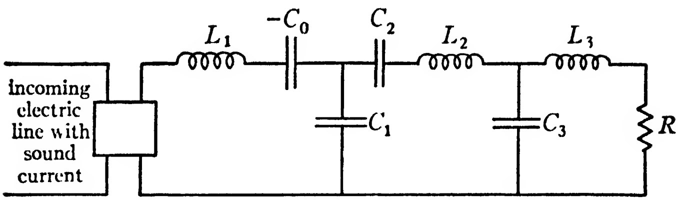
Certain assumptions are implied in the equations of both systems. Just as it has been assumed that electrical circuit elements may exist as pure entities, and in particular that a capacitor may be free from inductance in its leads and from electrical losses of any kind, so it is assumed that mass, mechanical resistance, and compliance may exist as pure entities, and in particular, that a spring may be without mass and mechanical losses of any kind. These ideal conditions may be approximated to about the same extent in the two systems. It should also be noted that mechanical resistance is assumed to develop a retarding force which is proportional to velocity. Sliding friction, in which the retarding force is substantially constant once motion is started, introduces a nonlinear element which is not considered here.



Courtesy Bell Telephone Laboratories

(a)

Detailed drawing of the mechanical filter of an electromagnetic recorder



(b)

Equivalent electric circuit of the electromagnetic recorder.

L_1 Inductance representing mass of armature $M1$
L_2 Inductance representing mass of cutting stylus $M2$
L_3 Inductance representing mass of coupling disk $M3$
$-C_0$ Capacitance representing negative compliance of magnetic field
C_1 Capacitance representing compliance $C1$ of coupling shaft
C_2 Capacitance representing compliance $C2$ of balancing springs
C_3 Capacitance representing compliance $C3$ of coupling shaft
R Resistance representing mechanical resistance of rubber line

FIG. 3. Electromagnetic (Orthophonic) phonograph recorder mechanism and equivalent circuit

This type of analogy has been recognized as useful for many years. The Bell Telephone Laboratories made momentous use of it in the analysis and design of the Orthophonic phonograph recorders and reproducers. The entire vibrating mechanical system was designed on the basis of an equivalent electrical system, with results which, in comparison with anything prior to that time, were very remarkable. This development is described in a paper by Maxfield and Harrison, published in 1926. Figure 3 shows the mechanical elements and the equivalent electrical circuit of the recorder as given in that paper, to which the student is referred for a detailed discussion of the analogy between these two systems.[1]

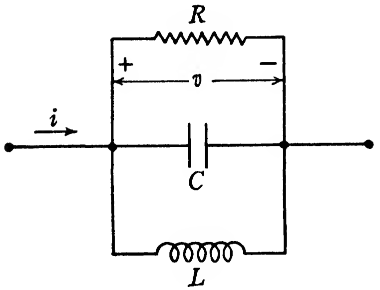


FIG. 4. Electric circuit having resistance, inductance, and capacitance in parallel.

The analogy in this example, where force is represented by voltage, velocity by current, mass by inductance, and so on, is only one of two equally valid possibilities. Just as the series electrical circuit of Fig. 1 has its dual in parallel form as shown in Fig. 4, so the mechanical circuit of Fig. 2 has its dual as illustrated in Fig. 5. In Fig. 4 the total current at the point where the voltage is acting is the sum of the currents through the capacitance, the resistance, and the inductance. In Fig. 5 the velocity at the point where the force is acting is the sum of the velocity of the mass and the relative velocities between the ends of the resistance and the spring. It should be understood that when a force or voltage is said to act at a point it is implied that the action is between two points, one of which is a reference at a fixed, or "ground," potential.

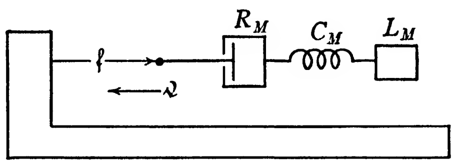


FIG. 5. Mechanical system having mechanical resistance (friction), mechanical inductance (mass), and mechanical capacitance (compliance) in series. Here $\mathfrak{f}$ represents the reaction force at the point.

The relations in the circuits of Fig. 4 and Fig. 5 are given by the equations:

$$i = C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int vdt \qquad [6]$$

$$\mathfrak{v} = C_M\frac{d\mathfrak{f}}{dt} + \frac{1}{R_M}\mathfrak{f} + \frac{1}{L_M}\int \mathfrak{f}dt. \qquad [7]$$

[1] J. P. Maxfield and H. C. Harrison, "Methods of High Quality Recording and Reproducing of Music Based on Telephone Research," *B.S.T.J.*, V (1926), 493–523.

Thus the original analogies are shown to hold for the duals of the systems. Comparison of these equations with those for the circuits of Figs. 1 and 2, however, shows that all four equations are identical in form and therefore that all four of the systems are analogous. In particular, the series mechanical system of Fig. 5 is analogous to the series electric circuit of Fig. 1, and the parallel mechanical system of Fig. 2 is analogous to the parallel electric circuit of Fig. 4. For these cases, velocity becomes analogous to voltage, force to current, mass to capacitance, compliance to inductance, and mechanical resistance to conductance. This type of analogy in many cases greatly simplifies the formulation of equivalent circuits in that the general appearance of the analogous systems is far more similar, a fact which is at once apparent from the figures.

3. Analogies used in the study of electrostatically and electromagnetically coupled systems

To determine an electric circuit which is the mathematical equivalent of a mechanical system, either analogy that has been presented may be employed. An electrical system coupled to a mechanical system through the medium of an electromagnetic or an electrostatic field, however, as illustrated respectively in the D'Arsonval galvanometer and the condenser microphone, presents an important type of problem wherein there is no longer a free choice of the analogy to be used, because the physical laws of magnetic and electric fields dictate the relations that must be employed. These two cases are now discussed.

3a. Electromagnetic coupling

As an aid in formulating by the velocity-voltage analogy the equilibrium equations of an electric circuit equivalent to a given mechanical system, it should be recalled that

$$e = \mathcal{B}l\mathcal{v} \tag{8}$$

and

$$f = \mathcal{B}li \tag{9}$$

when a conductor of projected length l moves with a velocity $\mathcal{v}$ in a direction perpendicular to the magnetic flux having a density $\mathcal{B}$. Under these conditions, the electromotive force e generated in the conductor is directly proportional to the velocity $\mathcal{v}$, and the force f acting on the conductor is directly proportional to the current i in it. Equations 8 and 9 demonstrate the physical relations which dictate the analogy that must be used whenever magnetic coupling exists between an electrical and a mechanical system. The student may wonder why the equivalent circuit,

Fig. 3b, representing the phonograph recorder mechanism, is not based on the analogies of Eqs. 8 and 9. The reason is that the equivalent circuit represents only the mechanical system itself and does not (as indicated by the vacant coupling box) include the effect of the mechanical load on the electrical input. If the apparent impedance of the combined mechanical and electrical system as seen from the input terminals were to be represented, the mechanical system would necessarily be converted to an electric circuit through the electromagnetic method of analogy.

The advantages of the electromagnetic analogy described above are treated in detail in a paper by Professor F. A. Firestone.[2] The student who desires to study these analogies further and to understand their limitations with regard to reactance, susceptance, impedance, and admittance concepts should study Professor Firestone's article.

3b. Electrostatic coupling

When an electrical system is electrostatically coupled to a mechanical system, the displacement current equals the rate of change of charge in the coupling field, and the mechanical velocity of the moving parts equals the rate of change of distance. These relations therefore fix the type of analogy that must be used in determining an electric circuit that shall be equivalent to the combined mechanical and electrical systems which are electrostatically coupled. The analogy in this case is the familiar one represented by Figs. 1 and 2 and Eqs. 1 and 5.

4. Summary of analogies

In order to summarize clearly the two types of analogy that have been discussed, a table is presented on the opposite page which shows the comparison between them.

Useful and valid electrical analogies exist for other mechanical systems besides the simple translational type which has been considered. The correspondence between quantities can generally be formulated from a consideration of energy changes without the use of mathematical equations.

In practical applications, the basic quantities of an electrical system are current and potential difference. The power which is transmitted is found in terms of the product of these basic quantities and the general behavior of the system is found in terms of their ratio, that is to say, impedance or admittance. In the translational mechanical system, the quantities corresponding to current and voltage are force and velocity, but in other types of systems other quantities are more convenient. In

[2] F. A. Firestone, "The Mobility Method of Computing the Vibration of Linear Mechanical and Electrical Systems," *J. App. Phys.*, IX (1938), 373–387.

TABLE OF ANALOGIES*

Electrical analogy for electrostatic coupling	*Mechanical system*	*Electrical analogy for electromagnetic coupling*
Voltage	*Force*	*Current*
$v = L\dfrac{di}{dt}$	$f = L_M\dfrac{du}{dt}$	$i = C\dfrac{dv}{dt}$
$v = Ri$	$f = R_M u$	$i = Gv$
$v = \dfrac{1}{C}\int i\,dt$	$f = \dfrac{1}{C_M}\int u\,dt$	$i = \dfrac{1}{L}\int v\,dt$
Current $i = \dfrac{dq}{dt}$	Velocity $u = \dfrac{dx}{dt}$ or $r\dfrac{d\theta}{dt}$	Voltage $v = \dfrac{d\lambda}{dt}$
Charge q	Displacement x or $r\theta$	Flux linkage λ
Capacitance C	Mechanical capacitance or compliance C_M	Inductance L
Inductance L	Mechanical inductance or mass L_M	Capacitance C
Resistance R	Mechanical resistance R_M	Conductance G

* The table is based on the assumption that each element is pure and that all elements are linear

rotating machinery, for example, torque and angular velocity are to be preferred; and in hydromechanical systems, pressure and volumetric flow. In each, power is measured in terms of the product of the two basic quantities. Obviously, correct energy relations can be maintained in any analogy in which voltage represents one of these quantities and current represents the other. Thus electrical impedance can properly represent any of the ratios between the pairs of quantities mentioned:

$$\frac{\text{force}}{\text{velocity}}, \quad \frac{\text{velocity}}{\text{force}};$$

$$\frac{\text{torque}}{\text{angular velocity}}, \quad \frac{\text{angular velocity}}{\text{torque}};$$

$$\frac{\text{pressure}}{\text{flow}}, \quad \frac{\text{flow}}{\text{pressure}}.$$

When one correspondence is specified either by choice or by the requirements of the problem, all others are uniquely determined.

5. Electromechanical conversions in general

All electromechanical energy-conversion devices have one feature in common: A displacement of an element, or elements, in the system produces a change in the energy of the system. The coupling in the system is electromagnetic or electrostatic according to whether the change in

energy produced by displacement is in an electromagnetic field or an electrostatic field.

Specific examples of these two types of conversion are afforded by two elementary circuits: an isolated inductance L carrying current i; and an *isolated capacitance C across* an electromotive force e. These circuits are illustrated in Fig. 6. The energy stored is:

$$W_m = \tfrac{1}{2}Li^2 \tag{10}$$

in the magnetic field, and

$$W_e = \tfrac{1}{2}Ce^2 \tag{11}$$

in the electric field.

Here L and C each may depend on a space relation x. The inductance, for example, may represent a telephone receiver in which the value of the inductance depends on the distance x between a diaphragm and a pole

FIG. 6. Simple circuit elements of inductance (a) and capacitance (b).

face. The capacitance may represent a condenser microphone in which the value of the capacitance depends on the distance x from a diaphragm to a fixed plate.

In each example the power involved may be found by differentiating with respect to time. It is assumed that i, e, and x are all time functions. Then,

$$\frac{dW_m}{dt} = \frac{d}{dt}(\tfrac{1}{2}Li^2) = Li\frac{di}{dt} + \frac{i^2}{2}\frac{dL}{dx}\frac{dx}{dt}, \tag{12}$$

$$\frac{dW_e}{dt} = \frac{d}{dt}(\tfrac{1}{2}Ce^2) = Ce\frac{de}{dt} + \frac{e^2}{2}\frac{dC}{dx}\frac{dx}{dt}. \tag{13}$$

Each term in each equation represents a rate in change of energy. In the first case, $L(di/dt)$ is recognized as the voltage induced by virtue of the variation in the current. The term $Li(di/dt)$ evidently represents the electric power which is associated with the change in energy in the magnetic field. Since the next term must also represent power, and since dx/dt is mechanical velocity, $(i^2/2)(dL/dx)$ must represent force. A similar analysis of the second case shows that $(e^2/2)(dC/dx)$ represents force.

If dL/dx is constant, and if i is made up of a constant component I_0 and a relatively small variable component i_v, the force includes a fixed

component $(I_0^2/2)(dL/dx)$ and a component $I_0 i_v(dL/dx)$ which is proportional to the variable current i_v. The term in i_v^2 is here assumed to be negligibly small. The variable component of force is therefore given by the relation

$$f_v = \left(I_0 \frac{dL}{dx}\right) i_v. \qquad [14a]$$

A similar analysis of the second system, where the electromotive force e is made up of a fixed component E_0 and a relatively small variable component e_v, shows that there is a variable component of force proportional to e_v according to the relation

$$f_v = \left(E_0 \frac{dC}{dx}\right) e_v. \qquad [14b]$$

This analysis is too general to be readily applicable to specific devices, but it serves to show in a concise manner those relationships which are characteristic of electromagnetic and electrostatic coupling. For electromagnetic coupling, an electromotive force is generated by mechanical motion and force is produced by current. For electrostatic coupling, current is generated by motion and force is produced by voltage. Therefore, when an equivalent electrical system is to be found for each of the two systems, force should obviously be represented by current if electromagnetic coupling is used and by voltage if electrostatic coupling is used.

In addition to the two types of coupling which have just been considered, two other types are of interest: magnetostrictive and piezoelectric. These types are really special cases of electromagnetic and electrostatic coupling, respectively, but since they depend on particular properties of certain materials they may properly be classed separately.

6. Electromagnetic conversion devices

Electromechanical conversion based on electromagnetic coupling is of chief importance from an engineering point of view whether judged by the number of applications or by the cost of equipment in which it is used. This type of conversion is therefore considered first and in greatest detail. The moving element in electromagnetic conversion devices in general comprises both magnetic material and current-carrying conductors, but in the simpler types of mechanisms only one or the other is associated with the moving parts. It is convenient for purposes of analysis to subclassify these devices as moving-conductor and moving-iron types. The fundamental principles of both types are discussed and practical examples are considered.

Since in either type of mechanism the magnetic field acts as the connecting link between the electrical and the mechanical systems, two inter-

changes of energy are involved: between the electric circuit and the magnetic field, and between the magnetic field and the mechanical system. If a separate electric circuit is used to produce the magnetic field, there may also be an interchange of energy between this circuit and the field.

7. Moving-conductor mechanism

The nature of the energy interchanges between the electric circuit and the magnetic field and between the field and the mechanical system may be demonstrated by an analysis of the simple moving-conductor device shown in Fig. 7. This system comprises the essential features of an important group of mechanisms such as moving-coil microphones and loud-speakers, and all instruments of the D'Arsonval type. In all these appli-

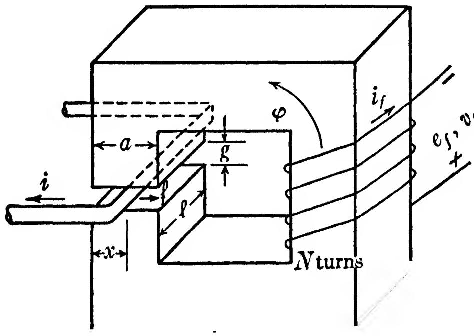


Fig. 7. Moving-conductor mechanism In normal operation, the field current i_f is substantially constant and the armature current i is a function of time.

cations the response of the device should be proportional to the actuating cause. In an ammeter, for example, to have the indication of the instrument proportional to the current in the winding is usually an advantage. In a loud-speaker, the pressure produced in the air should be proportional to the voltage applied at the terminals. For the loud-speaker, the requirements are severe, in that the factor of proportionality should be independent not only of amplitude but also of frequency over wide ranges. Moreover, the device should introduce no frequency which does not exist in the actuating voltage. Any departure from these conditions causes *distortion*, which means a lack of proportionality between cause and effect. The device to be analyzed comprises a conductor which has length ℓ in the field between the poles of an electromagnet. The conductor is free to move in a path perpendicular to its length and parallel to the pole faces. To avoid possible confusion with the field winding, the moving element is called the *inductor*.

For the configuration shown in Fig. 7 the force f on the inductor is

$$f = \mathcal{B}li. \tag{9}$$

Any consistent system of units may be used. This relation is derived on the assumption that there is no magnetic material in the field. In nearly every practical device of this type, magnetic material is present, but its effect on the simple relationship in Eq. 9 may be quite negligible, as demonstrated by the following analysis of the device illustrated in Fig. 7. The effects of hysteresis and eddy currents are not included in this analysis. The omission is justified because the effects are unimportant in devices of the type considered and can very easily be measured if a value is desired.

The energy stored in the magnetic circuit is the summation of $(\mathcal{H}\mathcal{B}/8\pi)dv$ over the entire volume. Since $\mathcal{B}$ is approximately the same both in the iron and in the air gap, and since the permeability of the iron may be assumed to be very large with respect to that of the air gap, $\mathcal{H}$ in the iron is considered to be negligible with respect to $\mathcal{H}$ in the air, and the entire stored energy may be considered to be concentrated in the air gap. Since μ has the constant value μ_0 in air, the energy is

$$W_m = \frac{\mathcal{B}^2 v}{8\pi\mu_0} \tag{15}$$

where v is the volume of the air gap. Fringing effects are assumed to be negligible.

The flux density due to the field winding is

$$\mathcal{B}_f = \frac{4\pi N i_f \mu_0}{g}. \tag{16}$$

In addition is the flux due to the current in the inductor. This flux appears only in that part of the gap which is linked by the inductor circuit, that is to say, to the left of the inductor in Fig. 7. The flux density due to the inductor current i is

$$\mathcal{B}_i = \frac{4\pi i \mu_0}{g}. \tag{17}$$

The energy in terms of these two components is then

$$W_m = \frac{lg}{8\pi\mu_0}\left[(\mathcal{B}_f + \mathcal{B}_i)^2 x + \mathcal{B}_f^2(a - x)\right], \tag{18}$$

where x is the distance between the left-hand edge of the pole face and the inductor, and a is the total width of the pole face. This energy is a function of both current and position of inductor.

The voltage drop between the terminals of each circuit is made up of

three parts: the drop through the resistance, the drop caused by changes *in current, and the electromotive force generated by the motion of the* inductor. It is convenient to consider the first two drops as being external to the actual conversion mechanism, that is to say, as a resistance and an inductance between the terminals and the mechanism proper.

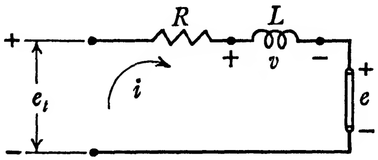


FIG. 8. Inductor circuit of moving-conductor mechanism.

The inductor circuit may be represented as in Fig. 8.

The rate at which the energy in the field changes with the displacement of the inductor is

$$\left.\begin{aligned}\frac{\partial W_m}{\partial x} &= \frac{\ell g}{8\pi\mu_0}\left[(\mathcal{B}_f + \mathcal{B}_i)^2 - \mathcal{B}_f^2\right] = \frac{\ell g}{8\pi\mu_0}(2\mathcal{B}_f + \mathcal{B}_i)\mathcal{B}_i \\ &= \left(\mathcal{B}_f + \frac{\mathcal{B}_i}{2}\right)\ell i.\end{aligned}\right\} \quad [19]$$

The rate of transfer of energy with respect to x into the field winding from its electrical source is

$$\frac{\partial W_f}{\partial x} = \frac{\dfrac{\partial W_f}{\partial t}}{\dfrac{dx}{dt}} = \frac{1}{\mathfrak{v}} e_f i_f, \quad [20]$$

where e_f is the electromotive force generated in the field winding as a result of the motion of the inductor. Since

$$e_f = N\mathcal{B}_i\ell\mathfrak{v}, \quad [21]$$

then

$$\frac{\partial W_f}{\partial x} = \mathcal{B}_i N i_f \ell = \mathcal{B}_f \ell i. \quad [22]$$

In like manner, the rate of transfer of energy to the inductor from the electrical source is

$$\frac{\partial W_i}{\partial x} = \frac{ei}{\mathfrak{v}} = (\mathcal{B}_f + \mathcal{B}_i)\ell i. \quad [23]$$

Finally, the rate at which energy is supplied to the device from the mechanical source is

$$\frac{\partial W_M}{\partial x} = \frac{\partial W_M}{\mathfrak{v}\,\partial t} = \mathfrak{f}, \quad [24]$$

where f is the external force exerted on the inductor. The effects of friction, stiffness, and inertia are considered to be segregated externally to the conversion mechanism just as the resistance and inductance of the inductor circuit and field winding are segregated externally.

The rate at which energy is delivered to the device must equal the rate of increase in energy in the field; hence

$$\frac{\partial W_f}{\partial x} + \frac{\partial W_i}{\partial x} + \frac{\partial W_M}{\partial x} = \frac{\partial W_m}{\partial x}. \qquad [25]$$

Putting in the values found for the energy-transfer terms gives

$$f = -\left(\mathfrak{B}_f + \frac{\mathfrak{B}_i}{2}\right) l i = -\mathfrak{B}_f l i - \frac{\mathfrak{B}_f l i^2}{2N i_f}. \qquad [26]$$

The first term is the force available to do work on the external mechanical system. The second term is a unidirectional force which tends to move the inductor to link more field flux. If the current is a sinusoidal function of time, this unidirectional force varies sinusoidally about a mean value at twice the frequency of the current. In order to maintain the position of the inductor in the field, a force must be exerted by the inductor support to counteract the steady pull of the unidirectional force. The residual double-frequency component introduces distortion.

The ratio between the undesirable and the useful components of force is $i/(2Ni_f)$, in other words, half the ratio between the magnetomotive forces of the inductor circuit and the field winding. It is quite practical to have the ratio of magnetomotive forces as low as 1 to 100 in devices such as loud-speakers.

If the field and its winding are considered as a single entity, the rate at which energy is transferred to this entity as x varies is

$$\frac{\partial W_m}{\partial x} - \frac{\partial W_f}{\partial x} = \frac{\mathfrak{B}_i l i}{2} = \frac{\mathfrak{B}_f l i^2}{2N i_f}, \qquad [27]$$

which is equal in magnitude to the unidirectional component of force resulting from the energy transfer which integrates to zero over a cycle. In so far as the unidirectional-force term is negligible,

$$\frac{\partial W_i}{\partial x} + \frac{\partial W_M}{\partial x} = 0, \qquad [28]$$

and the rate at which energy is supplied to the conversion mechanism from the induction circuit is equal at every instant to the rate at which the conversion mechanism supplies energy to the mechanical system, and the process is completely reversible. For this condition the following

simple relations hold:

$$f = \mathcal{B}li, \tag{9}$$

$$e = \mathcal{B}l\Omega, \tag{8}$$

where $\mathcal{B}$ is the flux density in the field produced by the magnetomotive force of the field winding. Equations 8 and 9 are related through the expression for conservation of energy:

$$ei = f\Omega. \tag{29}$$

The energy interchanges associated with changes in current remain to be investigated. If x is held constant, the time rate at which the energy in the field is increasing because of change in the current i is, from Eq. 18,

$$\frac{dW_m}{dt} = \frac{lgx}{4\pi\mu_0}(\mathcal{B}_f + \mathcal{B}_i)\frac{d\mathcal{B}_i}{dt}. \tag{30}$$

Expressing $\mathcal{B}_f$ and $\mathcal{B}_i$ in terms of the currents gives

$$\frac{dW_m}{dt} = xl(Ni_f + i)\frac{d\mathcal{B}_i}{dt}. \tag{31}$$

The rate of change in flux through both windings is

$$\frac{d\varphi}{dt} = xl\frac{d\mathcal{B}_i}{dt}. \tag{32}$$

Hence the induced voltage drops are

$$v = xl\frac{d\mathcal{B}_i}{dt} \tag{33}$$

and

$$v_f = Nxl\frac{d\mathcal{B}_i}{dt} \tag{34}$$

The power supplied to the field from the two circuits is

$$(vi + v_f i_f) = xl(Ni_f + i)\frac{d\mathcal{B}_i}{dt} = \frac{dW_m}{dt}. \tag{35}$$

The energy changes therefore do not involve the mechanical system. Provided, as assumed, that the field current is maintained constant, the interchange of energy between the field circuit and the field affects no other part of the system. The energy interchanged between the inductor circuit and the field is simply energy associated with the self-inductance of the inductor circuit. This part of the inductance, which depends only

on the inductor flux which links the field winding, has a magnitude

$$L = \frac{v}{\dfrac{di}{dt}} = xl\frac{4\pi\mu_0}{g}. \qquad [36]$$

If now x varies about a mean position x_0, there are a fixed component of inductance

$$L_0 = x_0 l\frac{4\pi\mu_0}{g} \qquad [37]$$

and a variable component

$$L_r = xl\frac{4\pi\mu_0}{g}. \qquad [38]$$

If x and i vary sinusoidally, there is a self-inductance drop which varies at double frequency and which therefore tends to cause distortion. The magnitude of the distorting voltage is

$$v_r = xl\frac{d\mathcal{B}_i}{dt} \qquad [39]$$

Since x and $\mathcal{B}_i$ are assumed to vary sinusoidally with time,

$$v_r = \omega x l\mathcal{B}_i = \mathcal{V} l\mathcal{B}_i. \qquad [40]$$

The electromotive force due to the motion of the inductor in the flux of the field winding is

$$e = \mathcal{B}_f l\mathcal{V}. \qquad [41]$$

The ratio between the distorting voltage v_r and the electromotive force e resulting in useful power transfer is $\mathcal{B}_i/\mathcal{B}_f$, twice the ratio between the undesirable and the useful force. Hence the general criterion to insure negligible distortion from these two causes is that the flux density $\mathcal{B}_i$ due to the inductor circuit shall be negligible with respect to the flux density $\mathcal{B}_f$ due to the field winding.

If a permanent magnet is used to maintain the field, an analysis is difficult since there is no general agreement among authorities as to the exact functioning of permanent magnets. Experience has shown, however, that devices of the kind under consideration do operate successfully if a permanent field magnet is used. It follows that there must be a sufficient storage of energy to take care of the reversible interchanges which are known to occur between the field and its supply. The reserve stored energy may reside in the magnetic material, in the leakage flux field, or in both.

In practical devices which employ an electromagnet, the assumption that the field current is constant can hardly be justified. Voltages are

induced in the field winding by the motion of the inductor and by changes in current in the inductor circuit through the mutual inductance. Unless these voltages can be counterbalanced, changes in field current result. Changes in the flux density resulting from these and other causes can be minimized if the pole faces are saturated.

The undesirable effects of the magnetomotive force of the inductor can be nearly eliminated by the use of a compensating winding connected in series with the inductor and designed to produce a magnetomotive force which balances that of the inductor in its mean position. The effect is to reduce greatly both the self-inductance of the inductor circuit and the mutual inductance with the field winding. The elimination cannot be complete because the coupling varies with the position of the inductor.

8. Moving-iron mechanism

Figure 9 illustrates a device typical of the group in which electrical energy can be converted to, and from, mechanical form by virtue of the displacement of an element which varies the reluctance of a magnetic circuit.

The main features in the functioning of moving-iron mechanisms are brought out in the brief analysis given in Art. 5 for an isolated induct-

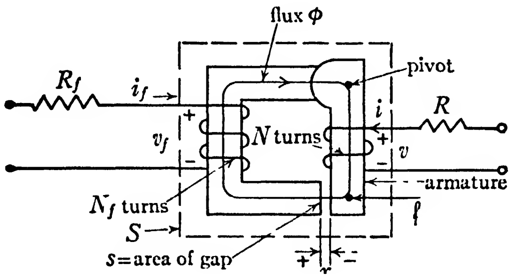


Fig. 9. Moving-iron mechanism.

ance whose magnitude varies with a displacement. A more detailed analysis follows. The procedure is similar to that used for the moving-conductor device.

The device shown in Fig. 9 includes a magnetic circuit and two windings. The magnetic circuit comprises a path formed by material of high permeability, presumably iron, and an air gap whose length x is variable. The field winding has N_f turns and carries current i_f which in normal operation is maintained substantially constant. The second winding, which is called the armature winding, consists of N turns and forms part of the electrical system whose energy is to be converted to or from

mechanical energy. The current i in this winding is variable in normal operation. The resistance of each winding is assumed to be external to the device as indicated in the figure. Only those energy changes which take place inside the dotted line, S, are considered. Electrical energy may be supplied to the device through either of the windings by virtue of the current in the winding and the voltage generated by changes in flux. Mechanical energy may be supplied to the system by the application of a force f which changes the length of the air gap.

The magnitude of the force tending to shorten the air gap can readily be determined from the rate of change of energy in the system as x varies under such conditions that only mechanical energy can be applied to the system. This condition is fulfilled if the current in each winding is varied in such a manner that there is no change in flux through either winding. No electromotive force is then generated in the winding, and consequently no electrical energy can be supplied.

For this condition,

$$\frac{dW_M}{dx} = f = \frac{dW_m}{dx}, \tag{42}$$

where W_M is the mechanical energy supplied to the system and W_m is the energy in the system, that is, the summation of $(\mathcal{H}\mathcal{B}/8\pi)dv$ throughout the volume of the magnetic circuit. But since the flux through the windings is maintained constant, $\mathcal{B}$ is constant except for possible changes in leakage flux (which are assumed to be negligible) and $\mathcal{H}$ is constant in the iron. Hence the only changes are in the air gap. If the distribution of flux in the air gap is not uniform or is modified by changes in length, the calculation of the energy change may be a complicated process. But if the air gap is short in comparison to the dimensions of the cross section and if the gap has parallel faces, it is substantially correct to assume that the flux is uniformly distributed in the gap and that the flux goes in straight lines across the gap without appreciable fringing at the edges. The energy in the gap is then restricted to a volume sx, where s is the area of the gap and

$$f = \frac{dW_m}{dx} = \frac{d}{dx}\left(\frac{\mathcal{H}\mathcal{B}}{8\pi}sx\right) = \frac{s\mathcal{B}^2}{8\pi\mu_0}. \tag{43}$$

The force thus is shown to be independent of anything in the magnetic circuit outside the air gap provided leakage flux is negligible.

All other changes in energy in the magnetic circuit are due to changes in flux and are supplied by the electrical systems. These changes result either from changes in magnetomotive force or from changes in reluctance, that is, from changes in the variable current i or the variable gap length x, both of which are functions of time.

The rate at which energy is taken from the field supply is

$$\frac{dW_f}{dt} = v_f i_f = N_f i_f \frac{d\varphi}{dt}, \tag{44}$$

where v_f is the voltage induced by change of flux due to motion of the armature or change of i.

Since i_f is constant,

$$W_f = N_f i_f \varphi, \tag{45}$$

and there is no net transfer of energy if the flux φ varies about a constant mean value.

In like manner, the rate at which energy is taken from the electrical system associated with the armature winding is

$$\frac{dW_a}{dt} = vi = Ni\frac{d\varphi}{dt}. \tag{46}$$

Since i is a function of time, there may be a net transfer of energy to, or from, the electrical system. The problem is idealized by the assumption that no energy is lost in this transfer. Actually, changes in flux in the iron involve losses due to eddy currents and hysteresis. The effect of these losses can be approximated by an increase in the apparent resistance of the winding, but this expedient does not take account of the fact that the flux in the iron lags the magnetomotive force producing it.

The flux φ depends on the variable current i and the variable air gap x. The relation involves the permeability of the iron μ. If the magnetomotive force of the armature winding is small with respect to that of the field winding, and if the change in air gap is small with respect to its mean length, the changes in flux are small with respect to the mean value, and μ may be considered to be constant. The effect of the reluctance of that part of the magnetic circuit which is in the iron may then be represented as an increase in the mean length of the air gap. Since the permeability of the iron is high, this increase is small if not entirely negligible.

The air-gap length is then given by

$$x = x_0 + \Delta x, \tag{47}$$

where x_0 includes the effect of the reluctance in the iron and Δx is the displacement from the mean position. The flux is

$$\varphi = \frac{4\pi(N_f i_f + Ni)s\mu_0}{(x_0 + \Delta x)}. \tag{48}$$

If the ratios $Ni/(N_f i_f)$ and $\Delta x/x_0$ are so small that their squares are

negligible with respect to the first powers, then

$$\varphi \approx \frac{4\pi s\mu_0 N_f i_f}{x_0}\left(1 + \frac{Ni}{N_f i_f} - \frac{\Delta x}{x_0}\right). \tag{49}$$

The voltage induced in the armature winding is

$$v = N\frac{d\varphi}{dt} = \frac{4\pi s\mu_0}{x_0}N^2\frac{di}{dt} - \frac{4\pi s\mu_0 N_f N i_f}{x_0^2}\frac{d}{dt}(\Delta x). \tag{50}$$

The term in di/dt is evidently the voltage of self-induction in an inductance of magnitude

$$L_0 = \frac{4\pi s\mu_0 N^2}{x_0}. \tag{51}$$

The second term is the electromotive force generated by the mechanical motion. This term may be written

$$e_M = \mathcal{B}_{f0}\frac{sN}{x_0}\dot{x}, \tag{52}$$

where $\mathcal{B}_{f0}$ is the flux density in the air gap due to the field winding when the gap has its mean value x_0.

The force is given by

$$f = \frac{s\mathcal{B}^2}{8\pi\mu_0} = 2\pi s\mu_0\frac{(N_f i_f)^2}{x_0^2}\left(1 - \frac{2\Delta x}{x_0} + \frac{2Ni}{N_f i_f} + \cdots\right). \tag{53}$$

Taken in order these terms represent:

(a) A force of fixed magnitude which tends to close the gap.
(b) A force proportional to the displacement which tends to increase the displacement. This component acts as negative stiffness.
(c) The useful component of force which depends on the current. This component is

$$f_M = \mathcal{B}_f\frac{sN}{x_0}i. \tag{53a}$$

In order for this device to function, the mechanical system must contain a spring which acts on the moving element. The force resulting from the mean deflection of the spring must be that required to maintain the desired mean length of air gap. The stiffness of this spring must be at least sufficient to supply the component of force which depends on the displacement Δx. It is convenient to consider the fixed component of magnetic force and the negative stiffness component as part of the mechanical system that is to result from a force acting on the armature but placed inside the dotted lines in Fig. 9, and hence to consider only the

component depending on the current i to be supplied from outside. From this point of view, the relations which determine the behavior of the device are

$$\mathfrak{f}_M = \mathfrak{B}_f \frac{sN}{x_0} i \qquad [53a]$$

and

$$e_M = \mathfrak{B}_{f0} \frac{sN}{x_0} \mathfrak{v}. \qquad [52]$$

Except for the factor of proportionality, these relations are the same as those found for the moving-conductor mechanism.

There are, however, very definite restrictions on the mechanical system that can be coupled to the moving-iron type of mechanism. For effective energy conversion, it is found in practice that the mean air-gap length should not exceed about $\frac{1}{64}$ of an inch. To minimize the distorting effects of the second-order terms which have been neglected, the deflection of the armature must be small with respect to the mean air-gap length; hence the amplitude of motion is very greatly restricted. Furthermore, if the mechanical system is to have small net stiffness, it is difficult to design a spring member which has stiffness just sufficient to compensate for the negative magnetic stiffness and which maintains accurately the mean length of air gap. As a result of these limitations, the use of devices of this type is practically restricted to telephone receivers. A permanent magnet is usually employed in place of an electromagnet to supply the unidirectional component of flux.

Moving-iron mechanisms are sometimes operated unpolarized, that is to say, with no unidirectional component of flux. An example is the "Nautophone," a type of fog signal. The mechanical force in such devices has a frequency twice that of the current producing it. The behavior of the unpolarized device depends on the terms in i^2 which have been neglected in the foregoing analysis.

9. Analysis of the moving-coil telephone

The method for finding the equivalent network for electromechanically coupled systems can best be shown by an analysis of actual devices. On account of its relative simplicity, the moving-coil telephone (Fig. 10) is an excellent introductory example.

Figure 10a shows the positions of the mechanical parts when assembled. The cylindrical coil is mounted in a radial field of flux produced by the magnet, which may be permanent or electrically excited. The coil is rigidly fastened to a flexible diaphragm in a way allowing axial motion of the coil against the elastic restraint of the diaphragm. Slightly modified

devices of this type mounted with the diaphragm in the throat of a long horn, are widely used as sound reproducers where good efficiency and relatively high output are required.

Figure 10b is a free-body diagram of the moving parts of the system, in which the mechanical parameters and reference directions for force

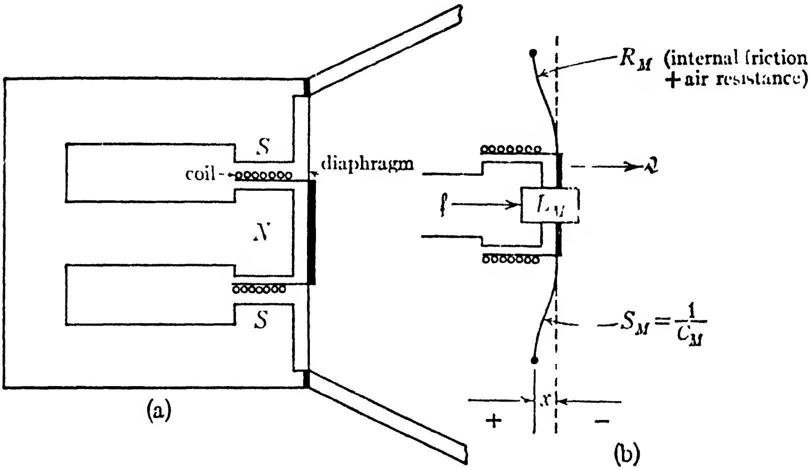


FIG. 10. Cross-sectional view of moving-coil telephone.

and velocity are clearly shown. Since there is a fixed length of conductor in the winding which is located in a uniform radial field, the electromotive force e generated by an axial velocity of the coil is given by Eq. 8a:

$$e = \mathcal{B}l v = K v, \tag{8a}$$

where K is constant.

The following relations are also important:

$$ei = f v \tag{29}$$

and

$$f = \mathcal{B}l i = K i, \tag{9a}$$

where f is the axial force acting on the winding due to the current i.

The actual electric circuit is represented by the diagram of Fig. 11. The resistance and inductance of the winding are represented by R_w and L_w, and the electromotive force generated by the motion of the winding is represented by e, between the points a and b.

Both e and i can be expressed in terms of mechanical quantities according to Eqs. 8a and 9a. Also, there is a relationship between the force f applied to the mechanical system and the velocity v at the driven point,

which may be expressed by

$$\mathfrak{f} = F(\mathfrak{v}), \tag{54}$$

where the function F is determined by the parameters of the mechanical system. Substituting e and i for $\mathfrak{v}$ and $\mathfrak{f}$, respectively, gives

$$i = \frac{1}{K} F\left(\frac{e}{K}\right). \tag{55}$$

This relation indicates how the mechanical system which is coupled to the electrical system may be represented by an equivalent electric circuit.

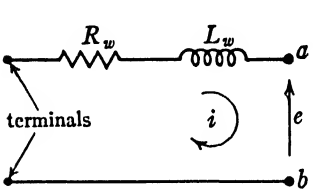


FIG. 11. Armature circuit of moving-coil telephone.

If the usual convention is followed, mechanical impedance is measured in units of force divided by velocity. In the moving-coil telephone, which is typical of many electromechanically coupled devices, mechanical impedance appears in the equivalent electric circuit as electrical admittance, as shown by Eq. 55. Specifically, if the mechanical system, including the air reaction, comprises a simple diaphragm which can be represented as a lumped mass, a concentrated stiffness, and concentrated frictional force, then the total mechanical force $\mathfrak{f}$ acting on the system is

$$\mathfrak{f} = F(\mathfrak{v}) = L_M \frac{d\mathfrak{v}}{dt} + R_M \mathfrak{v} + S_M \int \mathfrak{v}\,dt, \tag{56}$$

where L_M is the mechanical inductance or mass of the moving parts, R_M is the mechanical resistance or frictional force per unit velocity, and S_M is the mechanical elastance or stiffness. Hence, from Eq. 55

$$i = \frac{1}{K^2}\left[L_M \frac{de}{dt} + R_M e + S_M \int e\,dt\right]. \tag{57}$$

But this is the expression for the current in a circuit containing a capacitance C_{eq}, a conductance G_{eq}, and a reciprocal self-inductance Γ_{eq}, all in parallel, in terms of the voltage across them, provided:

$$C_{eq} = \frac{1}{K^2} L_M, \tag{58}$$

$$G_{eq} = \frac{1}{K^2} R_M, \tag{59}$$

and

$$\Gamma_{eq} = \frac{1}{K^2} S_M. \tag{60}$$

The subscript *eq* is used to denote circuit elements which represent the electrical equivalent of the mechanical load. The equivalent circuit diagram for the device is shown in Fig. 12. For the moving-coil telephone, the segregation of the air load from the rest of the mechanical system is desirable, since the mechanical impedance of the air load depends on the design of the horn into which the receiver works. If then the parameters in Eq. 56 represent the moving parts alone, there is an additional component of force to represent the reaction of the air on the face of the

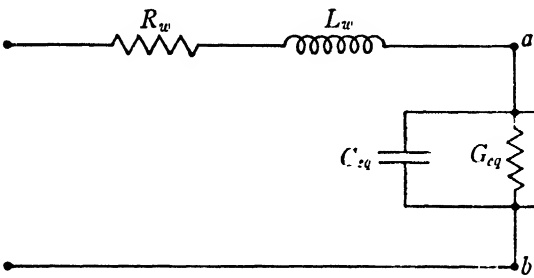


FIG. 12. Equivalent circuit of moving coil telephone, excluding effect of air load.

diaphragm, and there is a corresponding increase in the current given by Eq. 57, which requires another parallel element in the equivalent circuit. In general, the electrical admittance Y'_{eq} of the element representing the air load is $1/K^2$ times the mechanical impedance Z_a of the air load, or

$$Y'_{eq} = \frac{1}{K^2} Z_a. \tag{61}$$

It happens, however, that the expression for the mechanical admittance Y_a at the throat of a horn is more simple than that for the mechanical impedance; hence the reciprocal of Eq. 61 is more useful:

$$Z'_{eq} = K^2 Y_a. \tag{62}$$

For example, the admittance at the throat of a long conical horn is given by

$$Y_a = G_a - j\frac{B_a}{\omega}, \tag{63}$$

where G_a is determined by the area at the throat, and B_a is a function of this area and the flare of the horn. In the equivalent circuit the air load appears as a resistance R'_{eq} in series with a condenser having the elastance S'_{eq}, where

$$R'_{eq} = K^2 G_a \tag{64}$$

and

$$S'_{eq} = K^2 B_a. \quad [65]$$

The final equivalent circuit, shown in Fig. 13, indicates how the performance of the device depends on the various parameters. The power dissipated in the resistance R'_{eq} represents the useful power which is radiated as sound. At very low frequencies the output is restricted by both the relatively high impedance of the elastance S'_{eq} and the high admittance of the reciprocal inductance Γ_{eq}. The elastance S'_{eq} can be made small if the conical horn is given a very gradual flare, and Γ_{eq} can be reduced if the stiffness of the diaphragm support is lessened. At high frequencies the

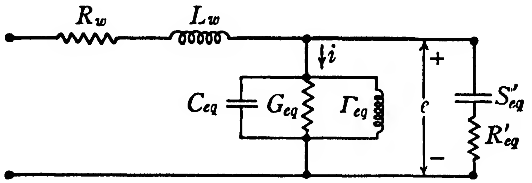


FIG. 13. Complete equivalent circuit of moving-coil telephone, including effect of air load.

output is restricted by the reactance of the winding ωL_w and the admittance of the capacitance C_{eq}. These effects can be minimized by reduction of the inductance L_w and the mass of the diaphragm which is represented by C_{eq}. Obviously, there is a maximum output at some intermediate frequency. Good response over a wide range of frequencies is obtained if the winding inductance, the mass, the stiffness, and the angle of flare are all made small.

In order that its mass may be kept small, the diaphragm is customarily constructed of thin sheet metal shaped to produce a relatively rigid central portion with a flexible annular section clamped at the periphery. In such a structure both the effective mass and the effective stiffness of the support vary with frequency, and hence the circuit elements which represent these properties in Fig. 13 are not constant. There is, however, a simple and practical means for measuring the electrical equivalent of the mechanical system at any one frequency. If the receiver works into a closed tube, the impedance of the air load may have any reactive value which is readily calculated in terms of the length of the tube. If this mechanical impedance of the air load is first made infinity and then zero, measurements of electrical impedance at the terminals give first the winding impedance alone and then the winding impedance in series with the equivalent circuit. When S'_{eq} and R'_{eq} are infinite, corresponding to a zero air load, the equivalent circuit of Fig. 13 has an open circuit in the right-hand branch.

10. Conversion of acoustical or mechanical energy to electrical energy

The moving-coil device discussed in the last article may also be used for converting acoustical or mechanical energy to electrical energy. Since the conversion is reversible, the circuit diagram already developed (Fig. 13) is applicable. The force applied to the diaphragm is represented

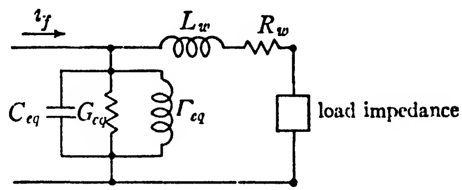


Fig. 14. Equivalent circuit of moving-coil microphone.

by a current from an outside source, and the electrical system to which power is delivered is connected to the terminals of the winding. The equivalent circuit is therefore that shown in Fig. 14. All the parameters have the same values as before.

The force f applied to the diaphragm is correctly represented by the current i_f when, according to Eq. 9a,

$$f = Ki_f. \qquad [9b]$$

11. Analysis of the D'Arsonval mechanism

Except for the fact that the mechanical motion of the D'Arsonval mechanism is rotary instead of translatory, the analysis of the devices which employ this mechanism is essentially identical with that for the moving-coil telephone instrument.

The moving element which is characteristic of the D'Arsonval devices is a rectangular winding free to oscillate about an axis in its plane against elastic restraint. Those sides of the rectangle which are parallel to the axis of rotation compose the active conductor. The magnetic circuit includes an iron drum of high permeability placed between the poles of a magnet shaped to form two air gaps of constant radial depth, as indicated in Fig. 15. The resulting fields under the poles have a substantially uniform radial distribution of flux. The winding is so mounted that the active conductor is under the poles and hence in a field of flux which is uniform over a fairly wide angular range.

The angular position of the winding may be indicated by a pointer and scale, as illustrated in Fig. 15, or by a beam of light reflected from a mirror mounted on the moving element.

The electromotive forces generated in the two sides of the winding because of the angular velocity of the winding add around the circuit; hence

$$e = \mathcal{B}\ell\mathcal{V}, \tag{8}$$

where $\mathcal{B}$ is the flux density, ℓ is the total length of active conductor, and

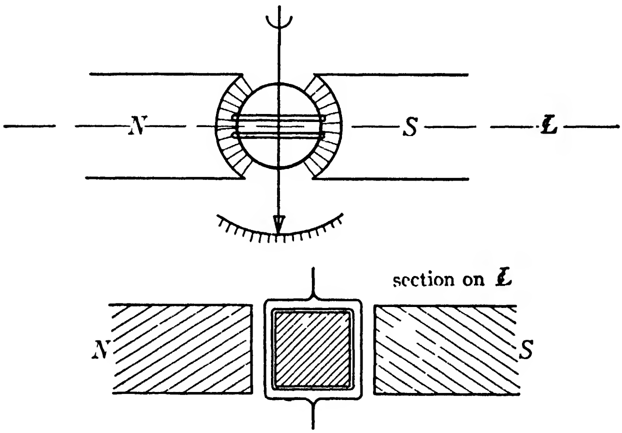


FIG. 15. Elementary diagram of D'Arsonval mechanism. (Reference should be made also to the plate on p. 281.)

$\mathcal{V}$ is the linear velocity of the active conductor. The total force f on the conductor produced by a current i in the winding is

$$f = \mathcal{B}\ell i. \tag{9}$$

This force, however, consists of two equal and opposite components in the two sides of the winding and consequently produces a pure torque

$$T = fr, \tag{66}$$

where r is the radial distance from the axis of rotation to the active conductor.

It is most convenient to analyze the device in terms of the angular displacement θ of the coil from its position of rest. Then

$$\frac{d\theta}{dt} = \frac{\mathcal{V}}{r}, \tag{67}$$

$$e = \mathcal{B}\ell r\frac{d\theta}{dt} = K\frac{d\theta}{dt}, \tag{68}$$

and

$$T = Ki, \tag{69}$$

in which K now is used for $\mathcal{B}\ell r$.

As for the telephone receiver, the electromotive force generated by the motion of the coil can be expressed as a function of the current in the coil and the parameters of the mechanical system. Thus, if

$$T = F\left(\frac{d\theta}{dt}\right), \tag{70}$$

then

$$i = \frac{1}{K} F\left(\frac{e}{K}\right). \tag{71}$$

In the conventional designs of the D'Arsonval mechanism, the mechanical circuit may properly be considered to have lumped parameters of constant value, since the coil structure is substantially rigid and the elastic member has negligible moment of inertia. It is, therefore, a simple matter to express the relation between torque and angular velocity:

$$T = F\left(\frac{d\theta}{dt}\right) = L_M \frac{d}{dt}\left(\frac{d\theta}{dt}\right) + R_M \frac{d\theta}{dt} + S_M \int \left(\frac{d\theta}{dt}\right) dt; \tag{72}$$

here L_M represents the moment of inertia; R_M represents the damping moment, or torque per unit angular velocity required to overcome friction; and S_M represents the moment of elastance, or torque per unit angular displacement.

Hence from Eq. 72,

$$i = \frac{1}{K^2}\left[L_M \frac{de}{dt} + R_M e + S_M \int e\,dt\right]. \tag{73}$$

This is the equation for the current through a capacitance, a conductance, and a reciprocal inductance in parallel, that is to say:

$$i = C_{eq} \frac{de}{dt} + G_{eq} e + \Gamma_{eq} \int e\,dt \tag{74}$$

where

$$C_{eq} = \frac{L_M}{K^2}, \tag{58}$$

$$G_{eq} = \frac{R_M}{K^2}, \tag{59}$$

and

$$\Gamma_{eq} = \frac{S_M}{K^2}, \tag{60}$$

as before.

The electrical circuit which is equivalent to the actual combined system seen from the terminals of the winding is shown in Fig. 16, in which R_w and L_w represent the resistance and inductance of the winding. The circuit is identical with that for an unloaded telephone receiver.

The D'Arsonval mechanism is used principally for the measurement of electrical quantities, however, and hence is to be treated from a point of view in which the efficiency of energy conversion is in itself unimportant. Moreover, the types of measurement to which the mechanism has been adapted are so diverse that a general treatment covering all applications is impractical. As shown in the following articles, this mechanism is used as an indicating instrument for direct currents and voltages, as a vibration galvanometer, as a ballistic galvanometer, and, in a highly specialized form, as the vibrating element of an oscillograph. The various types are

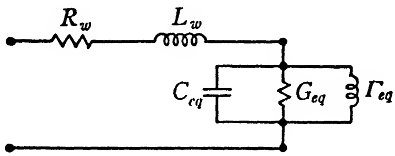


FIG. 16. Equivalent circuit of D'Arsonval mechanism.

distinguished principally by differences in their mechanical systems. In the direct-current instruments, the natural period of vibration is short enough to insure a reasonably rapid response, and the damping is close to the critical value. In the ballistic galvanometer, the natural period is relatively long and the mechanical damping may be low since it is usually convenient to obtain the desired damping in the electrical circuit. In the vibration galvanometer, the natural period is adjustable since this instrument is to be tuned to the frequency at which measurements are made; and mechanical damping is kept low, since sharp tuning is desirable. The mechanical system of a direct-current instrument can be adapted to serve effectively either as a ballistic galvanometer or as a vibration galvanometer. In the first the moment of inertia would be increased by the addition of bars or the like, and in the second the elastic restraint would be increased to raise the natural frequency to the desired value.

The vibrating element in an oscillograph may properly be treated as a specialized form of the D'Arsonval mechanism. In this application, the amplitude of vibration should be independent of the frequency, a condition which is practically fulfilled if the natural frequency of the mechanical system is considerably above the highest frequency at which measurements are made. In order that a high natural frequency may be obtained without the use of an excessively stiff elastic member, the device is

designed with as small a movable mass as is practical. These instruments are therefore quite different in appearance from the others mentioned, but the electromechanical relations involved are identical.

12. The vibration galvanometer

Of these instruments, the vibration galvanometer is treated first because analysis of it is the simplest. The vibration galvanometer is an extremely sensitive detector of currents having the frequency to which the device is tuned, and is relatively insensitive at other frequencies. It may therefore be used to detect a small current of one frequency in the presence of relatively large currents of other frequencies. This feature is of particular value in connection with those impedance bridge circuits in which the balance depends on frequency, for by using the vibration galvanometer as a detector it is possible to obtain a delicate balance at the desired frequency when harmonics are present.

When a sinusoidal voltage is impressed across the terminals of the instrument, the amplitude of vibration can be found conveniently in terms of the power transmitted to the mechanical system, for when a steady state is reached all of this power is used in overcoming mechanical friction.

The notation and the equivalent circuit developed for the D'Arsonval mechanism in Art. 11 are used for analysis of the vibration galvanometer. An effective voltage V_T is impressed across the terminals of the instrument. The impedance of the winding is Z_w, or $R_w + j\omega L_w$. An effective current I is in the winding. An effective electromotive force E is across the parallel branches of the equivalent circuit. Average power P_M is transmitted to the parallel circuit or, in other words, to the mechanical system:

$$P_M = R_M \frac{(\omega\theta_m)^2}{2} = G_{eq}E^2 = \frac{R_M}{K^2} E^2. \tag{75}$$

Hence

$$\theta_m = \frac{\sqrt{2}E}{\omega K}, \tag{76}$$

which is the absolute value of the angular amplitude. The deflection for a given voltage is inversely proportional to the frequency. Under normal conditions of use the stiffness of the suspension, represented by Γ_{eq}, is adjusted to make the deflection a maximum. This evidently is attained

when E is maximum. Since

$$\mathrm{I} = E\left(j\omega C_{eq} + G_{eq} + \frac{\Gamma_{eq}}{j\omega}\right) = \frac{\mathrm{V}_T - E}{R_w + j\omega L_w}, \qquad [77]$$

$$E = \frac{\mathrm{V}_T}{1 + (R_w + j\omega L_w)\left(G_{eq} + j\omega C_{eq} + \frac{\Gamma_{eq}}{j\omega}\right)}. \qquad [78]$$

In practical devices, ωL_w is small with respect to R_w and may be neglected. On this assumption, E is a maximum when ωC_{eq} and Γ_{eq}/ω are equal, a condition which is the criterion for mechanical resonance. Hence, when the instrument is tuned,

$$j\omega\frac{\Theta_m}{\sqrt{2}} \approx \frac{\mathrm{V}_T}{K(1 + R_w G_{eq})} = \frac{\mathrm{V}_T}{\left(K + \frac{R_w R_M}{K}\right)}, \qquad [79]$$

or

$$\theta_m \approx \frac{\sqrt{2}V_T}{\omega\left(K + \frac{R_w R_M}{K}\right)}. \qquad [79a]$$

A consistent system of units must, of course, be used for the electrical and mechanical quantities.

The implied assumption that the voltage at the terminals is independent of the current is not tenable, since in any practical application there is impedance between the voltage source and the terminals. This impedance may be considered, however, as an addition to the winding impedance, and no change in the analysis is necessary provided the reactance remains negligible. Thus,

$$j\omega\frac{\Theta_m}{\sqrt{2}} = \frac{\mathrm{E}_0}{\left[K + \frac{RR_M}{K}\right]}, \qquad [79b]$$

where E_0 is a fixed source of voltage and R is the total actual resistance in the circuit.

The value for K which makes the deflection maximum is evidently

$$K^2 = RR_M, \qquad [80]$$

equivalent to

$$R_{eq} = R. \qquad [80a]$$

That is, the angular amplitude is greatest when the equivalent impedance of the mechanical circuit equals, or matches, the impedance of the electri-

cal circuit. The value of K depends both on the active length of conductor, hence the number of turns, and on the flux density. Best results are obtained if the flux density is made as high as possible and just enough turns are used to give K the value required for impedance match. Fewer turns involve less winding resistance and hence a lower value for R. It is, of course, obvious that a low damping torque gives high sensitivity.

The selectivity of the vibration galvanometer for frequency may conveniently be expressed as the ratio of the amplitude at resonant frequency to the amplitude at twice resonant frequency when the same voltage is applied

If there is an impedance match at resonance, as expressed in Eq. 80a, and if the reactance of the electrical circuit is negligible, then at resonance the current is given by

$$I_1 \approx \frac{E_0}{R + R_{eq}} = \frac{E_0}{2R}. \qquad [81]$$

At double this frequency the equivalent electrical impedance of the mechanical system becomes negligible with respect to R and

$$I_2 \approx \frac{E_0}{R} \approx 2I_1. \qquad [82]$$

The steady-state deflection in terms of the current at any frequency may be found from the relation

$$KI = (j\omega^2 L_M + \omega R_M - jS_M)\frac{j\Theta_m}{\sqrt{2}} \qquad [83]$$

At resonant frequency ω_1

$$S_M = \omega_1^2 L_M. \qquad [84]$$

Hence

$$\Theta_{m1} = \frac{\sqrt{2}KI_1}{\omega_1 R_M}. \qquad [85]$$

At twice resonant frequency ω_2 the damping term is negligible and

$$j\Theta_{m2} \approx \frac{\sqrt{2}KI_2}{\omega_2^2 L_M - S_M} = \frac{2\sqrt{2}KI_1}{3\omega_1^2 L_M}. \qquad [86]$$

The amplitude ratio is therefore

$$\frac{\theta_{m1}}{\theta_{m2}} = \frac{3}{2}\frac{\omega_1 L_M}{R_M}. \qquad [87]$$

Analogy with a series RLS electrical circuit shows that $2L_M/R_M$ represents the time constant of the system, which may be several seconds.

The sensitivity ratio between fundamental and second harmonic is $3\omega_1/4$ times as great as the time constant. At 60 cycles per second this ratio is nearly 300 times as great as the time constant.

Both the sensitivity and the selectivity of the instrument are increased by reduction in the mechanical damping, but unless the frequency of the source is exceptionally constant, use of the device with the minimum damping which can be obtained is impractical since the tuning becomes too critical.

13. The mechanical oscillograph element

In an ideal oscillograph, the deflection is at every instant proportional to the actuating current. From Eq. 68 the voltage generated by the angular velocity of a D'Arsonval coil is

$$e = K\frac{d\theta}{dt}; \tag{68}$$

hence

$$\theta = \frac{1}{K}\int edt. \tag{88}$$

From Eq. 74 the current taken by the reciprocal inductance, which is the electrical equivalent of the elastic restraint, is

$$i = \Gamma_{eq}\int edt, \tag{74a}$$

$$i = \frac{S_M}{K^2}\int edt. \tag{89}$$

If, therefore, all the force produced by the current through the winding can be utilized in overcoming elastic restraint,

$$\theta = \frac{K}{S_M}i, \tag{90}$$

which is the ideal relationship as far as current is concerned. The required conditions are fulfilled if the moment of inertia and the damping moment are negligible with respect to the moment of stiffness of the elastic restraint. There is, of course, an upper limit to the frequency at which the condition can be satisfied.

If the instrument is used in series with a relatively large resistance, both the current and hence the deflection are proportional to the voltage across the combination.

14. Direct-current instruments

The D'Arsonval mechanism carrying direct current in the winding is in equilibrium when the elastic torque due to deflection just balances the torque produced by the current. Thus:

$$S_M\theta_s = KI, \tag{91}$$

or

$$\theta_s = \frac{K}{S_M} I. \tag{92}$$

This is the same relation as that found for the deflection of the oscillograph element (Eq. 90) since the elastic restraint is the only mechanical force involved when the frequency becomes zero. The deflection is exactly proportional to the current, provided K and S_M are exactly constant and provided there is no static friction in the movable parts. By means of

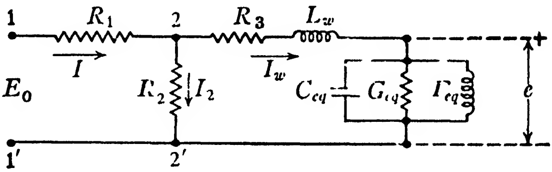


FIG. 17. Equivalent circuit for D'Arsonval instruments.

shunt and series resistors, the same mechanism may be adapted for use as either an ammeter or a voltmeter in a wide variety of systems.

The utility of direct-current instruments depends to a great extent on the transient behavior. For most purposes quick action and deadbeat damping are desirable. The behavior depends not only on the mechanical parameters but also on the electrical parameters of the circuit to which the instrument is connected.

Figure 17 shows the general circuit with the mechanical system of the instrument represented by its electrical equivalent. If the device is used as a voltmeter, the terminals are at 1–1′, R_2 is absent or is large compared with the winding resistance included in R_3, and R_1 is in effect part of the instrument. The value of R_1 should be large in relation to the resistance of the system whose voltage is to be measured, so that the current I taken by the instrument will cause a negligible voltage drop in the system.

If the instrument is used as an ammeter, the terminals are at 2–2′, R_2 is small compared with the winding resistance, and R_1 represents the resistance of the system in which the current is to be measured. For this use, the resistance of the instrument should be small with respect to R_1 so that the voltage drop through the instrument will have a negligible effect on the current. When a steady state exists,

$$I_w = \frac{R_2}{R_2 + R_3} I. \tag{93}$$

From Eq. 92,

$$\theta_s = \frac{K}{S_M} I_w. \qquad [92a]$$

Hence the relation between deflection and current is

$$\theta_s = \frac{K}{S_M} \frac{R_2}{R_2 + R_3} I. \qquad [92b]$$

But

$$I = \frac{E_0}{R_1 + \dfrac{R_2 R_3}{R_2 + R_3}} = \frac{R_2 + R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} E_0; \qquad [94]$$

so

$$\theta_s = \frac{K}{S_M} \frac{R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3} E_0, \qquad [92c]$$

which is the relation between deflection and voltage.

The exact analysis of the transient behavior is complicated by the presence of the winding inductance L_w. If the effect of this inductance is assumed to be negligibly small, the relation, and hence the conditions for critical damping, are exactly analogous to those in a series *RLS* circuit.

Typical initial conditions exist when a voltmeter is first connected to a system or when an ammeter already in circuit has a short circuit removed. In either situation the current I (Fig. 17) may properly be considered to attain its steady-state value instantaneously. The relations are:

$$I = i_2 + i_w, \qquad [95]$$

$$i_2 R_2 = i_w R_3 + e, \qquad [96]$$

$$i_w = C_{eq} \frac{de}{dt} + G_{eq} e + \Gamma_{eq} \int e\,dt. \qquad [97]$$

Hence

$$I = \frac{R_2 + R_3}{R_2} \left[C_{eq} \frac{de}{dt} + \left(G_{eq} + \frac{1}{R_2 + R_3} \right) e + \Gamma_{eq} \int e\,dt \right]. \qquad [98]$$

Substituting the values of C_{eq}, G_{eq}, and Γ_{eq} from Eqs. 58 to 60 and the value for $\int e\,dt$ from Eq. 88 gives

$$I = \frac{1}{K} \frac{R_2 + R_3}{R_2} \left[L_M \frac{d^2\theta}{dt^2} + \left(R_M + \frac{K^2}{R_2 + R_3} \right) \frac{d\theta}{dt} + S_M \theta \right]. \qquad [99]$$

The effective damping depends, therefore, not only on the mechanical damping torque R_M but also on the mechanical equivalent of the electric

resistance $R_2 + R_3$. From the analogy to the *RLS* circuit, the criterion for critical damping is

$$R_M + \frac{K^2}{R_2 + R_3} = 2\sqrt{L_M S_M}. \qquad [100]$$

15. The ballistic galvanometer

It has been shown that the D'Arsonval mechanism may be adapted by means of tuning to act as a very sensitive galvanometer at a single frequency, or by means of critical damping to act as a quick-reading direct-current instrument. A third important adaptation is that for measuring total quantities of electricity which are discharged through the instrument in a relatively short time. In this form the instrument is known as a ballistic galvanometer. An example of the method of use is to discharge a condenser through the instrument and read the maximum deflection reached by the coil. This should be proportional to the total charge initially on the condenser. Figure 18 shows the complete equivalent circuit of the ballistic galvanometer connected to a condenser whose capacitance C_x is to be determined. The procedure is to discharge the condenser C_x through the galvanometer after the condenser has been charged to a known voltage. The maximum deflection which the coil attains on its first swing is a measure of the total quantity of electricity discharged. The value of the unknown capacitance can then be determined in terms of the known voltage and the measured charge.

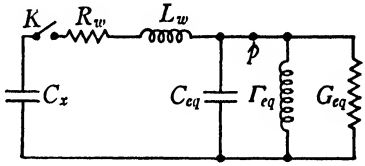


FIG. 18 Equivalent circuit of ballistic galvanometer, including condenser whose charge is to be measured.

The relation between the maximum deflection and the quantity of electricity discharged can be determined from the mechanical constants of the galvanometer and the magnitude of the electrical-circuit elements. In general, this relation is by no means a simple one. It is possible, however, to design an instrument with a deflection almost directly proportional to the quantity of electricity discharged through it. The requirements which must be fulfilled to obtain this simple relation can be deduced from consideration of the energy conversions which take place.

Initially, the energy is stored in the condenser C_x. If it were possible to open the circuit (Fig. 18) at the point marked p and then close switch K, the charge originally on the condenser C_x would be divided between the condensers C_x and C_{eq}. When equilibrium is established, the proportion of the initial charge which is in each condenser depends only on the ratio of the capacitances. The relations are as follows: If condenser C_x is charged

to an initial voltage V_0, the charge Q becomes V_0C_x. When the charge is divided, there is a voltage V on each condenser, and the total charge is unchanged. Thus:

$$Q = V_0C_x = V(C_{eq} + C_x). \tag{101}$$

Hence the charge is divided in the ratio of capacitances. The initial energy stored in condenser C_x is:

$$W_0 = \frac{1}{2} C_x V_0^2 = \frac{1}{2} \frac{Q^2}{C_x}. \tag{102}$$

After the charge is divided, the energy in the system is

$$W = \frac{1}{2} (C_x + C_{eq}) V^2 = \frac{1}{2} \frac{Q^2}{(C_{eq} + C_x)}. \tag{103}$$

The energy has therefore been reduced by the factor $C_x/(C_{eq} + C_x)$. The balance of the energy has been dissipated in the resistance R_w. The magnitudes of the resistance R_w and of the inductance L_w do not affect the proportion of energy that is dissipated, but do affect the time required for equilibrium to be reached.

If C_x is small compared with C_{eq}, it is approximately true that the energy in the system has been reduced to C_x/C_{eq} of the original energy. Furthermore, the energy finally residing in C_x is only C_x/C_{eq} of that in C_{eq}. The energy remaining in C_x may therefore be considered negligible. The energy transferred to C_{eq} is thus

$$W = \frac{1}{2} \frac{Q^2}{C_{eq}}. \tag{103a}$$

Inasmuch as C_{eq} represents the inertia of the mechanical system or, more specifically, since C_{eq} equals L_M/K^2, energy in the equivalent condenser is actually kinetic energy due to the angular velocity imparted to the coil of the instrument. The initial angular velocity is such that the generated electromotive force is equal to V.

The foregoing analysis is based on the supposition that the electrical circuit representing the mechanical system is opened at the point marked p in Fig. 18, an assumption equivalent to stating that all the energy delivered to the mechanical system is used in accelerating this system. This assumption is entirely justified, provided the mechanical damping is small, the restoring torque is weak, and the maximum angular velocity is attained in a time that is small with respect to the period of vibration of the instrument. Under these conditions, when the angular velocity is maximum, negligible elastic energy is stored in the suspension because of the small deflection and small torque, and negligible energy is dissipated

in friction because of small damping constant and small time of action. The resulting behavior of the mechanical system can then be found on the assumption that the system starts with an initial angular velocity when the displacement is zero. Further consideration of the coupling to the electrical system is not necessary since the energy in it is now negligible unless the galvanometer is shunted.

The following analysis shows that the amplitude of the first swing is proportional to the quantity of electricity Q discharged through the instrument. When the deflection has reached its maximum value, all the energy in the system is stored in the suspension, but, during the swing, energy is dissipated because of friction. Hence the initial kinetic energy less the dissipated energy equals the energy stored in the suspension. At any instant t the kinetic energy is proportional to $(d\theta/dt)^2$. The energy stored in the suspension is proportional to θ^2, and the energy dissipated is proportional to $\int_0^t \left(\frac{d\theta}{dt}\right)^2 dt$. Since the system is linear and the initial conditions are fixed, the angular velocity can be expressed as the initial angular velocity $(d\theta/dt)_{max}$ multiplied by a function of time. Since the time required to attain the maximum deflection θ_{max} does not depend on the initial angular velocity, the dissipated energy at the end of the swing is proportional to $(d\theta/dt)^2_{max}$ and hence is a fixed part of the initial kinetic energy.

It follows that θ^2_{max} is proportional to $(d\theta/dt)^2_{max}$ and hence to Q^2, and therefore θ_{max} is proportional to Q. The factor of proportionality can be found from the deflection occurring when a known quantity of electricity is discharged through the instrument.

The linear relation between deflection and charge, it should be noted, depends on establishing a definite initial condition in the mechanical system, the condition being that all the energy received by the system is used in accelerating the system to a maximum angular velocity before there is appreciable deflection. The two conditions which must be satisfied to obtain this result are: The time required to establish equilibrium in the electrical circuit must be small with respect to the mechanical period of the galvanometer, and the energy remaining in the electrical circuit when equilibrium is established must be negligibly small with respect to that which is delivered to the mechanical system. These conditions are fulfilled if the moment of inertia of the mechanical system is sufficiently large and the stiffness of the suspension sufficiently small. In a typical instrument, the capacitance of the condenser which represents the moment of inertia in the equivalent electrical circuit is of the order of 200 microfarads. This means that if a condenser of about two microfarads is to be measured, the energy remaining in the condenser after the initial discharge is about one per cent of that delivered to the mechanical system. This

fact does not imply that the measured capacitance will be in error by one per cent, since part of this residual energy is delivered to the mechanical system during its swing.

With any reasonable value of resistance in the electrical circuit, the time required for establishment of electrical equilibrium after the discharge of a condenser of two microfarads capacitance is of the order of 1/1,000 of a second or less. The period of the mechanical system is usually of the order of 15 seconds. Hence the condition regarding relative periods is well fulfilled.

The factor of proportionality between deflection and charge may be found in terms of the constants of the instrument by solution of the equation of motion, which is

$$L_M \frac{d^2\theta}{dt^2} + R_M \frac{d\theta}{dt} + S_M\theta = 0. \tag{104}$$

The initial conditions are

$$\theta_0 = 0, \tag{105}$$

$$\left(\frac{d\theta}{dt}\right)_0 = \left(\frac{d\theta}{dt}\right)_{max}. \tag{106}$$

Since the initial kinetic energy in the mechanical system is $\frac{1}{2}(Q^2/C_{eq})$,

$$\frac{1}{2} L_M \left(\frac{d\theta}{dt}\right)^2_{max} = \frac{1}{2}\frac{Q^2}{C_{eq}}. \tag{107}$$

Hence from Eqs. 107 and 58

$$\left(\frac{d\theta}{dt}\right)_{max} = \frac{K}{L_M} Q. \tag{108}$$

The solution for the differential equation 104 for the initial condition of Eqs. 105 and 106 is

$$\theta = \left(\frac{d\theta}{dt}\right)_{max} \frac{\epsilon^{-\alpha t}}{\omega_d} \sin \omega_d t = \frac{KQ}{L_M}\frac{\epsilon^{-\alpha t}}{\omega_d} \sin \omega_d t, \tag{109}$$

where

$$\alpha = \frac{R_M}{2L_M}, \tag{110}$$

$$\omega_d^2 = \frac{S_M}{L_M} - \alpha^2. \tag{111}$$

The maximum deflection occurs when $d\theta/dt$ is zero, as it is when $\tan \omega_d t$ is ω_d/α, or when

$$\sin \omega_d t = \frac{\omega_d}{\sqrt{\alpha^2 + \omega_d^2}}, \tag{112}$$

or when

$$t = \frac{1}{\omega_d} \tan^{-1} \frac{\omega_d}{\alpha}. \tag{113}$$

Substituting this value of time from Eq. 113 into Eq. 109 gives the maximum deflection

$$\theta_{max} = \frac{KQ}{\sqrt{L_M S_M}} \epsilon^{-(\alpha/\omega_d)\tan^{-1}(\omega_d/\alpha)}. \tag{114}$$

The usual custom is to express this relation in terms of the undamped period of oscillation T_0. If R_M is zero, the natural angular frequency of the system is given by Eq. 111 when α is zero. This is

$$\omega_0 = \sqrt{\frac{S_M}{L_M}}, \tag{115}$$

and

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{L_M}{S_M}}. \tag{116}$$

Hence

$$\sqrt{S_M} = \frac{2\pi\sqrt{L_M}}{T_0}. \tag{117}$$

Then

$$Q = \left[\frac{2\pi L_M}{KT_0} \epsilon^{(\alpha/\omega_d)\tan^{-1}(\omega_d/\alpha)}\right]\theta_{max} \tag{118}$$

$$= K'\theta_{max}. \tag{118a}$$

The factor of proportionality K' between maximum deflection and charge is thus evident from Eq. 118.

16. Electrostatic coupling illustrated by condenser transmitter

Just as electromechanical energy interchange is possible in an electromagnetic system if a mechanical displacement changes the energy in the magnetic field, so energy interchange is possible in an electrostatic system if a mechanical displacement changes the energy in the electric field.

Figure 19 shows a system representative of the usual types of condenser transmitters and similar devices, in which one plate of a capacitor is movable in such a way that its displacement changes the capacitance.

The analysis of this mechanism parallels very closely the analysis of the moving-iron system given in Art. 8.

The diaphragm is assumed to be a plane rigid piston, parallel to the fixed electrode and moving perpendicularly to it. The spacing is assumed to be small compared to the cross-sectional dimensions of the electrodes, so that fringing can be neglected, and the dielectric is assumed to be air. The energy in the electric field is then uniformly distributed and equal to

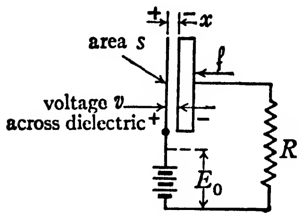


FIG. 19. Circuit diagram of condenser transmitter.

$$W_e = sx\frac{\mathfrak{D}^2}{8\pi\varepsilon_0}. \qquad [119]$$

The force acting on the diaphragm can be found from the energy balance as the diaphragm moves, under the condition that the electric flux remains constant in the capacitance, for as long as the electric flux remains constant there can be no current in the electric circuit and therefore no energy can be supplied from the electric system. The force is

$$f = \frac{dW_e}{dx} = \frac{d}{dx}\left(\frac{s\mathfrak{D}^2x}{8\pi\varepsilon_0}\right) = \frac{s\mathfrak{D}^2}{8\pi\varepsilon_0}. \qquad [120]$$

Any other changes in the energy in the electric field are associated with changes in electric flux and are supplied from the electric circuit. These changes are caused either by a change in voltage applied to the capacitor or by a change in the capacitance which accompanies motion of the diaphragm.

The electric flux density $\mathfrak{D}$ can be expressed in terms of the voltage applied to the capacitor and the spacing between the electrodes:

$$\mathfrak{D} = \frac{\varepsilon_0 v}{x}. \qquad [121]$$

In practice, the variable voltage v is the sum of a fixed biasing voltage E_0 and a relatively small variable component Δv. The spacing also is the sum of a constant initial displacement x_0 and a relatively small variable component Δx.

In terms of these components, the electric flux density is

$$\mathfrak{D} = \frac{\varepsilon_0 E_0}{x_0}\left(1 + \frac{\Delta v}{E_0} - \frac{\Delta x}{x_0}\right), \qquad [122]$$

analogous to Eq. 49. The current in the electric circuit is

$$i = \frac{s}{4\pi}\frac{d\mathfrak{D}}{dt} = \frac{s\varepsilon_0}{4\pi x_0}\frac{d\Delta v}{dt} - \frac{s\varepsilon_0}{4\pi x_0^2}E_0\frac{d\Delta x}{dt}. \qquad [123]$$

The first term is a displacement current into a capacitance of magnitude

$$C_0 = \frac{s\varepsilon_0}{4\pi x_0}. \qquad [124]$$

The second term is a current proportional to velocity in the mechanical system. This term may be written

$$i_M = \frac{C_0 E_0 \mathfrak{v}}{x_0}. \qquad [125]$$

The external force applied to the movable plate is

$$\mathfrak{f} = \frac{s\mathfrak{D}^2}{8\pi\varepsilon_0} = \frac{sE_0^2\varepsilon_0}{8\pi x_0^2}\left(1 + \frac{-2\Delta x}{x_0} + \frac{2\Delta v}{E_0} + \cdots\right), \qquad [126]$$

analogous to Eq. 53. The three terms on the right represent, respectively:

(a) A constant force which tends to close the air gap;

(b) A force proportional to the displacement which tends to increase the displacement;

(c) A force proportional to the variation in voltage applied to the capacitance. This third term can be rewritten

$$\mathfrak{f}_M = \frac{C_0 E_0}{x_0}\Delta v. \qquad [127]$$

It is necessary to balance the first two components of force mechanically if the device is to work, just as was done for the moving-iron mechanism. It is likewise convenient to consider these terms as an internal part of the mechanical system, and to treat the third term as the means of energy conversion. From this point of view, the relations which determine the behavior of the device are

$$\mathfrak{f}_M = \frac{C_0 E_0}{x_0}\Delta v, \qquad [127]$$

and

$$i_M = \frac{C_0 E_0 \mathfrak{v}}{x_0}. \qquad [125]$$

If this type of mechanism is used to convert electrical energy to mechanical energy and if a large mechanical force is desired, the capacitance must be large and the voltage gradient E_0/x_0 must be large. A large capacitance means either a large area, in which case the force is distributed widely, or very small spacing, in which case the allowable amplitude is restricted. The force per unit area is limited by the voltage gradient at which the dielectric breaks down. The dielectric must be compressible, and if air is used, the limiting gradient is about 30,000 volts per centi-

meter. At this gradient, the constant force of attraction, given by the first term in Eq. 126, is approximately 60,000 dynes per square centimeter, and the useful component, given by the third term of that equation, must be small compared to this if distortion is to be avoided. Such a mechanism is therefore impractical if large amounts of power are to be handled, although it is very satisfactory as a microphone and for similar low-power applications.

A conventional condenser transmitter has a diaphragm approximately $1\frac{5}{8}$ inches in diameter, a spacing between electrodes of about 0.002 inch, and an output of $\frac{1}{300}$ volt per dyne per square centimeter for a biasing voltage of 200 volts. The natural frequency of the diaphragm for such a transmitter may be about 6,000 cycles per second.

The proportionality between force and voltage and between current and velocity determines the analogue that must be used for the equivalent electrical circuit of an electrostatically coupled mechanical system, as pointed out in Art. 3. No electrostatic mechanism has been developed which is the exact dual of the moving-conductor electromagnetic device, although the Van de Graaff generator is a very close approach.[3]

17. Piezoelectric and magnetostrictive coupling

The operation of piezoelectric and magnetostrictive devices as means for energy conversion between an electric system and a mechanical system can be introduced very briefly. The physical phenomena involved in piezoelectric or magnetostrictive action are not the concern of this chapter, any more than were the physical phenomena of ferromagnetism or dielectric behavior.

In a piezoelectric crystal, voltage applied between a pair of electrodes gives rise to a reasonably proportional stress in the crystal. If the crystal is deformed, a charge appears on the electrodes proportional to the strain. This is equivalent to a current proportional to rate of change of strain. The operation of the piezoelectric element is therefore identical with the operation of the electrostatic coupling element described in the last article:

$$\mathfrak{f}' = Ke, \qquad [128]$$

$$i = K\mathfrak{v}', \qquad [129]$$

where $\mathfrak{f}'$ represents stress and $\mathfrak{v}'$ represents rate of change of strain.

[3] R. J. Van de Graaff, K. T. Compton and L. C. Van Atta, "The Electrostatic Production of High Voltage for Nuclear Investigations," *Phys. Rev.*, XLIII (1933), 149–157; L. C. Van Atta, D. L. Northrup, C. M. Van Atta and R. J. Van de Graaff, "The Design, Operation, and Performance of the Round Hill Electrostatic Generator," *Phys. Rev.*, XLIX (1936), 761–776.

Similarly, the magnetostrictive device is found to be identical in its operation to the electromagnetically coupled mechanism. Current in a coil surrounding a magnetostrictive bar gives rise to a stress in the bar proportional to the current. Strain in the bar produces a change in flux through the coil, so that a voltage proportional to rate of change of strain is induced in the coil:

$$f' = Ki, \qquad [130]$$

$$e = K\vartheta'. \qquad [131]$$

Beyond this point, the analyses that have been applied to the electrostatic and electromagnetic systems can be applied respectively.

PROBLEMS

1. A spring motor is "wound up" when the torque of the spring reaches 7.5 lb-in. The moment of elastance is 0.035 lb-in/radian. The moment of friction is 1.75 lb-in/radian/sec. A constant torque of 10 lb in is applied to the spring shaft for winding.

(a) The equivalent parallel electric circuit and the equivalent series electric circuit are to be drawn.
(b) How long does it take to wind the motor?

2. A contact-making pendulum consists of a 100-g ball suspended by a thread so that the center of the ball hangs 1 m below the point of suspension. A fine wire projects from the bottom of the ball and makes contact with a mercury drop each time the pendulum swings through the vertical. The pendulum is released from a position 5° from the vertical and is allowed to swing freely. At the end of the first return swing, the pendulum fails to reach the starting point by 1% of the initial amplitude; at the end of the second return swing the pendulum comes within 1% of reaching the amplitude of the first return swing, and so on. Amplitudes are measured from the position of rest.

(a) What are the numerical values of moment of inertia, moment of friction, and moment of elastance?
(b) Two electric circuits mathematically equivalent to the mechanical system are to be shown.
(c) How many times does the pendulum make contact before its amplitude is decreased to 20% of the initial amplitude?

3. A steam turbine and an alternator have their rotors on the same shaft.

Moment of inertia of turbine rotor	2940 lb-ft^2
Moment of inertia of alternator rotor	5180 lb-ft^2
Torsional constant of shaft	42.1×10^6 lb-ft/radian

A short circuit on the alternator suddenly impresses a torque of 150,000 lb-ft on that rotor.

(a) What is the maximum torque on the shaft?
(b) Two equivalent electrical circuits are to be drawn.

4. A dynamic loud-speaker element such as is schematically represented in Fig. 10, when operating without the horn attached, has the following constants:

R_w	L_w	R_M	L_M	S_M	$\mathcal{B}$	l
10 ohms	0.1 mh	100 dyne cm^{-1} sec	0.2 g	5×10^5 dyne cm^{-1}	10^4 gauss	100 cm

What is the apparent electrical impedance of the winding at a frequency of 250 ~

(a) without the horn but with the diaphragm free to move?

(b) with the diaphragm clamped so that the winding cannot move?

5. A vibration galvanometer has the following constants when tuned to 60 ~

R_w	L_w	R_M	L_M	$\mathcal{B}$	l	r
700 ohms	10^{-4} h	0.018 dyne cm radian^{-1} sec	0.015 g cm^2	10^4 gauss	50 cm	0.25 cm

(a) What is the current sensitivity of this galvanometer expressed as effective amperes per radian of angular displacement of the coil at resonance?

(b) What is the current sensitivity to a third harmonic of the tuned frequency?

The equivalent electrical circuit should be drawn, giving the values of electrical parameters in the practical system of units.

6. The following questions refer to the vibration galvanometer of Prob. 5.

(a) What is the impedance of the galvanometer at resonance?

(b) What is the impedance of the galvanometer at the frequency of the third harmonic?

(c) What is the impedance of the galvanometer at zero frequency?

7. A vibration galvanometer is tuned to 60 ~. It has a winding resistance of 815 ohms. The voltage sensitivity at resonance is 0.47×10^{-3} v (effective) per millimeter deflection (total width of light band). The current sensitivity at resonance is 2.2×10^{-8} amp (effective)/mm. The direct-current sensitivity is 40×10^{-6} amp/mm. The scale distance is 100 cm.

(a) What are the values of the elements in the equivalent circuit for this galvanometer?

(b) What is the efficiency of the galvanometer in converting electrical energy into mechanical energy?

(c) What is the constant K for this galvanometer?

8. The following data are for a vibration galvanometer:

R_w	L_w	R_M	L_M	S_M	$\mathcal{B}$	l	r
5 ohms	5×10^{-5} h	4π dyne cm radian^{-1} sec	4 g cm^2	5.68×10^5 dyne cm radian^{-1}	20,000 gauss	50 cm	1.0 cm

The scale distance is 50 cm. A voltage source having electromotive force of $\mathcal{R}_e[E_m\epsilon^{j\omega t}]$ and internal resistance of 75 ohms is applied to the galvanometer.

(a) What is the width of the light beam on the scale as a function of the magnitude E of the source voltage? How does this width vary with frequency in the vicinity of resonance? Reasonable approximations in the calculations are permissible.

(b) If the source voltage is suddenly applied, what time is required for the amplitude of oscillation to build up to 95% of its final value?

(c) How are the sensitivity and frequency selectivity of this instrument affected by the moment of friction and by the internal resistance of the source? How is the time of build-up as determined in (b) affected by the frequency selectivity of the device?

9. The following questions relate to the vibration galvanometer of Prob. 8.

(a) At approximately what frequency do the half-power points of the galvanometer circuit occur?

(b) If the galvanometer is fed from a voltage source having negligible internal resistance, is the sensitivity of the system increased or decreased? Does the system become more selective or less selective with regard to frequency?

10. The movement of a D'Arsonval galvanometer has the following constants:

R_w	L_w	L_M	S_M
5 ohms	0.05 mh	5.0 g cm^2	5,000 dyne cm radian^{-1}

It has a sensitivity which yields a full-scale deflection of $\pi/2$ radians when 10 ma pass through its winding. The mechanical damping of the movement is adjusted until a steady full-scale deflection is reached in a minimum time interval when full rated current is suddenly applied to the terminals.

(a) What is the resulting moment of friction of the movement?

(b) What time is required for the deflection to reach 99% of its final steady value?

(c) If the same movement is provided with a series resistance so that the instrument is suitable for use as a voltmeter, and the value of this resistance is adjusted to make full-scale deflection correspond to 150 v, how long does it take for the reading to reach 99% of its final value when rated voltage is suddenly applied?

11. The data for the movement of a D'Arsonval galvanometer follow:

R_w	L_w	L_M	S_M	$\mathcal{B}$	l	r
20 ohms	negligible	5 g cm^2	5,000 dyne cm radian^{-1}	5,000 gauss	400 cm	1.0 cm

Full-scale deflection corresponds to an angular displacement of $\pi/2$ radians.

(a) What is the series resistance R_S necessary so that the instrument may be used as a voltmeter with full-scale deflection corresponding to 10 v?

(b) What value of the moment of friction R_M makes the instrument critically damped when used as a voltmeter with the series resistance calculated in (a)?

12. A ballistic galvanometer having the following constants is to be used to measure the capacitance of some condensers:

Flux density in the region of the coil	10,000 gauss
Effective radius of the coil about its axis of rotation	2 cm
Number of turns in the coil	200
Effective length of each turn	4 cm
Moment of inertia of the movable system	15.07 g cm^2
Moment of elastance of the movable system . .	1.77 dyne cm/radian
Moment of friction of the movable system . . .	negligible

A condenser of 0.5 μf capacitance charged to 2 v is suddenly connected across the terminals of the moving coil.

(a) What is the maximum angular velocity of the coil in radians/sec?

(b) What is the maximum angular displacement of the coil in radians?

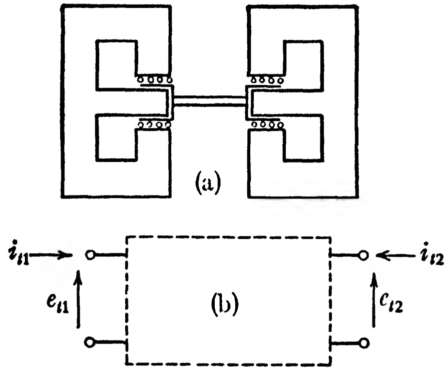


FIG. 20. Electromechanically coupled device for Prob. 13.

13. Figure 20a shows an arrangement of two coils situated in radial magnetic fields as in the moving-coil telephone. The coils are rigidly connected by means of a rod and supported so that both coils and the rod may move as a unit in the horizontal direction. There is no elastic suspension.

The windings have effective lengths ℓ_1 and ℓ_2 and the field strengths are $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The winding resistances and inductances are R_{w1}, L_{w1}, and R_{w2}, L_{w2}, respectively. The net mass of the moving system is L_M, and the mechanical friction constant is R_M.

What equivalent electrical circuits can be interposed between the two pairs of winding terminals in the form indicated in Fig. 20b?

CHAPTER XIII

Transients in Nonlinear Circuits

1. General considerations

Circuits which are *nonlinear* in that their parameters are not constant are frequently encountered in the practice of electrical engineering. They include, for example, iron-cored magnetic devices and electronic devices which are indispensable components of much essential apparatus. The methods for solving nonlinear circuits which are developed in this chapter are predicated upon a knowledge of the terminal characteristics or *external characteristics* of the circuit apparatus, such as a curve of current versus terminal voltage for nonlinear resistance. The theories of internal behavior which explain these external characteristics are not, however, a concern of this chapter.* Understanding of the problems offered by nonlinear circuits is most logically built up from consideration of linear-circuit theory — discussed in earlier chapters — which treats circuits wherein the component resistance, inductance, and elastance parameters may be assumed to be constant. That is, the values of these parameters may be assumed to be independent of current and time, as well as of conditions that might arise as a result of, or in the course of, the operation of a circuit of which they are a part.

The general mathematical formulation of the electrical relationships of a linear circuit yields linear differential equations with constant coefficients, hence the term *linear* is applied to these circuits. The solution of a linear differential equation of a physical system demonstrates what one might intuitively surmise; namely, that the resultant effect is linearly proportional to the cause. For example, in a linear mechanical system the displacement is proportional to the force. Also, the analyses presented in previous chapters demonstrate that when a voltage whose time variation is sinusoidal is applied between two terminals of a linear network, the time variation of the resultant steady-state current is sinusoidal. The ratio between the effective value of the voltage and the current or between the maximum value of the voltage and the current is a constant, and is expressed by a constant quantity Z, termed the impedance. The impedance is a function of the frequency, the resistance, the inductance, and the elastance involved but is independent of the voltage or the current amplitude as well as of the time.

* Theories of the behavior of iron-cored and electronic devices are presented in this series in the volume on magnetic circuits and transformers, and in the volume on electronics, respectively.

The representation of the circuit elements in terms of constant parameters such as resistance R, inductance L, and elastance S for purposes of analysis is an idealization of a condition that never occurs exactly in electrical-engineering practice. In many physical circuits, although the magnitudes of the parameters actually do vary for any of several reasons explained later, the degree to which they vary from an average value is so small that to assume the parameters constant leads to results sufficiently accurate for most purposes. The assumption makes possible a differential equation that is relatively simple to solve, whereas taking the small variations of the parameters into account in analyzing these circuits involves needless mathematical complication and labor.

Decision as to when the parameters may be assumed to be constant depends, of course, on how large a departure may be considered negligible. Here the engineer must exercise judgment. Frequently, the magnitudes of the parameters R, L, and S vary with any of several quantities; for example, temperature, pressure, current density, voltage, time, age of the apparatus, or its past history. Usually, the answer to the question whether or not to assume the parameters as constant in specified problems depends on the relative value of the more nearly accurate results that would be obtained if the variations of the parameters were not ignored. To secure this answer the complexities in the analysis, and hence the cost of the added time required by the more nearly exact solution, must be weighed. Unfortunately, the question is not always a clean-cut one, and it is for this reason that considerable judgment is often necessary when making an answer.

An idea of the complexities involved in the analysis when parameter variations are considered may be gained from a brief review of the mathematical formulation of a simple problem. The differential equation which describes the circuit behavior mathematically in terms of the variables i and t, or e and t, is readily formulated. For example, for a series RLS circuit with an applied voltage $e(t)$, the equation is

$$Ri + \frac{d(Li)}{dt} + S\int i dt = e(t). \qquad [1]$$

This equation can be solved readily only when the coefficients or parameters R, L, and S are constants; that is, when the equation is a linear differential equation with constant coefficients. When the parameters are functions of time, the equation is still called a linear differential equation, and is then said to have time-varying coefficients. Under these conditions it can be solved with reasonable ease only for a few special problems. The principle of superposition can be applied to the analysis only of circuits with elements whose parameters are constant, or vary as functions of time. Thus, the currents produced by two independent voltages acting

simultaneously can be obtained if first the current produced by each voltage acting alone is determined, and then these are added to obtain the total current. Evidently, when the parameters R, L, and S are assumed constant, any effects of factors which might cause them to vary do not enter into the solution, since they do not enter into the formulation of the circuit equation. Hence, when the parameters vary, their variations must be expressed as functions of the current, the voltage, or the time, or of current or voltage and the time, before the problem can be properly expressed in differential-equation form Except when the variations occur only as functions of time, the resulting differential equation is a nonlinear one. Even though it can be formulated, its solution frequently necessitates considerable expenditure of effort unless use is made of some form of machine aid such as the differential analyzer whose operation is described briefly in Art. 10.

In circuits where, among other factors, temperature, current-density, voltage, or age produces variations that appreciably modify the behavior of the circuit from the behavior that is indicated if the parameters are assumed to be constant, the parameters affected are said to be variable. When the magnitudes of the parameters can be expressed as explicit functions of current or voltage, the elements are said to be nonlinear because the mathematical formulation of the problem involves a nonlinear differential equation. When such variations occur, it is necessary to distinguish clearly among those elements that are constant with respect to time, current, or voltage; those that vary only with the time; those that vary only with current or voltage and not explicitly with the time; and, finally, those that vary with the voltage, or the current, and explicitly with the time; so that the problem may be properly formulated. For convenience in the discussions of circuits which follow, circuit elements are classified — in the next article — into four broad groups, namely:

(a) circuit elements whose parameters are constant,
(b) circuit elements whose parameters vary only with time,
(c) circuit elements whose parameters vary only with voltage or current,
(d) circuit elements whose parameters vary with the voltage or current and with the time.

Typical examples of circuit elements within each group are given. Selected methods for analyzing circuits comprising some of these various classes of element are treated in this chapter, and in the subsequent volumes on magnetic circuits and transformers and on electronics. However, the problems in this field are so numerous and so varied that many present special features of their own. Other examples can be found in the bibliographical material, page 764.

2a. Classes of Nonlinear Circuit Elements: Circuit Elements Whose Parameters Are Constant

All circuit apparatus is so constituted that it exhibits simultaneously the properties of resistance, inductance, and elastance to a large or small degree. Frequently, one or the other of these parameters predominates, so that under particular conditions of operation the apparatus may be represented by resistance, inductance, or elastance elements, respectively, depending on which property predominates.

A resistor is said to have constant resistance when the ratio of voltage to current is substantially constant. More generally, it is convenient to describe a resistor as having constant resistance when the ratio of a direct voltage to direct current, or the ratio of sinusoidal alternating voltage to the component of steady-state sinusoidal alternating current in phase with the voltage, is substantially constant throughout the range of voltage, current, and time entering into the problem under consideration.

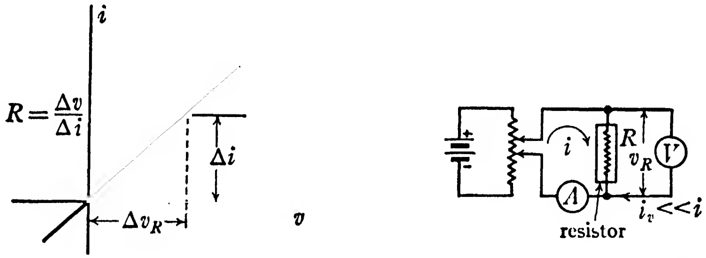


Fig. 1a. Volt-ampere characteristic for a constant-resistance element. The value of R is independent of v_R, i, and t.

Fortunately, a large number of resistors are constructed so that their resistance does not change appreciably over the whole range of operation occurring in their practical application. In practice, a metallic resistor is used generally under conditions for which its resistance is essentially constant throughout reasonable periods of time when it is carrying its rated current. Familiar exceptions are, of course, those elements in which heat produced by current characterizes the functioning of the device; as, for example, tungsten-filament lamps, electric heaters, ballast lamps,* and so on. The volt-ampere† characteristic for a resistor is the curve showing the relation between the voltage across the resistor and the current through it. For a resistor of constant resistance the curve is a straight line passing through the origin, as in Fig. 1a. The reciprocal of the slope of the line is constant and equal to the resistance R of the element.

* Plates on pp. 76 and 77.

† Although it is considered that usage justifies the term *volt-ampere*, as applied herein, the term *current-voltage* is more consistent with the terminology used to describe other types of parameter characteristic.

An element is said to have constant inductance when the component of voltage developed across its terminals by magnetic induction is linearly proportional to the rate of change of current in the element, and is independent of any other factors; that is, the inductance parameter is then definable by the relation

$$v_L = L\frac{di}{dt}, \tag{2}$$

where v_L is the inductive voltage, and L is the constant inductance. Alternatively, the inductance may be defined by the relation

$$\left.\begin{aligned} v_L &= N\frac{d\varphi}{dt}, \\ &= N\frac{d\varphi}{di}\frac{di}{dt}, \\ &= L\frac{di}{dt}, \end{aligned}\right\} \tag{2a}$$

where N represents the number of turns in the element that link the flux φ, and i is the current which establishes the flux φ. Thus, an element may

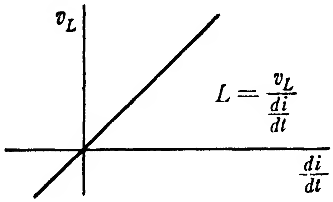


FIG. 1b. Characteristic showing voltage as a function of rate of change of current for a constant inductance element. The value of L is independent of v_L, i, and t.

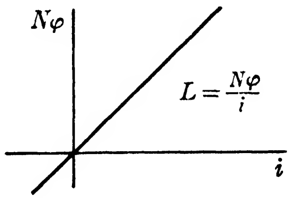


FIG. 1c. Characteristic showing flux linkages as a function of current for constant inductance element. The value of L is independent of v_L, i and t.

be said to have constant inductance L when the number of flux linkages $N\varphi$ per unit of current i is a constant. Many practical inductors are constructed so that their inductance is essentially constant over wide ranges of current or frequency encountered in their practical operation. The inductive voltage across their terminals for a given rate of change of current is in no way dependent upon the magnitude of the current, its frequency, its direction, the age of the inductance, or the extraneous magnetic fields that may surround the inductor. A volt-ampere characteristic for an inductance element evidently does not exist, but the characteristic showing the inductive voltage as a function of the rate

of change of current does exist and is important. Alternatively, the characteristic showing the flux linkages $N\varphi$ as a function of current i also exists. Figures 1b and 1c represent such characteristics for a linear inductance element. The slope of the line in each figure is constant and is equal to the inductance.

An element is said to have constant elastance or capacitance when the component of voltage appearing across its terminals as a result of its elastance property is a constant times the electric charge stored in it; that is, the capacitive or elastance parameter is defined by the relation

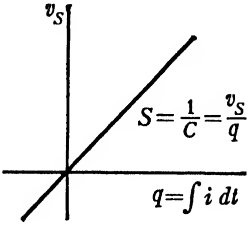


FIG. 1d. Characteristic showing the voltage as a function of the charge for a constant elastance element. The value of S or C is independent of v_S, q, i, or t.

$$v_S = S\int idt = Sq = \frac{q}{C}, \tag{3}$$

where v_S is the elastance voltage, S is the constant elastance equal to the reciprocal of the constant capacitance C, i is the current at its terminals, and q is its stored charge. The relation is in no way dependent upon the voltage, the time, the age of the capacitor, or increasing or decreasing of the charge. Elastance elements whose parameters are essentially constant over the range of their practical operation are encountered frequently in engineering. The voltage-charge curve for a capacitor is a convenient way of expressing its terminal characteristic. For a constant value of elastance the line representing the characteristic is straight and its slope is equal to the value of the elastance S. Figure 1d represents such a characteristic.

2b. Circuit elements whose parameters vary only with time

Circuit elements whose parameters vary explicitly with the time but are independent of changes in voltage or current are called *time-varying parameter elements*. At any instant of time

$$v_R = R_t i, \tag{4}$$

$$v_L = \frac{d(L_t i)}{dt} = L_t\frac{di}{dt} + i\frac{dL_t}{dt}, \tag{5}$$

$$v_S = S_t\int idt, \tag{6}$$

or, alternatively,

$$i = \frac{d(C_t v_S)}{dt} = C_t\frac{dv_S}{dt} + v_S\frac{dC_t}{dt}, \tag{7}$$

where v_R, v_L, and v_S are, respectively, the resistance, inductance, and elastance voltages at time t, and R_t, L_t, S_t, C_t are, respectively, the resistance, inductance, elastance, and capacitance at time t. As time increases, the parameters R, L, and S of these elements assume different magnitudes independently of v or i.

Many electrical devices may be represented by time-varying parameter elements. For example, the carbon-granule microphone has a resistance that is varied by the pressure of a diaphragm, the movement of which is effected by a source of pressure -- the sound wave -- separate from the electrical system. Hence, for analysis of the electrical system, described in terms of i or v, and t, the resistance R must be treated as varying with t independently of v or i. The apparent plate-circuit resistance of

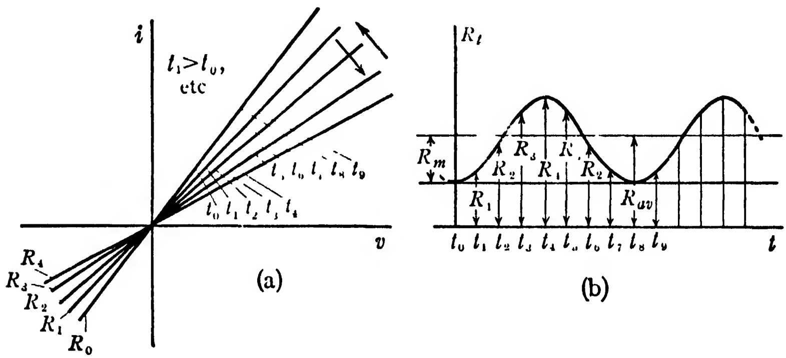


FIG. 2. Volt-ampere characteristic for a resistance which varies sinusoidally with time and the corresponding curve of resistance versus time.

a screen-grid vacuum tube under certain conditions is substantially a function of the grid voltage alone. Consequently, a variation of the grid voltage effects a time variation of this apparent plate resistance. Similarly, the elastance or capacitance of a condenser microphone is varied by the sound wave and therefore experiences a time variation. The inductance of the alternating-current winding of a synchronous motor or generator also is varied as the angular position of the rotor varies, and therefore has a time variation. Similar conditions arise in other devices frequently encountered in engineering practice.

A typical volt-ampere characteristic of a time-varying resistance element is depicted in Fig. 2a. At any instant of time t_0, t_1, $\cdots$, the characteristic is a straight line through the origin, but the slope of this line varies from instant to instant. The characteristics for time-varying inductors or capacitors are similar in shape but have different quantities as dependent and independent variables. A sinusoidal time variation of this resistance is illustrated in Fig. 2b, as shown by the plot of R_t as a function of t.

A circuit containing time-varying parameter elements and constant-parameter elements is called a *time-varying parameter circuit.* Its analysis leads to a linear differential equation with time-varying coefficients. For the element of Fig. 2 the resistance coefficient is a sinusoid plus a constant; specifically,

$$R_t = R_{av} - R_m \cos t. \qquad [8]$$

Even if the circuit is relatively simple, containing only a resistance whose time variation about a mean value R_{av} is sinusoidal, and having a constant voltage impressed across its terminals, the time variation of current is not sinusoidal:

$$i = \frac{E}{R_{av} - R_m \cos t} = \frac{E}{R_{av}}\left[1 + \frac{R_m \cos t}{R_{av}} + \frac{R_m^2 \cos^2 t}{R_{av}^2} + \frac{R_m^3 \cos^3 t}{R_{av}^3} + \cdots\right]. \qquad [9]$$

The principle of superposition holds for time-varying parameter circuits. Exact solutions may be obtained in some problems by strictly analytical methods; approximation methods or methods using mechanical calculating aids may be used with success in many others. For numerous problems, however, solutions have not yet been satisfactorily obtained.

An important distinction must be noted between the effect of a time-varying resistor on the energy relations of a circuit and that of a time-varying inductor or elastor. A time-varying resistor may be thought of as a valve, by means of which an external agency of some kind can control the current in an electric circuit; thus, a change in the pressure applied to a microphone can control or alter the current in the electric circuit. This control is sometimes termed modulation. The relation between an external force and the resistance variation that it produces is not mutual, however, since a resistance can convert electrical energy only into heat, never directly into mechanical energy. On the other hand, for a time-varying inductor or a time-varying elastor there is a mutual relation between the inductance or capacitance[1] variation, respectively, and the mechanical force giving rise to this variation, since these elements have the property of being able to store energy. By means of these elements energy can be transferred from the electrical to the mechanical system, giving rise to motor operation; or from the mechanical to the electrical system, giving rise to generator operation. In a time-varying resistor, however, energy cannot be transferred from the seat of the control to the electric circuit; only the energy relations within the electric circuit can be controlled or modulated.

[1] W. L. Barrow, "On the Oscillations of a Circuit Having a Periodically Varying Capacitance," *I.R.E. Proc.*, XXII (1934), 201 212.

2c. Circuit elements whose parameters vary only with voltage or current

Circuit elements whose parameters depend on the voltage or current in a manner that is characteristic of the particular element but do not vary explicitly with the time are called *nonlinear parameter elements.* In order that the terminology be consistent with that used for elements of class b, these elements should be called current- or voltage-varying parameter elements. The term "nonlinear," instead of the terms "current-" or "voltage-varying," is customary terminology, however, because the solutions of circuits comprising elements of this class, no matter how simple, lead to nonlinear equations.

Important examples of apparatus whose resistance is nonlinear are the mercury-arc rectifier, the thermionic-vacuum tube, and the copper-oxide or iron-selenium barrier-layer rectifier. An inductor whose inductance is nonlinear is obtained when, in order to increase the inductance per unit

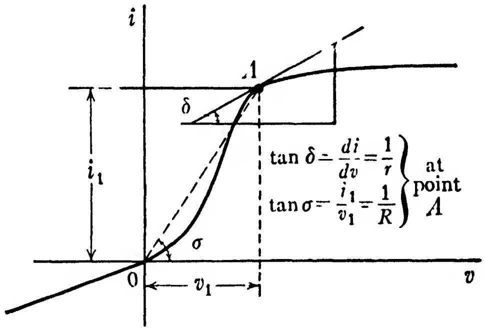


Fig. 3a. Volt-ampere characteristic of a nonlinear resistor.

volume or per unit weight, iron or an iron alloy is inserted in the field of the inductor. Because of the magnetic properties inherent in the iron, the inductance becomes dependent on the current in the coil, on whether the current is increasing or decreasing, and on other variables to a lesser degree. The power dissipated in the iron in such apparatus -- termed the core loss — is also a function of these variables. Three important devices whose inductance is nonlinear are the iron-cored inductor or choke coil, the iron-cored transformer, and the dynamo machine. Also, most present-day capacitors of the electrolytic kind or of the kind that use synthetic liquid dielectric compounds of high electric constant, such as those marketed under the trade names Pyranol and Inerteen, have a nonlinear elastance or capacitance.

A typical volt-ampere characteristic of a nonlinear resistor is shown in Fig. 3a. Usually the curve is unsymmetrical about the origin and has a

variable slope. Sometimes the curve of resistance as a function of current is convenient in depicting the behavior of the element. Where it is used, distinction must be made between the *static* resistance R, as defined by the ratio of total voltage across the terminals of the resistor to the current entering the resistor, and the ***dynamic*** resistance r, as defined by the ratio of incremental change in the voltage to the incremental change in current which it produces. Thus, if v is the voltage across the terminals of the resistor and i is the current at its terminals,

$$R = \frac{v}{i} \qquad [10]$$

and

$$r = \frac{\Delta v}{\Delta i} = \left.\frac{dv}{di}\right|_{\Delta v \to 0.} \qquad [11]$$

Figure 3b shows the shapes of the curves of static and dynamic resistance as functions of voltage for the nonlinear resistor typified by the curve of Fig. 3a.

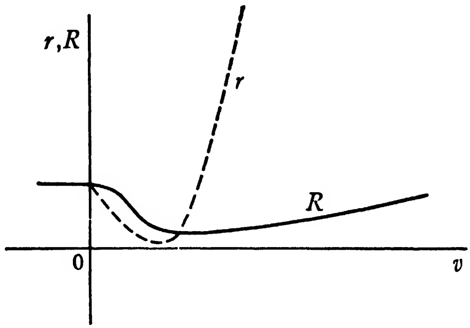


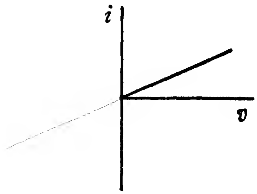
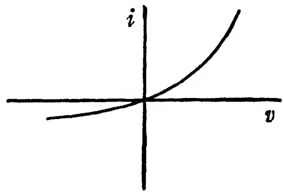
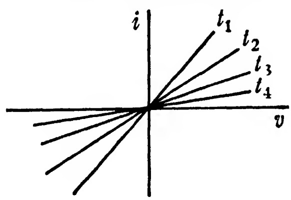
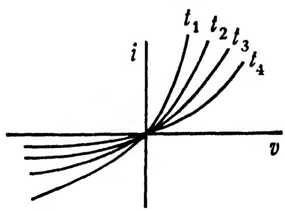
FIG. 3b. Curves of static resistance R and dynamic resistance r, as functions of voltage for the nonlinear resistor specified by Fig. 3a.

The characteristics of nonlinear inductors or capacitors are seldom susceptible of simple graphical representation. To a first approximation, however, the characteristic curve giving the apparent inductance as a function of the current for a nonlinear inductor would resemble that given in Fig. 3c, for inductance defined by the relation,

$$v_L = L\frac{di}{dt}. \qquad [2]$$

Inductance elements of this kind frequently introduce double-valued functions. The characteristics of the nonlinear elastance or capacitance parameter of many capacitors are even less susceptible of graphical representation than are the parameters of nonlinear inductors. Capacitance of this

CLASSIFICATION OF ELEMENTS

	Elements with linear characteristics	*Elements with nonlinear characteristics*
Independent of time	a. Constant Resistance  $\frac{v}{i} = R = \text{constant}$ Similarly, $L = \text{constant}$, $S = \text{constant}$. Superposition theorem applies. Linear differential equations have constant coefficients R, L, and S. Solutions in exact form relatively easy.	c. Nonlinear Resistance  $\frac{v}{i} = R(i)$ or $R(v)$. Similarly, inductance $= L(i)$, or $L(v)$ elastance $= S(q)$ or $S(v)$. Superposition theorem does *not* apply. Nonlinear differential equations have coefficients R, L, and S that are functions of current or voltage. Except in a few simple circuits or where mechanical calculating aids are available, only approximate solutions are possible.
Varying with time	b. Time-Varying Resistance  $\frac{v}{i} = R(t)$. Similarly, inductance $= L(t)$, elastance $= S(t)$. Superposition theorem applies. Linear differential equations have coefficients R, L, and S that are functions of time. Exact solutions possible but difficult. Mutual relations between outside agency and electric system for time-varying inductors and capacitors.	d. Time-Varying Nonlinear Resistance  $\frac{v}{i} = R(t, i)$ or $R(t, v)$ Similarly, inductance $= L(t, i)$, elastance $= S(t, q)$. Superposition theorem does *not* apply Nonlinear differential equations have coefficients R, L, and S that are functions of current or voltage and of time. Solutions obtained with difficulty, usually by use of mechanical calculating aid or by reduction to class a, b, or c.

class is rarely used in circuits where behavior of the capacitors must be accurately predictable. When precise performance or accurate prediction of behaviors is essential, capacitors whose parameters are essentially linear are used. They invariably involve a higher cost per unit of capacitance than do the nonlinear ones, but the added cost is usually not large compared with the economic worth of the device of which the capacitors form part.

When comparatively simple circuits, comprising, for example, a series connection of a nonlinear resistance and a linear or nonlinear inductance

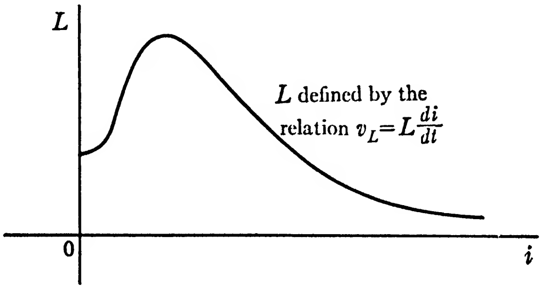


FIG. 3c. Characteristic showing apparent inductance L, as a function of current i, for an iron-cored inductor.

or capacitance, are to be analyzed, the analytical solution of the resulting nonlinear differential equation is only moderately difficult. In complicated circuit connections, however, the complexities of the solution become considerable. The difficulty of representing the nonlinear relationships either graphically or analytically makes some solutions impossible, even if mechanical calculating aids are available. The great convenience of the superposition principle cannot be appreciated fully until the solution of problems such as these wherein it cannot be used has been attempted.

2d. Circuit elements whose parameters vary with the current or voltage and with the time

Circuit elements whose parameters vary with the voltage or the current, as well as explicitly with the time, are called *nonlinear time-varying parameter elements*. If the properties of a large number of circuit elements are examined in sufficient detail, many of the elements are found to be of this class. Whenever possible, however, in engineering calculations these elements are assumed to belong to one of the three simpler classes already discussed, for solution of equations taking full account of the properties of parameters of this kind frequently becomes impractical.

Most multielectrode vacuum tubes have properties that classify them

as elements of this fourth kind. The plate-resistance parameter of such a tube, defined as

$$r_p = \frac{dv_p}{di_p} \qquad [12]$$

for constant grid voltage, where v_p is the plate voltage and i_p the plate current, is a nonlinear function of v_p, and undergoes a time variation

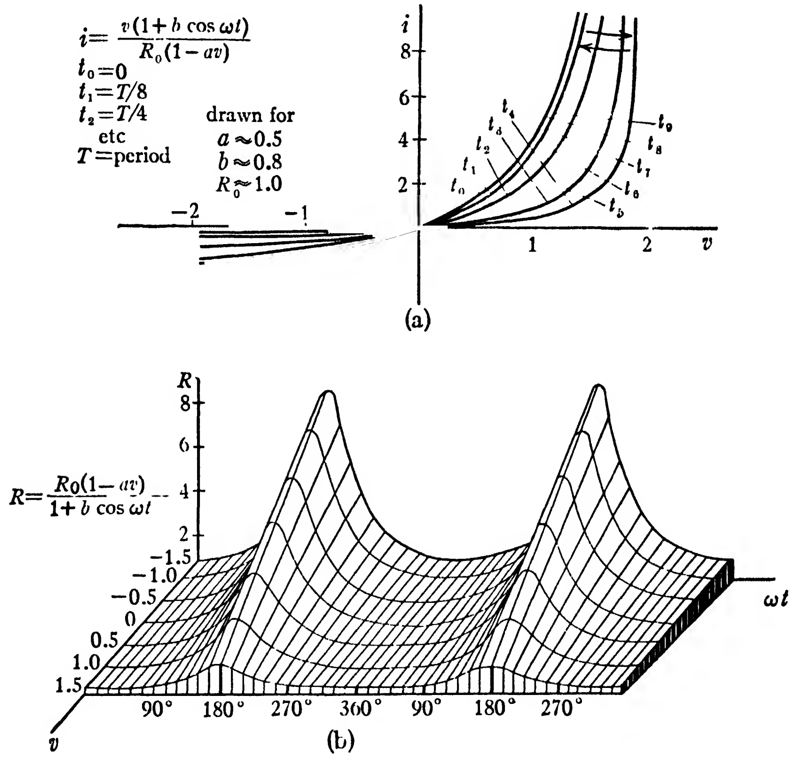


FIG. 4. Volt-ampere characteristic for a time-varying nonlinear resistance element, and the corresponding curve (b) of resistance versus angular frequency and voltage.

when the voltage of the grid experiences a time variation. Also, more detailed description of the behavior of a synchronous generator or motor is possible in terms of parameters of this class than in terms of parameters which vary only with time or only with current.

The general aspects of the volt-ampere characteristics of resistance elements of this class are indicated by the diagrams shown in Fig. 4a. The characteristic is curved, and it is different at different instants of time t_0, t_1, $\cdots$. A periodic time variation is illustrated in the figure in which

the current has the analytical expression

$$i = \frac{v(1 + b \cos \omega t)}{R_0(1 - av)}, \tag{13}$$

where a, b, and ω are constants, and v and t are independent variables Inasmuch as the resistance is a function of two independent variables,

$$R = \frac{R_0(1 - av)}{1 + b \cos \omega t}, \tag{14}$$

and a surface, a section of which is shown in Fig. 4b, is required to represent the dependence of the resistance on these variables. Similarly, a surface is required to represent the corresponding characteristic curves of this class of inductance or elastance parameter. Circuits containing elements of this class are often called *time-varying nonlinear parameter circuits*. Their behavior is given by nonlinear differential equations with nonlinear time-varying coefficients. Consequently, *the superposition principle is not applicable* and exact solutions are generally impossible. Approximate solutions for some of these problems may be obtained by reducing the circuit to one having a time-varying parameter, or by graphical, machine, or other methods such as those outlined in Arts. 4 to 10.

The summary on page 667 shows briefly the characteristics of the four classes of parameter, giving typical curves for the four classes of resistance parameters. The corresponding characteristic curves of the four classes of inductance and elastance parameters, when they are single valued, are similar except that different variables appear as co-ordinates.

3. Data needed for nonlinear circuit calculations

The functional relationships between the parameters of a circuit and the dependent or independent variables of the circuit, or both, must be known if an analytical or graphical solution of a problem is to be found; or if the performance of devices that compose the circuit is to be predicted. Sometimes, as demonstrated later, analytical expressions for these relationships can be obtained. For other relationships, only graphical expressions based on experimentation can be obtained.

The science of electronics has progressed to the stage where the relationships between the variables in a few relatively simple nonlinear conduction phenomena can be predicted within engineering accuracy by strictly analytical methods, provided the materials and the dimensions of the parts composing a device are known. The two-electrode vacuum diode with a pure-metal cathode may often be considered as in this category. However, the relationships involved in the majority of time-varying, or nonlinear, or time-varying nonlinear conduction-phenomena devices

such as most kinds of rectifiers, gas-discharge tubes, multielectrode vacuum tubes, and the like, can be expressed conveniently only by characteristic curves obtained from measurements performed on typical devices or on constituent parts of a particular device.

At present, the theory of magnetism is not well enough understood to make possible purely theoretical prediction of the behavior of magnetic

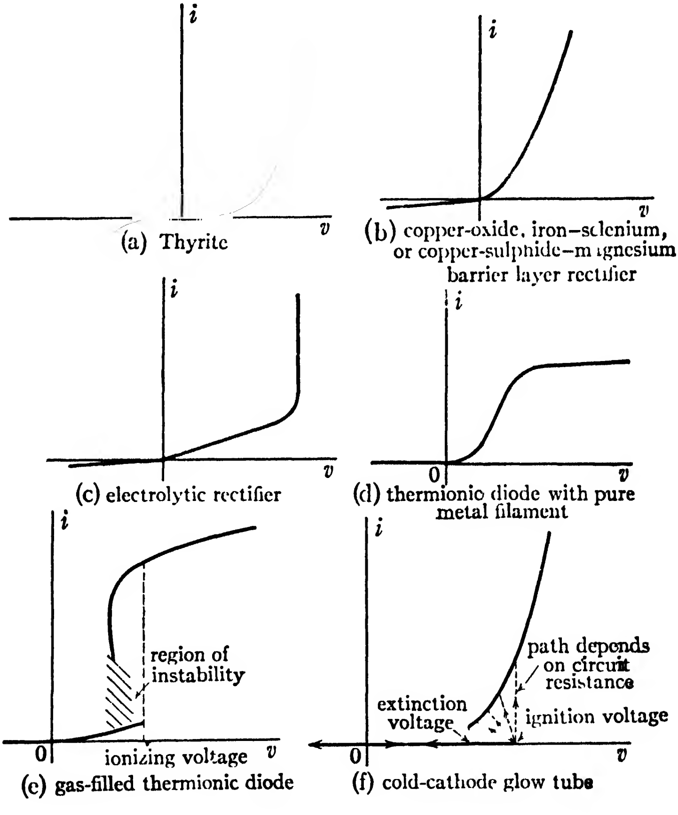
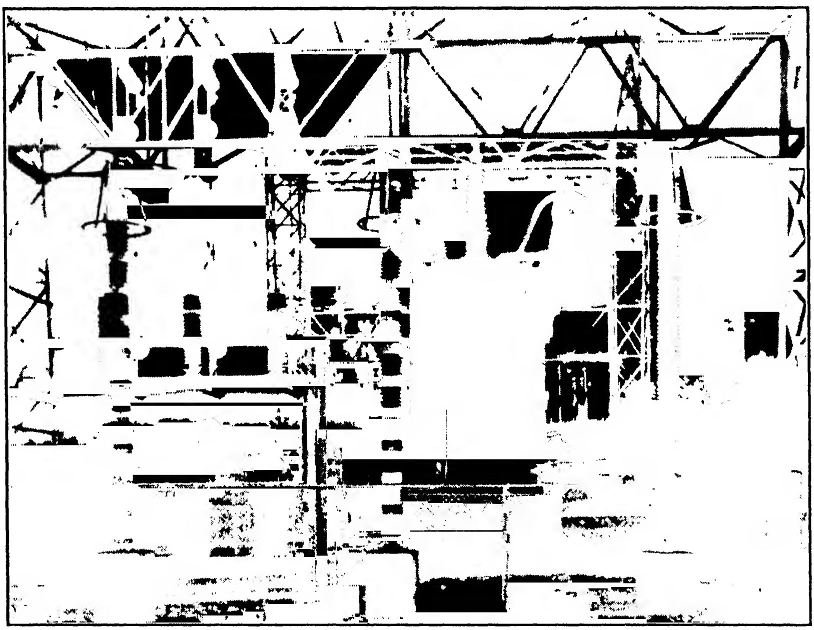


FIG. 5. Nonlinear volt-ampere characteristics.

materials. Data for determination of the magnitudes of the inductance parameter in devices involving magnetic-induction phenomena in the presence of ferromagnetic materials are hence obtained from measurements performed on samples of every batch of magnetic material manufactured. These data are expressed in the form of characteristic curves of the material. When the materials are fabricated as components of a particular device, the properties of the device can usually be computed from the characteristic curves.

Characteristic curves thus form a useful means for expressing the properties of all kinds of circuit elements encountered in engineering problems. For devices embodying nonlinear conduction phenomena, the relationships of interest are usually expressed graphically by the volt-ampere characteristics of typical two-terminal elements. For devices embodying nonlinear magnetic-induction phenomena, the relationships of interest are usually expressed graphically by characteristic curves which show, for example, the flux density as a function of the magnetizing force determined by means of a test solenoid whose core consists of a conveniently shaped sample of the magnetic material.

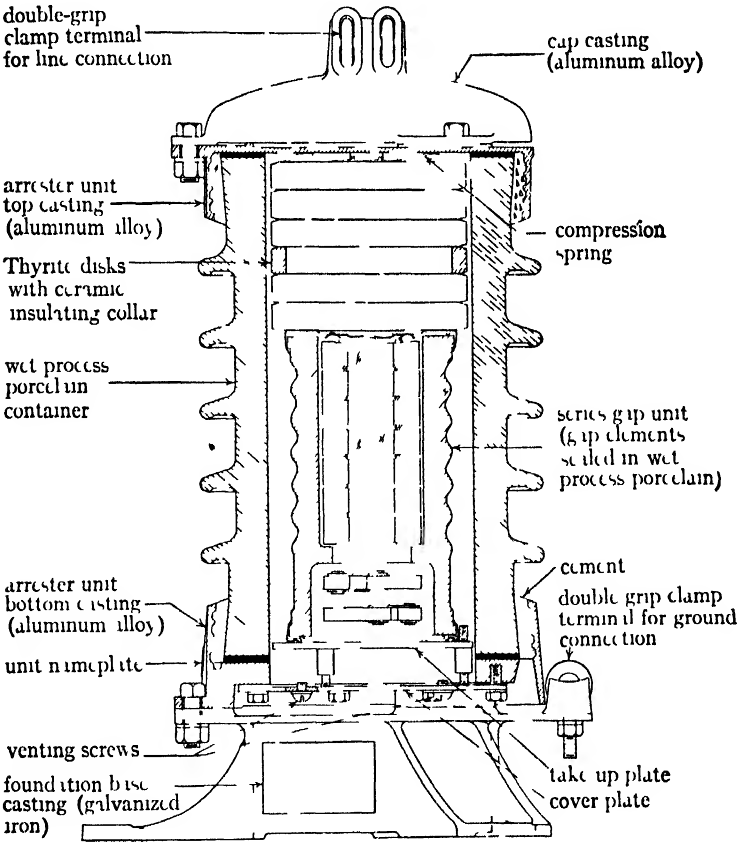
Frequently, the time-varying nonlinear characteristic of a device can be formed from a knowledge of the function of the device and the data given by a family of static characteristic curves which express the nonlinear properties of its parameters. Figure 5 shows the volt-ampere characteristics of a number of currently important resistors having nonlinear resistance. They offer different resistances to currents of different magnitudes. Most resistors of this kind also offer a different resistance to the same current when its direction is reversed; that is, their characteristics are nonsymmetrical with respect to the direction of current. The charac-



Courtesy General Electric Co.

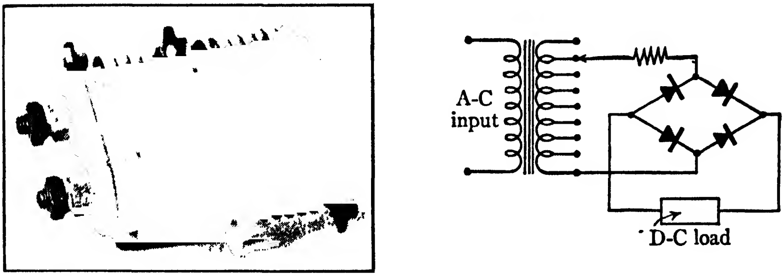
Installation of 138 kv and 34 5 kv Thyrite station-type lightning arresters protecting four 3333 kva single-phase transformers.

teristics of inductors having nonlinear inductance are best represented by magnetization curves the general appearance of which probably is familiar to the student. The characteristics of magnetic materials, including numerous plots of experimental data and further analysis of the performance of iron-cored devices, are presented in this series in the volume on magnetic circuits and transformers. For reasons stated in Art. 2c, devices having nonlinear elastance, though extremely useful and much used, are of minor concern in the following analysis and hence are given only slight mention.

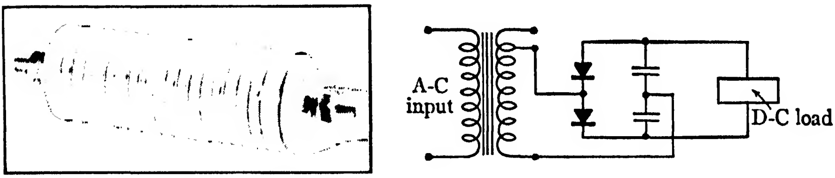


Courtesy General Electric Co.

Cutaway section of 12-kv Thyrite lightning arrester

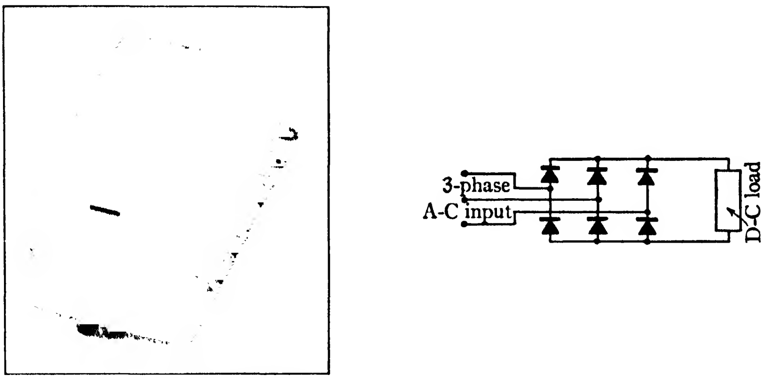


Copper-oxide rectifier assembled for use as a bridge circuit as shown in the diagram, for battery charging. Output rating: 30 75 v, 0.2 amp dc. (Manufactured by Westinghouse Electric and Manufacturing Co.)



Courtesy Federal Telephone and Radio Corpn.

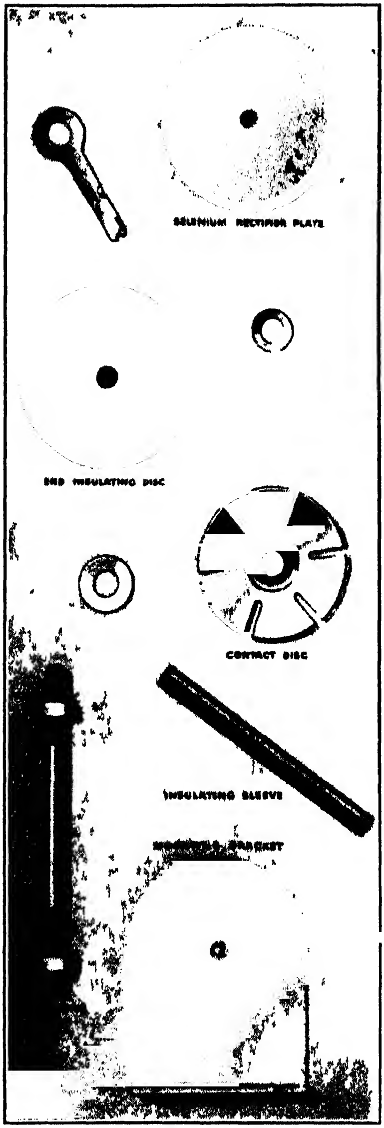
Iron-selenium rectifier assembled for use in a voltage doubling circuit as shown in the diagram. The transformer tap is provided so that the voltage can be stepped up about 5% to compensate for effects of ageing in the rectifier. Output rating: 180 v, 0.06 amp dc.



Courtesy Benwood Linze Co.

Copper-sulphide-magnesium rectifier assembled in one stack for three-phase operation as shown in the diagram. Output rating: 12 v, 25 amp dc.

Oxide film rectifiers are made in a variety of styles and ratings for use in battery chargers, rectifier-type instruments, modulators and demodulators, carrier-telephone systems, acoustic shock reducers in parallel with telephone receivers, protection of apparatus against voltage surges, and in many special applications. The plates or discs shown in the pictures are not representative of the rectifying area but are extended cooling fins. Sometimes forced draft is used for cooling.



Courtesy Federal Telephone and Radio Corpn.

Parts for iron-selenium rectifier.

4. Methods of computation: general discussion

The real usefulness of many of the nonlinear circuit devices encountered in engineering arises from the peculiar circuit behavior that results when they are combined with linear devices in particular circuits. For example, the two-electrode tube and the multielectrode tube are used in such combinations to rectify alternating current into unidirectional current, to invert direct current into alternating current, to produce sustained electric oscillations, and to perform numerous other useful functions. The iron-cored reactor is used frequently in combination with linear or nonlinear circuit devices in control circuits,[2] in relays, in current-limiting devices, and so on.

For successful use of nonlinear devices, knowledge of the consequences of certain circuit combinations of linear and nonlinear elements is important; and for speedier development of new devices embodying nonlinear parameters, the rapid prediction of their behavior is desirable. Many electrical devices already indispensable to some electric circuits, moreover, have properties that make them appear as nonlinear circuit elements, and often give rise to highly undesirable conditions in the circuit containing the device, or in near-by circuits. For these reasons and others that need not be cited here, means for analyzing the circuit behavior of a network including both linear and nonlinear elements are of primary importance.

The methods of treating circuits thus composed are by no means so straightforward as those of treating linear circuits. To describe the general behavior of any circuit comprising inductance or capacitance, or both, a differential equation is required. As a result of the nonlinearity of the differential equations necessary to represent a nonlinear circuit, its solution by formal processes is frequently difficult and involved, and sometimes entirely impracticable. Since the exponential functions, among the most readily manipulated functions of mathematics, do not in general satisfy nonlinear equations, the convenience which their use affords in linear differential equations is not here available. Most of the functions that satisfy nonlinear differential equations, in fact, have not been studied, named, or tabulated. Seldom do these functions lend themselves to simple or even practicable analytical expression; hence it frequently is necessary to resort to graphical representation or numerical tabulation. While such means may appear to lack the elegance of the analytical methods and may often be aptly described as " brute force " methods, they are by no means ineffective but, rather, can be made to yield extremely useful results in relatively little time when skillfully applied.

The chief purpose of the following articles is to illustrate the means

[2] " Reactance Amplifiers," *Electronics*, X (Aug., 1937), 28–30.

for transient analysis of typical nonlinear circuits. Though some of the circuit combinations used have practical significance and the results are often directly applicable in engineering, others are used chiefly for their effectiveness in illustrating the principles of the analysis. The treatment is restricted primarily to the transient behavior, first, because the relatively complex methods of analysis involved can then be treated with greater clarity, and, second, because the analysis of the steady-state behavior is much less complicated and is discussed more effectively, as occasion arises, in other volumes of this series. Steady-state analysis often amounts to little more than picking points from the characteristic curves. When the impressed force is not constant, the steady-state solution commonly can be obtained only by working through the transient solution anyhow.

The following articles include explicit graphical, analytical, near-analytical, and step-by-step methods of analysis. Where applicable, the explicit graphical methods are frequently very effective. The analytical methods generally are used to solve a problem sufficiently resembling the one under consideration to make the analytical results directly applicable when appropriate judgment is used in their interpretation. The most effective, although in many ways also the most cumbersome, is the step-by-step method of numerical integration, Art. 5c. Any ordinary differential equation arising from a physical problem can be solved numerically with particular initial conditions by this method, which, although laborious, is used frequently when others fail. When the solution of nonlinear ordinary differential equations is sought, it is important to realize that numerical solutions can always be obtained if the equations truly describe physical phenomena. The existence of the phenomena constitutes proof of the existence of solutions to the equations describing the phenomena but does not prove that obtaining the solutions is always a simple process. In fact, ingenuity and resourcefulness are usually necessary if a reasonably simple method of solving any given nonlinear problem is to be found.

5a. Nonlinear resistance in series with a linear inductance, constant voltage applied

The portion of the field circuit of an electric motor or generator comprising the field winding in series with the field rheostat or the field-discharge resistor, the winding of an iron-cored choke or an iron-cored transformer, and a two-element tube in series with a filter choke each represents a circuit for which either the resistance or the inductance parameter, or both, are functions of the current, and hence are nonlinear. For analysis, many circuits such as these can be represented closely enough by a single linear element connected in series with a single

nonlinear element. For instance, in the circuit comprising a generator field and a thyrite field-discharge resistor, the inductance parameter of the field winding may be considered to be essentially linear compared with the resistance parameter of the thyrite, even though the inductance parameter is seen to be nonlinear if examined closely. On the other hand, the winding of an iron-cored choke has an apparent resistance that is essentially linear compared with the inductance parameter of the winding, even though the resistance parameter is not truly linear. Representation of such a circuit with only one nonlinear element is a highly desirable simplification that usually can be made in the analysis of the circuit behavior without causing such errors in the results that the analysis is useless.

To illustrate methods of analysis applicable to calculation of the behavior of nonlinear series circuits including only one energy-storage element, a circuit comprising a two-electrode tube in series with an air-core coil and with a direct voltage applied is used. The circuit diagram is shown in Fig. 6. For this illustration, explicit graphical and semigraphical integration processes, linear approximations, and semianalytical processes are employed.

Since the inductance of the coil is linear, its circuit behavior can be described in terms of a constant inductance L. The resistance element must here be represented in terms of its volt-ampere characteristic and an equation in terms of the $v(i)$ relationship must be formulated from it. The concept of impedance has no significance, nor has the concept of time constant as applied to series RL circuits treated in Ch. III.

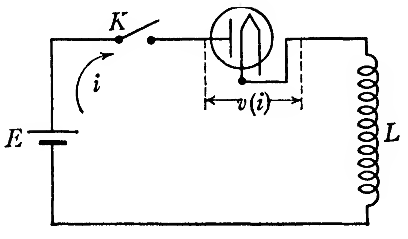


FIG. 6. Nonlinear resistance in series with linear inductance, constant source voltage.

It is not necessary that the inductor have negligible resistance. For the purposes of the analysis the resistance of the inductor may be added to that of the nonlinear resistor by *shearing* the resistance characteristic of the inductor into the resistance characteristic of the nonlinear element to form a new composite characteristic which represents the two resistances combined. The procedure for shearing the two characteristics into this single composite characteristic is to add the voltages which each current value produces in the elements.

The differential equation expressing the equilibrium of voltages for the circuit of Fig. 6 is

$$L\frac{di}{dt} + v(i) = E. \qquad [15]$$

Since v is not a linear function of i, Eq. 15 is nonlinear, with one linear coefficient. When the voltage v is given as an analytical function of i, the function sometimes can be integrated by analytical means as shown in Art. 5e. When v is given as a function of i only in graphical form, an analytical procedure first requires that an approximate analytical relation between v and i be derived from the graphical data. A way of making this derivation is presented in this series in the volume on electronics, but, where analytical expressions are used at this stage of the treatment, it is assumed that they are given data. However, a graphical process of integration may always be attempted and, when the variables are separable, is illustrated in Art. 5b. The principle of the solution is equally applicable for circuits in which the resistance elements may represent arcs, glow-discharge tubes, thyrite, or many other nonlinear devices – it is applicable, in fact, to any circuit for which one can write an equation with variables separable in a manner similar to the following procedure.

5b. Explicit graphical integration

For explicit graphical integration Eq. 15 is rewritten so as to separate the variables as follows:

$$L\frac{di}{dt} = E - v(i), \tag{15a}$$

or

$$\frac{dt}{L} = \frac{di}{[E - v(i)]}, \tag{15b}$$

from which

$$\frac{t}{L} = \int_{i=i_0}^{i=i_t} \frac{di}{[E - v(i)]}, \tag{16}$$

where i_0 is the value of i when t is zero, and i_t is the value of i at time t. Since the function $v(i)$ is the tube characteristic and is assumed to be given in graphical form, the function $E - v(i)$, and hence a curve of $1/[E - v(i)]$ may be determined graphically.

To illustrate the graphical procedure the initial conditions selected are that the voltage is applied when t is zero, and that the current is then zero. Then, in accordance with Eq. 16, the area under the curve of $1/[E - v(i)]$ plotted as a function of i, between the points where i has the values of zero and i_t, is the value of t/L. Thus, in this problem the usual procedure of integrating a time function to obtain a current is reversed. That is, *a function of current is integrated to give the value of time at which a particular*

value of current exists. This process is repeated for each value of current for which the corresponding time is desired. Hence, for a number of chosen values of the upper limit such as i_1, i_2, $\cdots$, the corresponding values t_1/L, t_2/L, $\cdots$, are determined by graphical integration, and the desired curve of i as a function of t/L is determined. Since di/dt is zero where t is infinite, the maximum value of i is that given directly by the volt-ampere characteristic for the applied source voltage E. Hence appropriate intervals of i for the upper limit of integration can be chosen.

By introduction of the proper scale factor, a single determination of i as a function of t/L may be made to apply to any numerical value of the inductance L. However, the determination holds for only the particular value of E used. The diagrams of Fig. 7 illustrate the various steps in the graphical procedure.

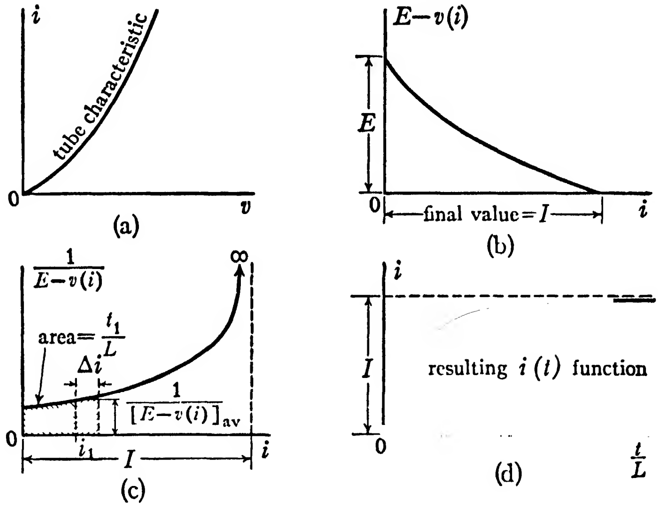


FIG. 7. Illustration of the various steps in the procedure for explicit graphical integration.

To make possible the comparison of results given by this method with those given by other methods treated in Arts. 5c, 5d, and 5e, the calculation is performed for a numerical example. For the comparison, E is taken as 100 volts and the $i(v)$ characteristic for the vacuum tube is assumed to be that given in columns 1 and 2 of Table I, and plotted as a solid curve in Fig. 8. The necessary calculations are carried out in tabular form in Table I. The curve of $1/[E - v(i)]$ is plotted as a function of i in Fig. 9. In Table I the first four columns are self-explanatory. The rows of figures in the fifth, sixth, and seventh columns are spaced between the rows of the remaining columns to indicate that the entries in these

TABLE I*

1	2	3	4	5	6	7	8
v *volts*	i *milli-amperes*	$E - v(i)$ *volts*	$\frac{1}{E - v(i)}$ *Reciprocal volts*	$\left[\frac{1}{E-v(i)}\right]_{av.}$ *Reciprocal volts*	Δi *milli-amperes*	$\frac{\Delta t}{L}$ *milliseconds per henry*	$\frac{t}{L}$ *milliseconds per henry*
0	0	100	0.0100				0
				0.0105	0.9	0.00945	
10	0.9	90	0.0111				0.0095
				0.0118	1.25	0.01475	
20	2.15	80	0.0125				0.0242
				0.0134	1.60	0.02143	
30	3.75	70	0.0143				0.0456
				0.0155	1.90	0.02945	
40	5.65	60	0.0167				0.0751
				0.0183	2.15	0.03935	
50	7.80	50	0.0200				0.1144
				0.0225	2.40	0.05400	
60	10.20	40	0.0250				0.1684
				0.0268	1.30	0.03485	
65	11.50	35	0.0286				0.2033
				0.0309	1.40	0.04328	
70	12.90	30	0.0333				0.2466
				0.0367	1.35	0.04957	
75	14.25	25	0.0400				0.2961
				0.0450	1.43	0.06440	
80	15.68	20	0.0500				0.3605
				0.0536	0.67	0.03590	
82.5	16.35	17.5	0.0571				0.3964
				0.0619	0.73	0.04520	
85	17.08	15	0.0667				0.4416
				0.0735	0.72	0.0529	
87.5	17.80	12.5	0.0800				0.4945
				0.0900	0.70	0.0630	
90	18.50	10	0.1000				0.5575
				0.1125	0.60	0.0675	
92	19.10	8	0.1250				0.6250
				0.1459	0.60	0.0876	
94	19.70	6	0.1667				0.7126
				0.2084	0.60	0.1250	
96	20.30	4	0.2500				0.8376
				0.3750	0.64	0.240	
98	20.94	2	0.5000				1.0776
				∞	0.66	∞	
100	21.60	0	∞				∞

* In general the calculations are carried out in practical units. For convenience in tabulation, the results of the calculations are frequently expressed here and subsequently in terms of multiples and submultiples of these units.

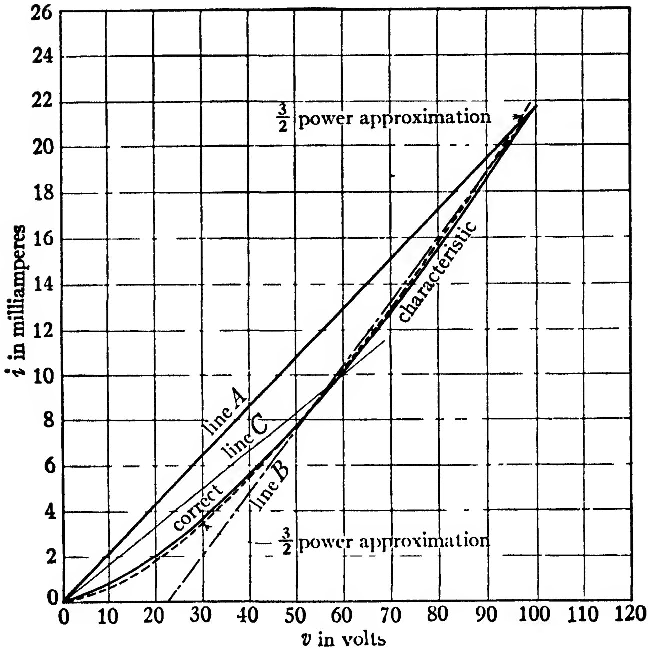


FIG. 8. Representations of the volt-ampere characteristic of the vacuum tube of Fig. 6.

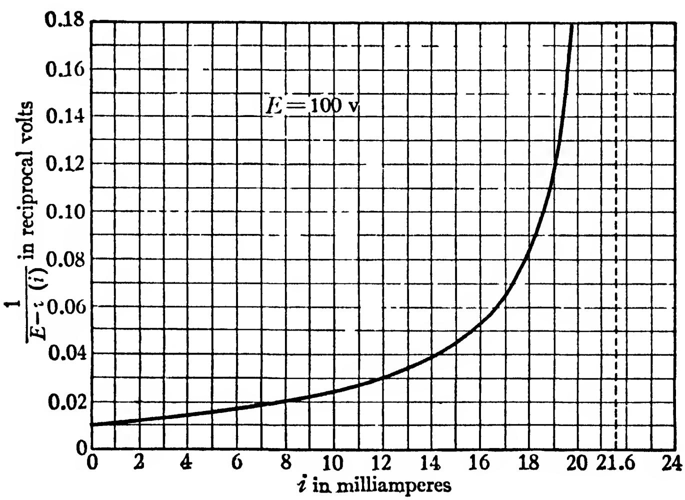


FIG. 9. Curve corresponding to Fig. 7c, for numerical example.

three columns apply to the intervals between the entries in the other columns. For example, an entry in column 5 is the average of the column 4 entries on the adjacent rows. A column 7 entry is the product of the adjoining column 5 and column 6 entries. A given entry in column 8 is the sum of the one above it and the intervening column 7 entry.

There are several simple methods of finding the area under a curve such as that in Fig. 7c when the equation for it is not known. Perhaps the simplest is to count the squares on cross-section paper and to estimate the fractions. Also, the average height of equally spaced ordinates is easily found, but care should be taken not to weight the end ordinates more than the others in taking the average. A procedure useful when the curve is jagged or very irregular is to cut the curve out of good-quality

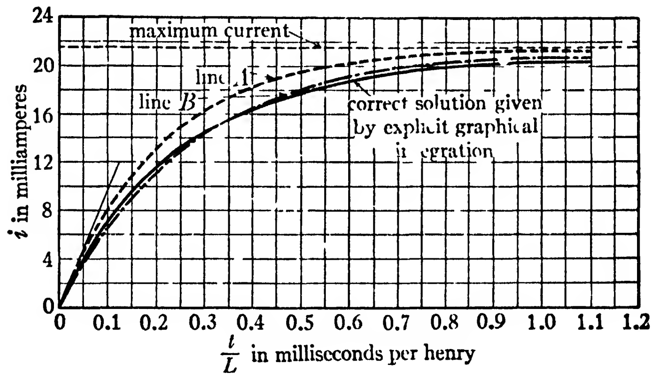


FIG. 10. Illustrating the solution for current in Fig. 6, as computed by different methods, Arts 5b, 5c, 5d, and 5e.

paper and to weigh it on a chemical balance. The weights are conveniently calibrated by weighing a piece of simple shape, such as a square or a circle, whose area is easily computed. Also, of course, a planimeter may be used.

The solution of this calculation in the form of a plot of i as a function of t/L is the solid curve shown in Fig. 10, and is seen to have the same general form as the curve of i as a function of time for a constant-R, constant-L circuit with a suddenly applied constant E.

5c. SEMIGRAPHICAL SOLUTION

In another method that is often useful for calculating the current in a circuit such as that of Fig. 6, the curve of the tube characteristic is approximated by a series of chords as shown by the construction of Fig. 11. On each of these chords the $v(i)$ relation is a linear one, and for corresponding intervals in v and i the differential equation may be

integrated analytically. The solution thus obtained is valid only within such an interval, but it is possible to connect the solution for the end of one interval with the solution for the beginning of the next by adjustment of the boundary conditions of each solution. In this manner, a solution throughout any desired range may be obtained. This form of procedure is essentially a step-by-step calculation, but it is distinguished from the step-by-step method given in Art. 9b by the fact that here the tube characteristic is assumed linear for each step, whereas in Art. 9b the rate of change of current, di/dt, is assumed constant for each step.

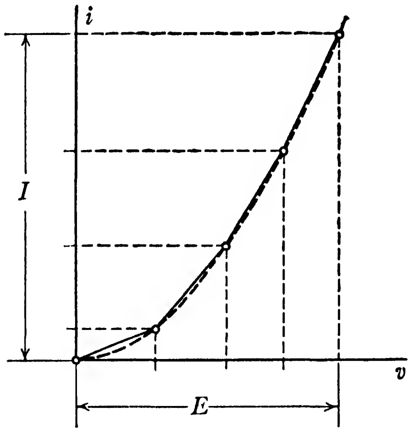


FIG. 11. Approximation to volt-ampere characteristic by means of chords.

For this semigraphical solution the tube characteristic is expressed as $1/r_p$, where $1/r_p$ is the slope of the chord in Fig. 11 for a particular interval in question. Equation 15 may then be written as

$$\frac{\Delta t}{L} = \frac{1}{r_p}\int_{v_1}^{v_2} \frac{dv}{E - v(i)} = \frac{t_2 - t_1}{L}. \qquad [17]$$

In Eq. 17, v_1 and v_2 are the boundary values for the linear interval within which r_p is constant, and t_1 and t_2 are the corresponding temporal boundaries. The integration of Eq. 17 results in

$$\frac{t_2 - t_1}{L} = -\frac{1}{r_p}\left[\ln (E - v)\right]\Big|_{v_1}^{v_2} \qquad [18]$$

$$\frac{t_2 - t_1}{L} = \frac{1}{r_p}\ln\left(\frac{E - v_1}{E - v_2}\right). \qquad [19]$$

The calculation of a numerical solution by this method for the circuit having the same elements as the circuit used in the calculation of Art. 5b may be carried out in the manner indicated in Table II. The results given in that table agree so closely with those secured by the graphical process of integration that if they were plotted on Fig. 10 the deviations would not be noticeable.

TABLE II

v volts	i milliamperes	Δv volts	Δi milliamperes	$\frac{\Delta i}{\Delta v} = \frac{1}{r_p}$ $\frac{\text{milliamperes}}{\text{volts}}$	$\left[\frac{1 - \frac{v_1}{E}}{1 - \frac{v_2}{E}}\right]$	$\ln\left[\frac{1 - \frac{v_1}{E}}{1 - \frac{v_2}{E}}\right]$	$\frac{\Delta t}{L}$ milliseconds per henry	$\frac{t}{L}$ milliseconds per henry	Error* in microseconds per henry
0	0							0	
		10	0.9	0.09	1.111	0.1059	0.00953		
10	0.9							0.00953	+0.08
		10	1.25	0.125	1.125	0.1175	0.01469		
20	2.15							0.02422	+0.02
		10	1.60	0.160	1.143	0.1335	0.02137		
30	3.75							0.04559	−0.01
		10	1.90	0.190	1.167	0.1543	0.02933		
40	5.65							0.07492	−0.08
		10	2.15	0.215	1.200	0.1825	0.03925		
50	7.80							0.11417	−0.23
		10	2.40	0.240	1.250	0.2235	0.05365		
60	10.20							0.16782	−0.58
		10	2.70	0.270	1.333	0.2880	0.0778		
70	12.90							0.24562	−0.98
		10	2.78	0.278	1.500	0.4060	0.1128		
80	15.68							0.35842	−2.08
		10	2.82	0.282	2.000	0.6935	0.1955		
90	18.50							0.55392	−3.58
		8	2.44	0.305	5.000	1.610	0.4910		
98	20.94							1.04492	−32.68
		2	0.66	0.330	∞	∞	∞	∞	
100	21.60								

* These errors are negligible except for v larger than 90 v where the curve is so flat that even appreciable errors in t are not noticeable in the resulting plot.

5d. Single straight-line approximations

The accuracy of the result obtained by the foregoing method suggests that a simpler procedure for approximating the volt-ampere characteristic might be tried with the expectation of a fairly accurate solution. Thus even an attempt to replace the actual tube characteristic by a single straight line might be made.

The simplest approximation of this kind is a single chord extending from the origin to the known final value of the current. Such a line is designated as line A in Fig. 8. For the given data the equation of this line is expressed by

$$v = \frac{100}{21.6} i, \qquad [20]$$

where i is in milliamperes and v is in volts. This approximation amounts to replacing the tube by a constant resistance of $10^5/21.6$ ohms. The problem is thereby reduced to that of a series RL circuit with a constant applied voltage E. When the applied voltage is 100 v, the solution to this simple circuit problem is

$$i = 21.6\ (1 - \epsilon^{-4620(t/L)}), \qquad [21]$$

where i is in milliamperes, t is in seconds, and L is in henrys. The dotted curve marked line A in Fig. 10 shows a plot of i as a function of t/L for this equation. The deviation from the correct result given by explicit graphical integration is as large as ten per cent, which, although appreciable, may in certain problems not render the result useless.

An appreciable improvement in the result may be obtained by reorienting the single straight line with respect to the tube characteristic. A guide as to the appropriate location of the line results from a consideration of the fact that for small values of t, and hence of i, the $v(i)$ term of Eq. 15 is small compared with the $L(di/dt)$ term, but, for large values of i, the converse is true. Therefore, the transient solution is expected to be closer to the correct one, if, so far as the transient is concerned, the tube characteristic is represented by a single line that closely approximates its slope at values of i for which the tube-voltage drop is appreciable, rather than at values where this voltage drop is small. Such a line is shown as B in Fig. 8, and departs appreciably from the correct characteristic only in the vicinity of the origin.

The equation of line B is

$$v = 22 + \frac{78}{21.6} i. \qquad [22]$$

This procedure amounts to replacing the tube by a constant counter-electromotive force of 22 volts in series with a constant resistance of

78,000/21.6 ohms. If, again, E is 100 volts, then the problem becomes identical with that for the series RL circuit comprising a resistance of 78,000/21.6 ohms and an applied constant electromotive force of 100 − 22, or 78 volts. The time constant of the approximate linear circuit is then 21 6L/78,000. The maximum value of i, however. is still 21.6 milliamperes. Hence the result is

$$i = 21.6(1 - \epsilon^{-3610(t/L)}). \qquad [23]$$

In Fig. 10 the curve of i as a function of t/L, as given by Eq. 23, is plotted as the dotted curve, line B. The approximation to the correct result is reasonably good. Linear approximations of this kind are useful in the treatment of vacuum-tube circuit problems discussed in Ch. X of the volume on applied electronics.

5e. Other analytic approximations

Although the process of replacing the nonlinear resistance characteristic by one or more straight lines often gives rise to a reasonably satisfactory approximation to the effect of the nonlinearity on the circuit behavior, it may sometimes lead to extremely cumbersome manipulations. In certain problems it may be more effective to attempt to approximate the actual tube characteristic throughout the desired range by an analytical expression with a single term. The volt-ampere characteristics of many types of tubes are often expressible to a satisfactory degree of approximation by the relation

$$i = Kv^n, \qquad [24]$$

in which K and n are constants, the values of which may be adjusted so that Eq. 24 represents the characteristic to a satisfactory degree.

A test for determining whether or not a given set of data is representable by the form of Eq. 24 is to plot log i as a function of log v, or, what is equivalent, to plot i as a function of v on logarithmic graph paper. When the logarithms of the factors in Eq. 24 are taken,

$$\log i = \log K + n \log v, \qquad [25]$$

which is the equation of a straight line in the variables log i and log v with slope n and an intercept log K for v equal to unity or log v equal to zero. Hence, if $v(i)$ is representable by the form of Eq. 24, the points of the plot for log i versus log v lie on a straight line from whose co-ordinates K and n can be found.

The substitution of Eq. 24 in Eq. 15 leads to

$$LKnv^{n-1}\frac{dv}{dt} = E - v. \qquad [26]$$

The separation of the variables gives

$$\frac{dt}{L} = nK\frac{v^{n-1}dv}{E - v}, \tag{27}$$

so that

$$\frac{t}{L} = nK\int_0^v \frac{v^{n-1}dv}{E - v}. \tag{28}$$

For certain values of the exponent n this integral may be evaluated without much difficulty.

If n is an integer, a method of evaluation that involves a change of variable may be used. For the change of variable,

$$E - v = x, \tag{29}$$

so

$$v = E - x, \tag{30}$$

$$dv = -dx. \tag{31}$$

The integral given by Eq. 28 then becomes

$$\frac{t}{L} = nK\int_{x=E-v}^{x=E} \frac{(E - x)^{n-1}}{x}dx. \tag{32}$$

By the binomial theorem $(E - x)^{n-1}$ may be expanded to give

$$E^{n-1} - \frac{n-1}{1\,!}E^{n-2}x + \frac{(n-1)\,(n-2)}{2\,!}E^{n-3}x^2 + \cdots. \tag{33}$$

For integer values of n the integrand of Eq. 32 is a polynomial in x with a finite number of terms and hence can easily be integrated.

For example, if n is 3,

$$\frac{t}{L} = 3K\int_{x=E-v}^{x=E} \frac{E^2 - 2Ex + x^2}{x}dx = 3K\left[E^2 \ln x - 2Ex + \frac{x^2}{2}\right]_{x=E-v}^{x=E} \tag{34}$$

which reduces to

$$\frac{t}{L} = -3KE^2\left[\ln\left(1 - \frac{v}{E}\right) + \frac{v}{E}\left(1 + \frac{v}{2E}\right)\right]. \tag{34a}$$

This may be put into the form of an expansion,

$$\frac{t}{L} = \frac{i}{E}\left[1 + \frac{3}{4}\left(\frac{v}{E}\right) + \frac{3}{5}\left(\frac{v}{E}\right)^2 + \frac{3}{6}\left(\frac{v}{E}\right)^3 + \cdots\right], \tag{34b}$$

which represents the behavior for small values of v more clearly than does Eq. 34a. For instance, it shows that the current starts out by satisfying the relation

$$i = \frac{Et}{L} \tag{35}$$

as it should, because initially all the voltage drop is across the inductance and none is across the tube, there being intially no current.

As a second example, if n is 2, the integral of Eq. 32 becomes

$$\frac{t}{L} = 2K\int_{x=E-v}^{x=E} \frac{E-x}{x}\,dx = -2KE\left[\ln\left(1-\frac{v}{E}\right)+\frac{v}{E}\right], \tag{36}$$

and the following expansion may be obtained:

$$\frac{t}{L} = \frac{i}{E}\left\{1+\frac{2}{3}\left(\frac{v}{E}\right)+\frac{2}{4}\left(\frac{v}{E}\right)^2+\cdots\right\}. \tag{36a}$$

When the exponent n in Eq. 24 is not an integer but an *odd multiple* of 1/2, a successful method of evaluation is that which makes use of the condition that

$$\frac{dv}{E-v} = \frac{d(v^{1/2})}{E^{1/2}-v^{1/2}} - \frac{d(v^{1/2})}{E^{1/2}+v^{1/2}}. \tag{37}$$

Now, if the change of variable is made to x and y, where

$$E^{1/2} - v^{1/2} = x, \tag{38}$$

whence

$$v^{1/2} = E^{1/2} - x, \tag{39}$$

$$dv^{1/2} = -dx, \tag{40}$$

$$v^{n-1} = (E^{1/2} - x)^{2(n-1)} \tag{41}$$

and

$$E^{1/2} + v^{1/2} = y, \tag{42}$$

whence

$$v^{1/2} = y - E^{1/2}, \tag{43}$$

$$dv^{1/2} = dy, \tag{44}$$

$$v^{n-1} = (y - E^{1/2})^{2(n-1)}, \tag{45}$$

the integral of Eq. 28 becomes

$$\left.\begin{aligned}\frac{t}{L} = -nK\Bigg[\int_{x=E^{1/2}}^{x=E^{1/2}-v^{1/2}} &\frac{(E^{1/2}-x)^{2(n-1)}dx}{x}\\ +\int_{y-E^{1/2}}^{y=E^{1/2}+v^{1/2}} &\frac{(y-E^{1/2})^{2(n-1)}dy}{y}\Bigg],\end{aligned}\right\} \tag{46}$$

which can be contracted into

$$\frac{t}{L} = nK\int_{x=E^{1/2}-v^{1/2}}^{x=E^{1/2}+v^{1/2}} \frac{(E^{1/2}-x)^{2(n-1)}}{x}\,dx. \tag{46a}$$

For example, if n is 3/2, then $2(n-1)$ is unity, and

$$\frac{t}{L} = \frac{3K}{2}\int_{E^{1/2}-v^{1/2}}^{E^{1/2}+v^{1/2}} \frac{E^{1/2}-x}{x}\,dx = \frac{3KE^{1/2}}{2}\left\{\ln\left[\frac{1+\sqrt{\frac{v}{E}}}{1-\sqrt{\frac{v}{E}}}\right] - 2\sqrt{\frac{v}{E}}\right\} \quad [46b]$$

or

$$\frac{t}{L} = 3KE^{1/2}\left\{\tanh^{-1}\sqrt{\frac{v}{E}} - \sqrt{\frac{v}{E}}\right\}, \quad [46c]$$

which has the expansion

$$\frac{t}{L} = \frac{i}{E}\left\{1 + \frac{3}{5}\left(\frac{v}{E}\right) + \frac{3}{7}\left(\frac{v}{E}\right)^2 + \cdots\right\}. \quad [46d]$$

The 3/2-power relation

$$i = Kv^{3/2}, \quad [24a]$$

is used frequently to approximate a tube characteristic. For example, if an expression of the form

$$i = 22 \times 10^{-6} v^{1.5} \text{ amp}, \quad [24b]$$

where v is in volts, is taken to represent the characteristic for the tube of Fig. 6, a good approximation is obtained as shown by the dotted curve labeled "3/2-power approximation" in Fig. 8. The solution for the

TABLE III

$E = 100$ volts, $\left[\frac{3KE^{1/2}}{2}\right] = 0.33 \times 10^{-3}$

v *volts*	$\frac{v}{E}$	$\sqrt{\frac{v}{E}}$	$\left[\frac{1+\sqrt{\frac{v}{E}}}{1-\sqrt{\frac{v}{E}}}\right]$	$\ln\left[\frac{1+\sqrt{\frac{v}{E}}}{1-\sqrt{\frac{v}{E}}}\right]$	$\frac{t}{L}$ *milliseconds per henry*	i *milli-amperes*	*Error in* $\frac{t}{L}$ *microseconds per henry*
0	0	0	1	0	0	0.0	0
10	0.1	0.316	1.925	0.655	0.00759	0.70	−1.86
20	0.2	0.447	2.617	0.962	0.02243	1.97	−1.77
30	0.3	0.548	3.427	1.232	0.0449	3.62	−0.70
40	0.4	0.632	4.435	1.490	0.0746	5.57	−0.50
50	0.5	0.707	5.828	1.762	0.1148	7.78	+0.40
60	0.6	0.775	7.890	2.066	0.1705	10.23	+2.10
70	0.7	0.837	11.268	2.422	0.2469	12.88	+0.30
80	0.8	0.894	17.86	2.883	0.3615	15.74	+1.00
90	0.9	0.949	38.22	3.642	0.5760	18.78	+18.50
98	0.98	0.990	198	5.288	1.092	21.35	+14.4
100	1.0	1.0	∞	∞	∞	22.0	—

current in the circuit of Fig. 6 carried out in accordance with Eq. 46b, and using the value for n given by Eq. 24b, is tabulated in Table III. The last column in this table shows the discrepancy between this method and the explicit graphical-integration method as tabulated in Table I. The discrepancies are so small that the curves of i as a function of t/L for both methods are not distinguishable from each other. The 3/2-power relation for approximating the tube characteristic in this example is therefore considered very satisfactory.

6a. Nonlinear resistance in series with a linear capacitance, constant voltage applied

For numerous electric-control devices, a direct voltage is applied to a circuit comprising a nonlinear resistor connected in series with a linear capacitor as shown in Fig. 12. Many electronic sweep circuits if examined carefully are found to involve such connections, usually employing as the nonlinear resistor a two- or three-element vacuum tube. For such circuits, the expression for the tube current as a function of the time after the voltage is applied, or the voltage across the condenser as a function of time (which can be obtained by integration of the expression for current), is usually desired. Fortunately, it is often permissible to represent the tube by a linear element and the analysis becomes a relatively simple problem. Sometimes, however, more exact results are necessary and a method of calculation that takes account of the nonlinear properties of the resistor is essential.

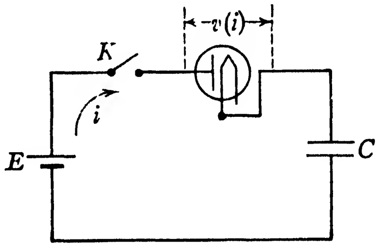


FIG. 12. Nonlinear resistance in series with linear capacitance, constant source voltage.

The methods of analysis applicable to problems of this kind are essentially the same as those given in Arts. 5b, 5c, 5d, and 5e. They are treated briefly in Arts. 6b to 6e, only to the extent necessary to show the basic procedures involved.

6b. Explicit graphical integration

The differential equation for the circuit of Fig. 12 is

$$v(i) + \frac{1}{C}\int i\,dt = E, \tag{47}$$

which, upon separation of the variables, can be written as

$$\frac{dt}{C} = \frac{d[E - v(i)]}{i}, \tag{48}$$

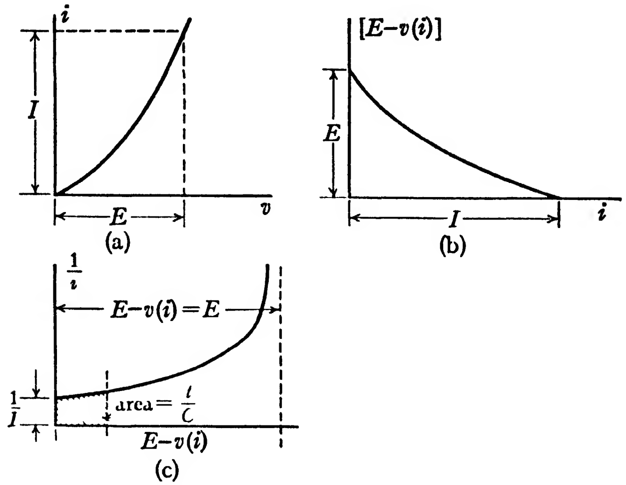


FIG. 13 Various steps in the procedure for explicit graphical integration

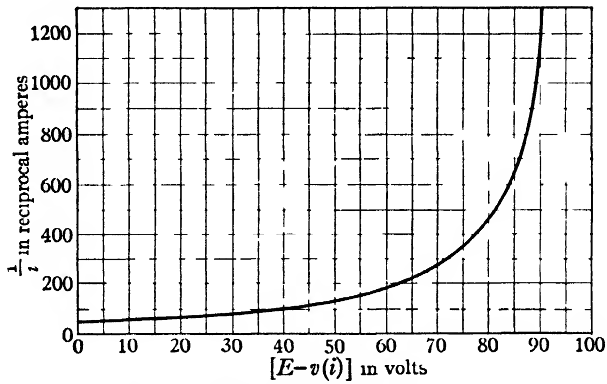


FIG. 14 Curve corresponding to Fig. 13c, for numerical example

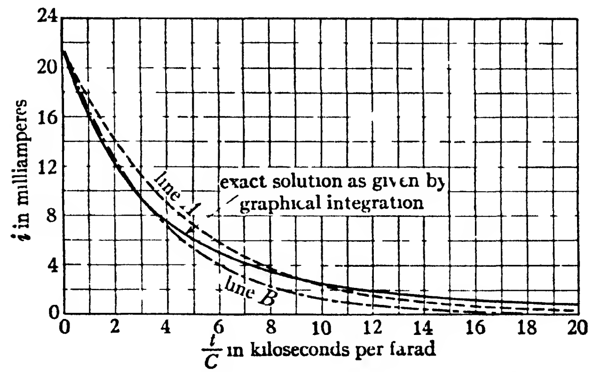


FIG. 15 Solution for current in Fig. 12 as computed by different methods, Arts. 6b, 6c, 6d, 6e

or

$$\frac{t}{C} = \int_{E-v(i)=E_{C0}}^{E-v(i)=E_{Ct}} \frac{d[E - v(i)]}{i}. \qquad [49]$$

The integral represents the area under the curve $1/i$ plotted as a function of $E - v(i)$ from a value of $E - v(i)$ of zero to a value of v corresponding to E_{Ct}, where E_{Ct} is the voltage across the condenser at time t. The manner in which this integration is carried out is indicated in the three sketches of Fig. 13. For the tube data of Table I (plotted in Fig. 8) and for a voltage of 100 volts, the numerical data of this calculation are summarized in Table IV. The plot of $1/i$ as a function of $[E - v(i)]$ is shown in Fig. 14 and the resulting plot of i as a function of t/C is given by the solid curve of Fig. 15.

TABLE IV

$[E - v(i)]$ volts	$\frac{1}{i}$ reciprocal amperes	$\left(\frac{1}{i}\right)_{av}$ reciprocal amperes	$\Delta[E - v(i)]$ volts	$\frac{\Delta t}{C}$ seconds per farad	$\frac{t}{C}$ kiloseconds per farad	i milliamperes
0	46.3				0	21.6
		50.18	10	501.18		
10	54.05				0.5012	18.5
		58.9	10	589.0		
20	63.75				1.0902	15.68
		70.63	10	706.3		
30	77.50				1.7965	12.90
		87.78	10	877.8		
40	98.05				2.6743	10.20
		113.12	10	1,131.2		
50	128.2				3.8055	7.80
		152.6	10	1,526.0		
60	177.0				5.3315	5.65
		194.9	5	974.5		
65	212.7				6.306	4.70
		239.7	5	1,199.0		
70	266.6				7.505	3.75
		305.8	5	1,529.0		
75	345.0				9.034	2.90
		405.0	5	2,025.0		
80	465.0				11.059	2.15
		565.9	5	2,829.5		
85	666.7				13.888	1.50
		787.9	3	2,363.7		
88	909.0				16.252	1.10
		1,010.0	2	2,020.0		
90	1,111.0				18.272	0.9

6c. Semigraphical solution

Another approach to this problem may be made by representation of the tube characteristic by a series of chords, as is done in Fig. 11. The detailed procedure takes the following form. Differentiation of Eq. 47 with respect to t gives

$$\frac{d[v(i)]}{dt} + \frac{i}{C} = \frac{dv}{di} \times \frac{di}{dt} + \frac{i}{C} = 0. \qquad [50]$$

On any chord,

$$\frac{dv}{di} = \text{constant} = r_p, \qquad [51]$$

so that Eq. 50 becomes, after separation of the variables,

$$\frac{dt}{C} = -r_p \frac{di}{i} \qquad [52]$$

TABLE V

v *volts*	i *milliamperes*	Δv *volts*	Δi *milliamperes*	r_p *ohms*	$\frac{i_2}{i_1}$	$\ln \frac{i_1}{i_2}$	$\frac{\Delta t}{C}$ *kiloseconds per farad*	$\frac{t}{C}$ *kiloseconds per farad*	*Error in seconds per farad*
100	21.6							0	0
		10	3.1	3,225	1.167	0.1542	0.4975		
90	18.5							0.4975	−3.7
		10	2.82	3,546	1.179	0.1650	0.585		
80	15.68							1.0825	−7.7
		10	2.78	3,597	1.215	0.1948	0.701		
70	12.90							1.7835	−13.0
		10	2.70	3,704	1.265	0.2357	0.873		
60	10.20							2.6565	−17.8
		10	2.40	4,167	1.308	0.2685	1.119		
50	7.80							3.7755	−30.0
		10	2.15	4,650	1.380	0.3224	1.499		
40	5.65							5.2745	−57.0
		10	1.90	5,263	1.506	0.4095	2.1552		
30	3.75							7.4297	−75.3
		10	1.60	6,250	1.744	0.5562	3.4764		
20	2.15							10.9061	−152.9
		10	1.25	8,000	2.390	0.8713	6.9704		
10	0.9							17.8765	−395.5
		10	0.9	11,110	∞	∞	∞		
0	0.0							∞	

which upon integration yields

$$\frac{\Delta t}{C} = -r_p \Big[\ln i \Big]_{i_1}^{i_2} = -r_p \ln \left(\frac{i_2}{i_1} \right), \qquad [53]$$

where i_1 is the current at t_1 and i_2 the current at $t_1 + \Delta t$. The data of Table V illustrate the numerical work for the solution for a source voltage E of 100 volts. The last column shows the deviations between the solution given by this method and that given by the graphical integration method. These deviations are too small to be noticed in the plot of Fig. 15.

6d. Single straight-line approximations

In this example, it is instructive to compare the results obtained when the two different straight-line approximations for the tube characteristic corresponding to lines A and B of Fig. 8 are used, and also to compare the relative accuracy of the results then obtained with those obtained for the corresponding inductive circuit treated in Art. 5d. For line A, as given by Eq. 20,

$$v = \frac{100}{21.6} i. \qquad [20]$$

The solution is therefore merely that for an RC circuit with R equal to $10^5/21.6$ ohms. From the treatment in Art. 10, Ch. III, the form of this solution is

$$i = 21.6\epsilon^{-2.16 \times 10^{-4}(t/C)} \qquad [54]$$

where i is in milliamperes, t is in seconds, and C is in farads. For line B, on the other hand, the solution is that for an RC circuit with R equal to 78,000/21.6 ohms and an applied voltage of $E - 22$ or $100 - 22$ or 78 volts, which gives

$$i = 21.6\epsilon^{-(0.216/780)(t/C)}. \qquad [55]$$

The solutions given by Eqs. 54 and 55 are shown dotted in Fig. 15, as lines A and line B, respectively. Neither is a very good approximation to the correct result. The solution using line B is more nearly correct, however, because as mentioned in the corresponding inductive problem it represents the tube characteristic better when the tube voltage is large compared with the capacitor voltage. However, the disagreement is appreciable in the vicinity of small i values where line B does not represent the $v(i)$ characteristic particularly well. In the inductive circuit the disagreement for small values of i is not apparent in the plot of i as a function of t in Fig. 10, but it would be apparent if di/dt were plotted as a function of t. A given result thus may appear to be fairly good on a plot because of the particular function chosen for that plot, but that result is not necessarily a generally good one. For example, if for the inductive

circuit the voltage across the coil as a function of time were desired, this voltage being proportional to di/dt, the result for line B would show the same deviations that are evident in the corresponding capacitive circuit.

6e. Other analytic approximations

In a manner analogous to that used in the treatment of the inductive circuit in Art. 5e, the integration of Eq. 47 for the capacitive circuit may be performed through use of the analytical approximation for the tube characteristic as given by Eq. 24. From Eq. 48,

$$\frac{dt}{C} = -\frac{dv}{i(v)}, \tag{48a}$$

so that

$$\frac{t}{C} = -\int_E^v \frac{dv}{i(v)} = \int_v^E \frac{dv}{i(v)}. \tag{49a}$$

The substitution of Eq. 24 in Eq. 49a gives

$$\left.\begin{aligned} \frac{t}{C} &= \frac{1}{K}\int_v^E v^{-n}\,dv = \left[\frac{v^{-(n-1)}}{-K(n-1)}\right]_v^E = \frac{E^{-(n-1)} - v^{-(n-1)}}{-K(n-1)} \\ &= \frac{E^{(n-1)} - v^{(n-1)}}{(n-1)KE^{(n-1)}v^{(n-1)}}. \end{aligned}\right\} \tag{56}$$

For n equal to 3/2, K equal to 22×10^{-6} ampere per volt$^{-3/2}$, and E equal to 100 volts, Eq. 56 gives

$$\frac{t}{C} = \frac{0.28 - i^{1/3}}{1.1 \times 10^{-4} i^{1/3}} \text{ sec/farad}, \tag{56a}$$

when i is in amperes. The calculated points for this relation lie substantially on the solid curve of Fig. 15 obtained by graphical integration, thus again indicating that the representation of the tube characteristic by the 3/2-power law is satisfactory in problems of this kind.

7. Circuits with iron-cored inductors and constant voltage applied

In the circuits used for illustration in Art. 5, it is assumed that the inductance parameter is essentially linear compared with the resistance parameter, even though the inductor may contain an iron core. In some circuits, however, this assumption is not valid and the element that has the predominant nonlinear parameter is the inductance. For example, the inductance parameter of the field circuit of a motor or generator, of either the direct- or alternating-current type, is essentially nonlinear compared with the resistance parameter of the field rheostat. If the field-discharge

resistor is of the type using such a material as thyrite, however, the inductance parameter may not be conspicuously nonlinear in comparison with the resistance parameter.

The build-up of flux or current in the field winding of an alternator when the field circuit is suddenly connected to a source of constant voltage is considered as a specific illustration. The connection diagram for such a circuit is as shown in Fig. 16. The core material is such that the nonlinear relation between the core flux φ and the current i is that given by a curve of the type shown in Fig. 17.

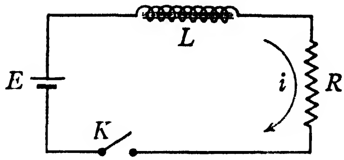


FIG. 16. Nonlinear inductance in series with linear resistance, constant source voltage.

The differential equation expressing the voltage equilibrium for the circuit at any instant of time is

$$N\frac{d\varphi}{dt} + Ri = E \qquad [57]$$

where $N\varphi$ represents the instantaneous flux linkages and R represents the total series resistance of the circuit. Since the data are given in terms of a curve of φ and i, it is anticipated that explicit graphical integration may

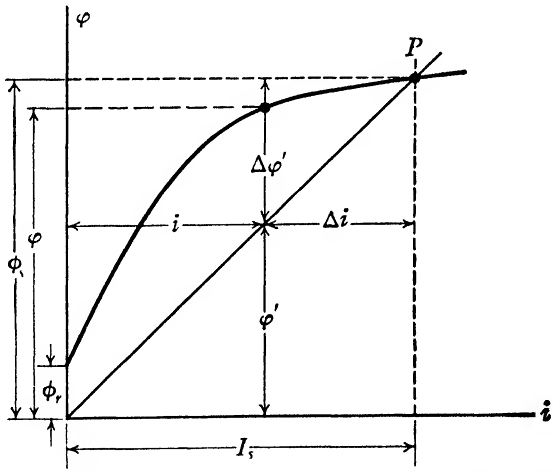


FIG. 17. Rising magnetization curve for inductance element, Fig. 16.

prove convenient for obtaining the result. To use this process, an attempt should be made to transform the equation into a form permitting separation of the variables in a manner similar to that of Arts. 5b and 6b.

The final steady value of the current is given by

$$I_s = \frac{E}{R}. \qquad [58]$$

The corresponding steady value of flux according to the curve of Fig. 17 is denoted by ϕ_s at the point P. A straight line from the origin of Fig. 17 to point P then evidently represents a fictitious linear relation between flux and current which results in the same steady-state value of flux for the given source voltage E. The instantaneous flux for this fictitious linear relation corresponding to the current i is denoted by φ'.

The first step in the graphical solution of Eq. 57 is to substitute in Eq. 57 the expression for E given by Eq. 58:

$$N\frac{d\varphi}{dt} = E - Ri = R(I_s - i). \tag{59}$$

From Fig. 17 it is seen that

$$(I_s - i) = \Delta i \tag{60}$$

and

$$\frac{\Delta i}{I_s} = \frac{\Delta\varphi'}{\phi_s}. \tag{61}$$

The utilization of this expression in Eq. 59 results in

$$N\frac{d\varphi}{dt} = RI_s\frac{\Delta\varphi'}{\phi_s}, \tag{62}$$

or

$$dt = \frac{N\phi_s}{RI_s}\frac{d\varphi}{\Delta\varphi'} = \frac{L'}{R}\frac{d\varphi}{\Delta\varphi'}, \tag{63}$$

where

$$L' = \frac{N\phi_s}{I_s} \tag{64}$$

is merely a fictitious inductance corresponding to the linear $\varphi(i)$ relation introduced and does not imply the existence of a linear circuit parameter. The quantity

$$\frac{L'}{R} = T' \tag{65}$$

is then the time constant of the fictitious linear circuit — the fictitious linear circuit being merely that which gives rise to the same steady values of current and flux.

Equation 63 may then be written as

$$\frac{t}{T'} = \int_{\phi_r}^{\varphi}\frac{d\varphi}{\Delta\varphi'}, \tag{66}$$

which is recognized as an integral of the same form as that occurring in Arts. 5b and 6b, and which can be evaluated readily either graphically or

numerically by the methods used there. The same types of approximations as those used in Arts. 5c and 6c, that is, replacement of the actual $\varphi(i)$ characteristic by one or a series of chords, can also be applied here. The detailed solution of this problem is not performed here since no new ideas or techniques are involved.

The use of a flux-current characteristic makes appropriate allowance for the effect of hysteresis provided the curve used represents the flux build-up conditions; that is, Fig. 17 should represent a rising magnetization curve for a noncyclic time-varying magnetomotive force. The eddy-current losses in the iron core are neglected in the treatment. Such neglect leads to a good approximation if the core is finely laminated, because the losses are generally then negligible. In the transient problem, this approximation is made because of the mathematical difficulties which are otherwise introduced. In the steady-state analysis for a cyclic magnetomotive force given in this series in the volume on magnetic circuits and transformers, the effect of the eddy-current component of the core loss is considered.

8. Circuits with iron-cored inductors and alternating voltage applied

The transients that occur when sinusoidal voltages are connected to nonlinear inductive circuits are often of considerable importance in engineering practice. The circuits sometimes comprise merely a nonlinear inductance, as is essentially the situation when an iron-core transformer winding is suddenly connected to its source of voltage. For certain conditions of residual core flux, core saturation, and angle on the wave of the sinusoidal voltage at which the switch is closed, the peak value of the exciting current may reach a value of the order of ten times the rated load current of the transformer. For these reasons means of computing the value of such current are desirable so that in the design of the transformer provision can be made for bracing the coils to enable them to withstand the stresses resulting from the combined action of the large currents and the magnetic fields. A method for computing the approximate maximum value of these transient exciting currents follows.

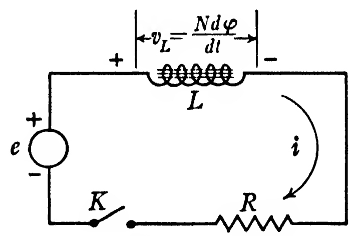


FIG. 18. Nonlinear inductance in series with linear resistance, sinusoidal source voltage.

The circuit to be considered is shown in Fig. 18. The differential equation for this circuit is

$$N\frac{d\varphi}{dt} + Ri = e, \tag{67}$$

in which N is the number of turns in the coil and φ is flux in the core linking these turns. On account of the nonlinear relationships existing between φ and i, eliminating one or the other from Eq. 67 is desirable. A guide as to which one to eliminate is obtained from the knowledge that for sinusoidal applied voltage, φ is essentially sinusoidal provided Ri is not large, whereas i bears no direct, simple relationship to e. It is at once evident, therefore, that a solution in terms of φ is easier to obtain than one in terms of i. Since in many applications of transformers or reactors the resistance drop is small compared with the reactance drop, the Ri term in Eq. 67 is small compared with the $N(d\varphi/dt)$ term; and hence an expression for the current that may be used with good approximation *in the Ri term only* is

$$i \approx \frac{N\varphi}{L_{av}}, \tag{68}$$

in which L_{av} is an average inductance coefficient; that is, L_{av} is given by the average slope of the $\varphi(i)$ curve for the coil. With this approximation Eq. 67 becomes

$$N\frac{d\varphi}{dt} + \frac{R}{L_{av}}N\varphi = e, \tag{69}$$

which can be solved readily for φ as a function of time t.

Since e is sinusoidal, it may be written as

$$e = \mathcal{R}_e[\mathrm{E}_m\epsilon^{j\omega t}], \tag{70}$$

where

$$\mathrm{E}_m = E_{m1} + jE_{m2}, \tag{71}$$

E_{m1} and E_{m2} being real constants. From previously solved similar equations, the steady-state solution of Eq. 69 is easily shown to be

$$\varphi_s = \mathcal{R}_e[\Phi_m\epsilon^{j\omega t}], \tag{72}$$

in which

$$\Phi_m = \frac{\mathrm{E}_mL_{av}}{N(R + j\omega L_{av})}. \tag{73}$$

The transient portion of the solution is evidently given by

$$\varphi_t = \phi_t\epsilon^{-(R/L_{av})t}, \tag{74}$$

so that the complete solution becomes

$$\varphi = \mathcal{R}_e[\Phi_m\epsilon^{j\omega t}] + \phi_t\epsilon^{-(R/L_{av})t}. \tag{75}$$

When t is zero, φ has the value ϕ_r if the circuit is initially at rest. This gives

$$\phi_t = \phi_r - \mathcal{R}_e[\Phi_m], \tag{76}$$

and finally

$$\varphi = \mathcal{R}_e[\Phi_m \epsilon^{j\omega t}] + \{\phi_r - \mathcal{R}_e[\Phi_m]\}\epsilon^{-(R/L_{av})t}. \qquad [77]$$

If at the switching instant E_{m1} is zero and E_{m2} has the value, $-E_m$, then from Eq. 73

$$\Phi_m = \frac{-jL_{av}E_m}{N(R + j\omega L_{av})}. \qquad [78]$$

Since it is assumed that ωL_{av} is large compared with R, Φ_m in Eq. 78 has the approximate value

$$\Phi_m \approx \frac{-E_m}{N\omega}\angle 0^\circ, \qquad [79]$$

which when substituted in Eq. 77 gives, for this switching instant,

$$\varphi = \frac{-E_m}{N\omega}\cos\omega t + \left(\frac{E_m}{N\omega} + \phi_r\right)\epsilon^{-(R/L_{av})t}. \qquad [80]$$

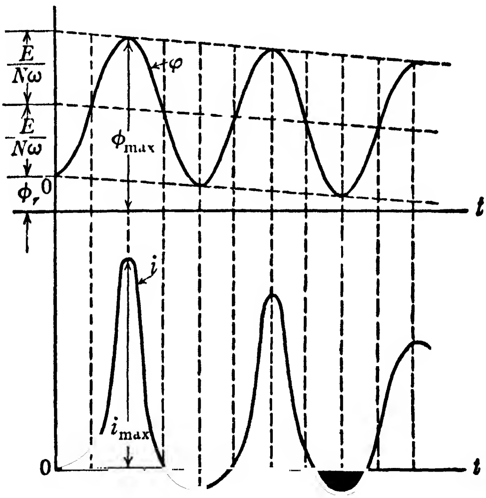


FIG. 19. Flux and current in inductance element, Fig. 18.

Plots of the wave forms of flux and the corresponding current according to Eq. 80 are shown in Fig. 19 for an assumed dissipation ratio of $R/(\omega L_{av})$ equal to 0.05, or 5 per cent. Since the first maximum of the flux occurs when t is near π/ω, this dissipation ratio gives

$$\varphi_{max} = \phi_r\epsilon^{-0.05\pi} + \frac{E_m}{N\omega}(1 + \epsilon^{-0.05\pi}) \approx 0.86\,\phi_r + 1.86\frac{E_m}{N\omega}. \qquad [81]$$

This maximum flux is almost twice the normal flux amplitude. Hence, even with an appreciable amount of power dissipation in the resistance, the flux may be so large that during the initial cycles following a switching operation the current must surge to very high values because of core saturation. In power transformers these initial transient currents may become severe enough to require special considerations in the design, as stated previously.

In order to illustrate the severity of such transient currents, the switching transient when voltage is applied to one winding of a particular transformer is considered. All other windings are open-circuited. Under these conditions the transformer can be treated as a reactor. The transformer name-plate data are: 100 kilovolt-amperes; 60 cycles per second; primary volts, 11,500; secondary volts, 2,300. For comparison with the transient current, the rated full-load primary current of the transformer is given by:

$$I_1 = \frac{100{,}000}{11{,}500} = 8.7 \text{ amp.} \qquad [82]$$

Under full-load operation this current is practically sinusoidal in form for a sinusoidal applied voltage. Hence the full-load peak value of the primary current is

$$I_{1m} = 8.7\sqrt{2} = 12.3 \text{ amp.} \qquad [83]$$

The foregoing theoretical considerations show that a large transient current may be expected when an appreciable residual flux is left in the core. A large residual flux occurs if the transformer is disconnected from the voltage source while the flux is at or near its maximum value, that is, if the flux has the value on the hysteresis loop corresponding to zero magnetomotive force. If the transformer is reconnected to the source at an instant corresponding to a voltage phase angle which causes a rate of change of flux such as to increase the flux from the residual value, the maximum value of the flux is considerably larger than normal. Because

TABLE VI

$N\varphi$ (*weber-turns*)	*i amperes*	$N\varphi$ (*weber-turns*)	*i amperes*
30.0	0	55.7	3.0
39.0	0.3	58.1	4.5
44.7	0.6	61.2	7.5
48.0	0.9	66.7	15.0
50.3	1.2	70.6	22.5
51.8	1.5	110.0	98.0
54.1	2.25		

of the saturation effect the current will rise to a peak value which is even greater in comparison with normal than is the flux.

Table VI gives the data for the rising magnetization curve of this transformer from a large positive residual flux. The curve plotted from these data is shown in Fig. 20. The transformer winding resistance is first neglected in order that an approximate idea of the maximum value of the

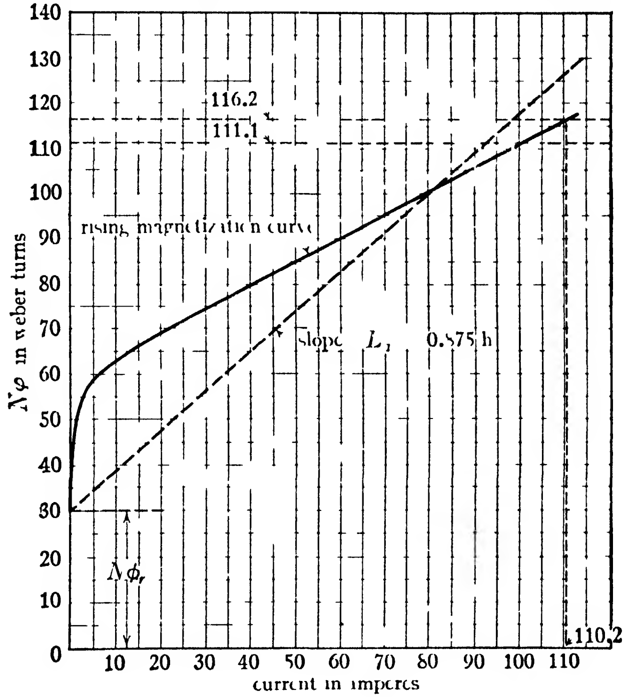


FIG. 20. Rising magnetization curve for transformer.

transient current surge may be obtained. The most unfavorable switching instant (with regard to the voltage phase) giving rise to a positive current surge is that at which the voltage passes through zero and increases positively. Equation 80 satisfies these initial conditions. Thus, for the given initial conditions and for R assumed to be zero, Eq. 80 becomes

$$\varphi = \frac{E_m}{N\omega}(1 - \cos \omega t) + \phi_r \qquad [84]$$

The numerical values for the transformer are

$$V_m = V\sqrt{2} = 11{,}500\sqrt{2} = 16{,}260 \text{ v}, \qquad [85]$$

$$\omega = 2\pi f = 6.28 \times 60 = 377 \text{ radians per sec}, \qquad [86]$$

$$N\phi_r = 30 \text{ weber turns}. \qquad [87]$$

Hence,

$$\left.\begin{aligned} N\varphi &= \frac{16{,}260}{377}(1 - \cos 377t) + 30 \\ &= 43.1\ (1 - \cos 377t) + 30 \text{ weber-turns.} \end{aligned}\right\} \qquad [88]$$

As indicated by the discussion in Art. 7, this solution is correct only up to the first flux maximum on account of the hysteresis effect, but, so far as winding and eddy-current losses are negligible, this first maximum is correctly evaluated. It evidently occurs when the cosine first reaches the value -1. Thus the first maximum value is given by

$$(N\varphi)_{max} = 2 \times 43.1 + 30 = 116.2 \text{ weber-turns.} \qquad [89]$$

From the magnetization curve of Fig. 20 the transient current at this value of flux linkage is seen to surge to the peak value of 110.2 amperes, which is almost nine times the normal full-load peak value of 12.3 amperes.

This transient-current peak now is recalculated with the primary winding resistance taken into account. For the same switching instant, the resulting flux relation is expressed by Eq. 80. Here L_{av} is the average slope of the magnetization curve of Fig. 20. Small variations in the value of this average inductance affect the value of the resultant transient peak but slightly, so that the average slope need be determined only approximately. The previous solution in which resistance was neglected is helpful as a guide to the range of $N\varphi$ for which L_{av} is to apply. In Fig. 20, the peak values of 116.2 and 110.2 for flux linkages and current, respectively, are indicated by dotted lines and represent approximately the range of the magnetization curve traversed. This range is approximately the same when resistance is considered, so that the sloping dotted line appears to be a reasonable average straight line. The slope of this line is 0.875, and this value is therefore chosen for L_{av}.

The resistance of the primary winding of this transformer is given as 7.7 ohms. The average reactance of this winding is

$$\omega L_{av} = 0.875 \times 377 = 330 \text{ ohms.} \qquad [90]$$

The assumption in the theoretical derivation of Eq. 80 that ωL_{av} is large compared to R is thus seen to be justified in this example. The substitution of R and L_{av} in Eq. 80 gives

$$N\varphi = -43.1 \cos 377t + (43.1 + 30)\epsilon^{-8.8t}. \qquad [80a]$$

The first maximum of this function occurs approximately at the instant when $377t$ equals π, when the cosine function equals -1. The substitution of this value of time gives

$$\left.\begin{aligned} (N\varphi)_{max} &= 43\ 1 + (43.1 + 30) \times 0.93 \\ &= 43.1 + 68.0 = 111.1 \text{ weber-turns.} \end{aligned}\right\} \qquad [80b]$$

From the magnetization curve of Fig. 20, the corresponding peak value of the transient current is 100.0 amperes, which is more than eight times the normal full-load peak value. The effect of the primary winding resistance is relatively small, so that the solution for the nondissipative case may ordinarily be used when an approximate result is sufficient.

Additional methods of analysis, such as are applied to the nonlinear resistance element in the preceding articles, are also useful in certain problems involving ferromagnetic phenomena. Such problems are likely to be more complicated, however, particularly if hysteresis and eddy-current effects are significant, because of the complicated nature of the actual $\varphi(i)$ characteristic. As the complication of the circuit is increased, the analytical and numerical methods rapidly become impracticable, and even the machine methods discussed in Art. 10 quickly reach their practicable limit. As a result, the theory of networks of these kinds is relatively incomplete, and such development as takes place in practical applications is primarily the result of experiment. In this development, the application of model theory as treated in this series in the volume on magnetic circuits and transformers can be very helpful since it greatly reduces the amount of experimental work required. The use of models perhaps offers one of the most promising means of simplifying such investigations and of stimulating development.

9a. Nonlinear circuits comprising resistance, inductance, and capacitance

A more complicated problem of circuit analysis arises when the nonlinear circuit contains both inductive and capacitive energy-storage elements. Numerous rectifier circuits, ferroresonant circuits used in relay and control devices, or protective circuits on transmission lines fall within this category. When calculations on such circuits are to be performed it is desirable, for reasonable simplicity, to have not more than one nonlinear parameter in any circuit. Even with such simplification, the explicit solution for the transient is generally difficult or impossible, and graphical or semigraphical or step-by-step processes must be adopted. Of the methods available, the step-by-step has the largest scope. For this reason its application is treated in considerable detail.

9b. Step-by-step calculation

For illustration of the step-by-step method the circuit represented diagrammatically in Fig. 21 is considered. Here a tube is used as the nonlinear resistor, and a direct voltage is applied only as a means for clarifying the explanation of the method of calculation. It is not to be implied that the applicability of the method is restricted to these conditions.

The $v(i)$ characteristic of the tube is assumed to represent the composite nonlinear resistance characteristic for the entire circuit. The problem is to find the current i as a function of time following the closing of

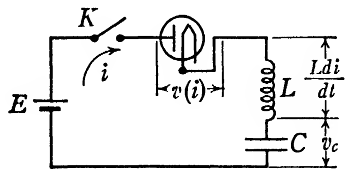


FIG. 21. Nonlinear resistance in series with linear inductance and capacitance, constant source voltage.

the switch K, the charge on the capacitor and the current in the inductor being zero prior to the closing of this switch.

The differential equation of the circuit is

$$L\frac{di}{dt} + v(i) + \frac{1}{C}\int i\,dt = E. \qquad [91]$$

If the voltage drop across the condenser is written as

$$v_C(t) = \frac{1}{C}\int i\,dt, \qquad [92]$$

and the functional notation is dropped for convenience, Eq. 91 can be rewritten as

$$L\frac{di}{dt} = E - v_R - v_C, \qquad [91a]$$

or

$$di = \frac{1}{L}(E - v_R - v_C)\,dt. \qquad [91b]$$

In Eqs. 91a and 91b, v_R and v_C are functions of current and of time. Hence di/dt is a function of current and time.

The essential feature of the step-by-step process of integration of di to obtain i is the utilization of the fact that if the interval Δt is short enough, a satisfactory solution can be obtained on the assumption that di/dt is constant throughout the interval. The application of this feature may be illustrated by consideration of the circuit behavior during a few successive short time intervals.

At the instant immediately following the closing of the switch K, from which instant time is measured, the current remains zero because of the presence of inductance; hence v_R is zero and v_C is zero. Therefore the value of di/dt when t is zero is E/L. If it is assumed that di/dt remains constant

at this value for a small time interval Δt, the current then rises linearly to a value Δi_1 given by $[di/dt]_0\Delta t$ at the end of the first interval. The values of v_{C1} and v_{R1} at the end of the first interval Δt_1, and hence $(E - v_{R1} - v_{C1})$, are then computed from the derived value of i. The value of $[di/dt]_1$ is then $(E - v_{R1} - v_{C1})/L$, and is assumed constant throughout the second interval Δt_2. The value of i_2 at the end of the second interval is $i_1 + \Delta i_2$, or $i_1 + [di/dt]_1\Delta t$. The procedure is continued in this way for as many values of Δt as are desired.

The selection of the size of the interval Δt is a matter that requires good judgment if large errors are to be avoided. The procedure becomes more exact as Δt approaches an infinitesimal interval, but the time required to perform the calculation increases proportionally. As a rough guide to the choice of size of interval, it usually is desirable to make Δt such that, were $[di/dt]_0$ maintained, several intervals would be required to reach an estimated maximum value of i. Since the choice of interval is somewhat arbitrary, a method for determining the order of magnitude of the error introduced by the finite interval or for extrapolating the data to give the correct solution is mentioned below.

For a numerical illustration, E is 100 volts, C is 1.0×10^{-8} farad, L is 1.0 henry, and the $v(i)$ characteristic is the same as that used for the tube in Arts. 5b and 6b, and given by Fig. 8. From an inspection of the tube characteristic, it appears that the maximum current for the particular applied voltage may be about 0.01 ampere, within a factor of two or three. Since $[di/dt]_0$ is 100 amperes per second, the time required for i to reach 0.01 ampere is $0.01/100$ or 10^{-4} second. The value of Δt is selected as one-tenth of this interval for the illustration, or 10^{-5} second. This rough estimating is permissible because the value of Δt is not particularly critical, provided it is sufficiently small.

The actual step-by-step calculation may now be made. At the end of the first time interval, t_1 is 10^{-5} second, and

$$\Delta i_1 = i_1 = \left[\frac{di}{dt}\right]_0 \Delta t = 100 \times 10^{-5} = 10^{-3} \text{ amp.} \tag{93}$$

During this interval, the condenser accumulates a charge equal to the product of Δt and the average i throughout the interval, or

$$q_1 = \frac{10^{-3}}{2} \times 10^{-5} = \frac{10^{-8}}{2} \text{ coulomb.} \tag{94}$$

The condenser voltage at time t_1 is

$$v_{C1} = \frac{q_1}{C} = \frac{10^{-8}}{2 \times 10^{-8}} = 0.5 \text{ v,} \tag{95}$$

and from the tube characteristic of Fig. 8, v_{R1} is 11.0 volts. Then, from Eq. 91b at time t_1,

$$\left.\frac{di}{dt}\right|_1 = \frac{100 - 0.5 - 11.0}{1.00} = 88.5 \text{ amp/sec.} \qquad [96]$$

Of course v_R and v_C increase continuously during the entire first time interval, and di/dt decreases continuously from its initial value of 100 amperes per second although it is assumed to be constant. This error is inherent in the step-by-step method, and is cumulative in its effect. However, it can be made small if a reasonably small value of Δt is used, and, at the expense of additional work, can be eliminated entirely, as is to be shown. When a differential equation is integrated analytically or by a machine such as the differential analyzer discussed in Art. 10, the interval Δt is reduced to an infinitesimally small value and the error is reduced to zero.

During a second time interval of 10^{-5} second between t_1 and t_2, $[di/dt]_1$ is assumed to be constant at 88.5 amperes per second, and on that basis i increases from its value of 1.0×10^{-3} ampere at time t_1 to its value i_2 of $(1.0 + 0.885)10^{-3}$ or 1.89×10^{-3} ampere at time t_2, which is 2×10^{-5} second. Also during this second time interval the condenser charge increases by

$$\Delta q = \frac{i_1 + i_2}{2}\Delta t = \left(i_1 + \frac{\Delta i}{2}\right)\Delta t = 1.44 \times 10^{-8} \text{ coulomb;} \qquad [97]$$

hence its total charge at time t_2 is

$$q_2 = q_1 + \Delta q = (0.50 + 1.44)10^{-8} = 1.94 \times 10^{-8} \text{ coulomb.} \qquad [98]$$

The condenser voltage at time t_2 is then

$$\frac{q_2}{C} = \frac{1.94 \times 10^{-8}}{1.00 \times 10^{-}} = 1.94 \text{ v.} \qquad [99]$$

At this instant the value of v_{R2} corresponding to the current of 1.89 milliamperes is 18.5 volts, from the tube characteristic.

For a third and any succeeding time interval the same process is repeated until the solution is carried as far as may be desired. Usually such calculations are carried out in a tabular form as, for example, that shown in Table VII, where a few lines of the calculation started in the foregoing paragraphs are given.

The errors arising in the step-by-step process are caused by the use of finite instead of infinitesimal increments of time. Perhaps the most practicable method of eliminating the error is to carry through the calculation three times, each time with a different value of Δt. If a dependent variable is plotted as a function of Δt at any given value of the independent vari-

TABLE VII*

t *micro-seconds*	i *milli-amperes*	v_R *volts*	v_C *volts*	$E - v_R - v_C$ *volts*	Δi *milli-amperes*	Δq *micro-coulombs*	Δv_C *volts*
0	0	0	0	100			
					1.0	0.005	0.5
10	1.0	11.0	0.5	88.5			
					0.885	0.0144	1.44
20	1.89	18.5	1.94	79.6			
					0.796	0.0229	2.29
30	2.69	23.5	4.23	72.1			
					0.722	0.0305	3.05

* Computations based on

$$\Delta i = \frac{E - v_R - v_C}{L}\Delta t, \quad \Delta q = \left(i + \frac{\Delta i}{2}\right)\Delta t, \quad \Delta v_C = \frac{\Delta q}{C}.$$

able, an extrapolation to the zero value of Δt gives the correct value of the dependent variable. The results obtained when this method is applied to the first portion of the example given are shown in Table VIII for the current i and the condenser voltage v_C.

TABLE VIII

Time	Current i (*Milliamperes*)			
t *micro-seconds*	$\Delta t = 20\ \mu sec$	$\Delta t = 10\ \mu sec$	$\Delta t = 5\ \mu sec$	*Extrapolation to* $\Delta t = 0$
20	2.00	1.89	1.78	1.66
40	3.77	3.41	3.28	3.13
60	5.01	4.63	4.50	4.38
	Condenser voltage v_C (*volts*)			
20	2.00	1.94	1.89	1.82
40	7.77	7.28	7.01	6.80
60	16.55	15.36	14.84	14.45

If only an estimate of the order of magnitude of the errors is wanted, it may be sufficient to make three calculations using different time intervals only for the first portion of the curve. Accurate results, however, can be obtained only if the extrapolation process is applied to the entire length of the result curve. The use of small increments Δt of time contributes to accuracy, and also to labor.

In Fig. 22 are shown the curves of i and v_C calculated with time intervals of 10^{-5}, 2×10^{-5}, and 0.5×10^{-5} second, and the extrapolation of the result to zero time interval.

In the preceding paragraphs, numerical values are given to all the parameters in order to illustrate, as simply as possible, the physical

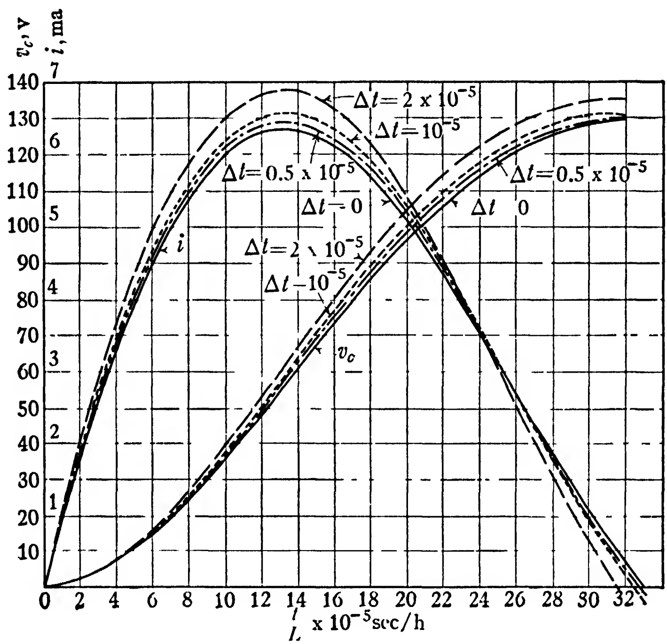


FIG. 22. Current in circuit of Fig 21 computed by step-by-step process.

reasoning associated with the step-by-step calculation. Actually this same numerical work is applicable to a somewhat more general case if Eqs. 91b and 92 are rewritten, respectively, as

$$di = (E - v_R - v_C)d\left(\frac{t}{L}\right) \qquad [91c]$$

and

$$v_C = \frac{L}{C}\int id\left(\frac{t}{L}\right). \qquad [92a]$$

In terms of finite increments and in a form for step-by-step calculation, Eqs. 91c and 92a become

$$\Delta i = (E - v_R - v_C)\Delta\left(\frac{t}{L}\right) \qquad [91d]$$

and

$$\Delta v_C = \frac{L}{C}\left(i + \frac{\Delta i}{2}\right)\Delta\left(\frac{t}{L}\right). \qquad [92b]$$

Equations 91d and 92b can be integrated numerically provided only the tube characteristic, the applied voltage E, and the ratio L/C are known, the independent variable being the quantity t/L instead of t. When this substitution is made, the results of Table VII are applicable not only for the specific values of L and C of the example but also for all values of L and C for which L/C is 10^8. The values of t in Table VII then become values of t/L which must be multiplied by a numerical value of inductance to be converted to time. For example, if the value of L is 12.0 henrys and of C is 12.0×10^{-8} farad, L/C is 10^8 henrys per farad and the calculations apply except that the time increment becomes 12.0×10^{-5} second and each of the values in the t column must be multiplied by the value of L, namely, 12.0, to be converted to seconds. In Fig. 22 are plotted the results of carrying Table VII to completion, expressed in terms of t/L as the independent variable.

From the curves for i and v_C it is seen that the current would be oscillatory except for the fact that the resistor used in the illustration is a tube that does not conduct in the reverse direction, so that the charge on the condenser is trapped. The condenser voltage of 129.6 volts exceeds the battery voltage by 29.6 volts and would produce a reverse surge of current were it not for the blocking action of the tube.

The step-by-step analysis can readily be extended to problems involving nonlinear inductance or nonlinear capacitance, but it must be recognized that an expression describing exactly the nonlinear relationship of the inductance or capacitance parameter may not be easily obtainable. Frequently, these parameters introduce double-valued functions, but in spite of these complications it is possible, by the use of carefully chosen approximations, to obtain results that are very useful in engineering. As an illustration of this situation, the many and varied analyses of ferroresonant circuits might be cited. These circuits, which comprise ferromagnetic inductive elements in combination with linear or nonlinear resistive and capacitive elements, have a response that is sensitive to voltage, current, or frequency. They are used extensively in voltage regulation, relay, and other control circuits.

9c. Straight-line approximations

The problem of Art. 9b can be analyzed through substitution of a series of chords for the volt-ampere characteristic of the nonlinear resistor, as was done in the two preceding examples. With the present circuit it is much more difficult, however, to join the solutions for the various straight-line portions. The effect upon the circuit of passage from one line to the next is that of a sudden alteration of the resistance. The determination of the transient response in the succeeding linear interval from the boundary values at the end of the preceding interval is a rather lengthy process,

which must be repeated frequently before the solution is completed. This method is less practical than the step-by-step procedure and is not discussed in detail here.

As in the preceding examples, an approximation of the $v(i)$ characteristic by means of a single straight line may also be attempted. Except when the range over which the nonlinear device is operated is so limited that a single straight line is a good approximation to the actual volt-ampere curve, the results of such a procedure are useful only where an approximate quantitative result is adequate.

10. Mechanical method of solution: the differential analyzer

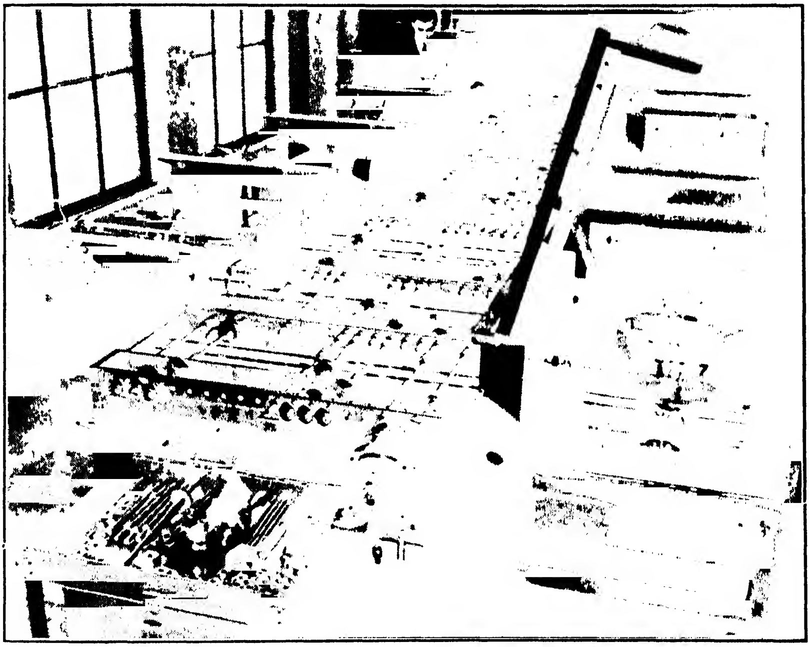
It is shown in Art. 9b that an ordinary nonlinear differential equation formulated in accordance with a physical problem can always be solved for given physically realizable boundary conditions by the step-by-step or numerical integration process. The error in such a solution can be made arbitrarily small if sufficiently small increments of the independent variable are used, but the labor involved may often be considerable.

Another valuable method of solution of rapidly growing importance in a wide variety of linear or nonlinear problems is one that uses mechanical calculating aids. Numerous mechanical devices have been developed to perform certain mathematical operations. For example, some electrical measuring instruments in effect give readings that are the result of performing various mathematical operations on certain electrical quantities, and can be connected in ways that make possible a machine for performing complicated mathematical processes. Several mechanical devices for performing the process of integration have also been developed. One such mechanism is the ball-disk-and-roller integrator invented by Lord Kelvin. It has been carried to a high degree of effectiveness in its application in a machine called the differential analyzer[3] which is the outstanding mechanism of the present era for solving linear or nonlinear differential equations. When the elements of a differential analyzer are properly connected, the machine performs *simultaneously and continuously* all the processes of *integration, multiplication, and addition* in accordance with the dictates of the equation for which the connections are made. Only those methods of solution that use such a machine are truly classified as machine methods and are to be distinguished from those methods that merely use the various familiar mechanical calculating aids such as planimeters or business calculating machines. While the machine method cannot be used to solve problems that cannot be solved numerically step by step, the economies in time and labor achieved by its use are frequently very great, particu-

[3] V. Bush, "The Differential Analyzer. A New Machine for Solving Differential Equations," *J.F.I.*, CCXII (1931), 447 488.

larly in complicated problems where step-by-step calculations become involved.

Unfortunately the availability* of a calculating machine of any appreciable scope is the exception rather than the rule on account of the high cost of such devices, so that recourse must often be had to the pencil-and-paper method of solution. Even if a machine is available, a thorough understanding of the mathematical processes involved in the solution is still essential if improper use of the machine is to be avoided. Familiarity with the methods outlined in Arts. 5, 6, 7, 8, and 9 is prerequisite to the successful use of the machine method.



Differential Analyzer at Massachusetts Institute of Technology (footnote 3, p. 712).

In its operation, the differential analyzer carries out a process which is similar to the step-by-step method but with the important difference that instead of using a finite increment of time, for example, a Δt of 10^{-5} second, it uses an infinitesimal increment, or in other words the time increases continuously instead of in finite steps, and all dependent quantities vary continuously. This procedure is the equivalent of a step-by-step

* Differential analyzers are located at the Massachusetts Institute of Technology; the Moore School of Engineering at the University of Pennsylvania, The University, Manchester, England, and The University Observatory, Oslo, Norway.

solution in which Δt is zero, and the error introduced by the finite increment is therefore made zero.

The use of such a machine is subject to two major limitations, however; first, the complexity of the equation that can be solved is limited by the number of available elements in the machine; and, second, the accuracy obtainable is limited by the degree of perfection of the mechanism.

The most important mechanism in the differential analyzer is the integrator. Figure 23a gives a schematic diagram of this device. In its usual form it consists of a flat disk of metal or glass that is rotated in a horizontal plane about a vertical axis. The amount of rotation in revolutions of its drive shaft represents a quantity, for example, V. A wheel of known

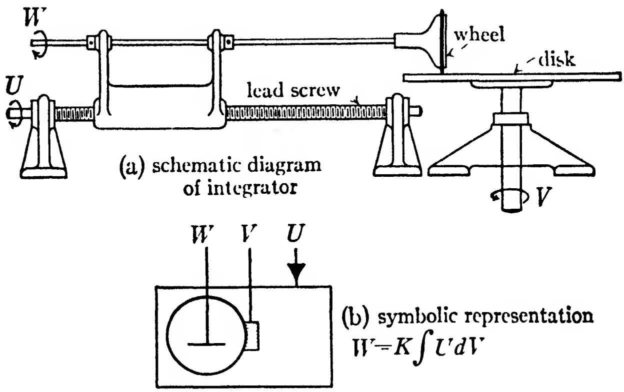


FIG. 23. Integrating unit, differential analyzer.

diameter, having a sharp edge, is supported by a horizontal shaft that is in the plane of the shaft that rotates the disk, and is rotated an amount W when the disk rotates an amount V. The necessary torque to rotate the wheel is supplied by friction between the wheel and the disk. The position of the edge of the wheel with respect to the center of the disk can be varied by means of a lead-screw mechanism, the turns of which are proportional to U.

If the wheel of the integrator is displaced from the center of the disk by a rotation U of the lead screw, and the horizontal disk is rotated an amount ΔV, the corresponding rotation ΔW of the vertical wheel (assuming no slipping between the wheel and the disk) is then equal to

$$\Delta W = KU\Delta V, \tag{100}$$

or, if ΔV is allowed to approach zero,

$$dW = KUdV, \tag{100a}$$

in which K is a constant determined by the pitch of the lead screw and the diameters of the wheel and disk. Since this relation holds for any in-

stantaneous values of U and V, both sides of Eq. 100a can be integrated, so that

$$W = K\int_{V_1}^{V_2} UdV. \qquad [101]$$

Equation 101 shows that the output of the integrator, as represented by the number of revolutions of the shaft W, is continuously proportional to the integral of the lead-screw rotation U with respect to the disk rotation V. By proper choice of scale factors, this device can be used to integrate any variable with respect to any other variable, provided the rotations of the shafts coupled to U and V are constrained in accordance with the functional relationships between the variables. Similarly, mechanisms are available in the machine for performing other mathematical operations on variables whose magnitudes are represented by the rotations of shafts, and a network of shafts is available for interconnecting the mechanisms. It is important to recognize that the magnitudes of all quantities are represented in terms of the rotation of a shaft or shafts in the machine.

Each integrator is equipped with a mechanical torque amplifier. This device is guided by the small torque produced by the friction between the wheel and the disk, and in turn duplicates the motion of the wheel and makes available at the output shaft a torque of approximately one pound-foot.

When the interconnections of the machine are being planned for a particular problem, representation of the various mechanisms by symbols is helpful. The symbol for an integrator is shown in Fig. 23b. The symbols

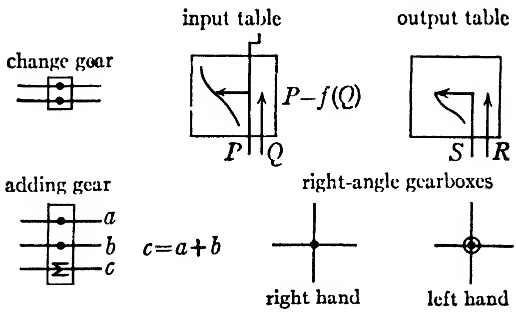


FIG. 24. Symbols for various mechanisms of differential analyzer.

for an adding or differential gear, an ordinary pair of change gears, right-angle gearboxes, an input table, and an output table are shown in Fig. 24. The input table is one on which a plotted curve representing a functional relation between variables appearing in the problem may be placed. The

shaft Q of the input table, Fig. 24, is coupled to the shaft in the machine that represents the independent variable of the function. As the machine rotates the independent-variable shaft, a pointer on the input table is moved parallel to the axis of abscissas. An operator turns a hand crank that moves the pointer at right angles to the first motion and at the same time rotates the shaft P. By adjusting the speed of cranking, the operator is able to make the pointer stay on the curve and in doing so introduces the desired functional relationship between the rotation of the shaft Q that represents the independent variable and the rotation of the shaft P that represents the dependent function.

The application of the differential analyzer to the problem discussed in Art. 9b is now made. The machine mechanisms are interconnected in accordance with Eqs. 91c and 92a, since these forms give a more general result. If T is set equal to t/L, and the left-hand side of Eq. 91c is expressed as a derivative, the equations combine as

$$\frac{di}{dT} = E - v(i) - \frac{L}{C}\int i\,dT. \qquad [102]$$

In terms of the symbols given in Figs. 23 and 24, and with the assumption that the integrator constant K is taken as unity for simplicity, the mechanical set-up for solving Eq. 102 is as shown in Fig. 25. In this figure,

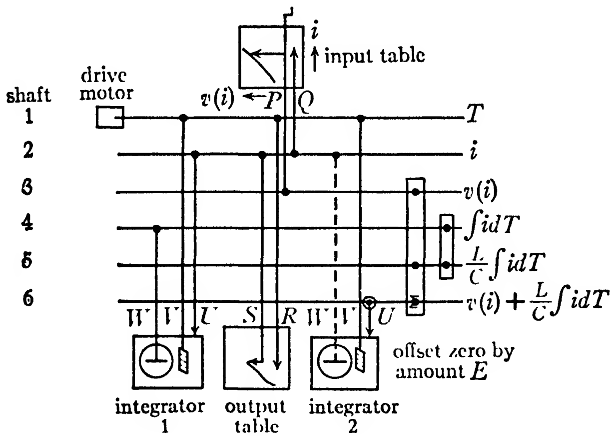


FIG. 25. Arrangement of differential analyzer for solution of Eq. 102.

shaft 1 represents the independent variable T and is driven by a main-drive motor. The number of turns through which shaft 1 is rotated is proportional to the value of T. Shaft 2 is considered to represent the current i, the dependent variable, which at this point is not known. For the present, however, the assumption is that a means to rotate shaft 2 can

eventually be found. Shaft 3 represents the relation $v(i)$ and its constraint is obtained by coupling shaft 2 through a right-angle gearbox to the abscissa drive, that is, shaft Q, of an input table on which is mounted a plot of voltage $v(i)$ as a function of current i. Then if the person operating the crank on the input table keeps the pointer on the curve and the shaft P of the input table is connected to shaft 3 through another gearbox, the rotation of shaft 3 is constrained in accordance with the function $v(i)$.

The quantity i is also connected to the lead-screw drive, shaft U, of an integrator; and the disk drive, shaft V, is driven in accordance with T from shaft 1. From Fig. 23, the rotation of the output shaft, that is, the W shaft, of this integrator evidently is then proportional to $\int idT$. The output shaft is connected to shaft 4. Equation 102 shows that the term $\int idT$ has the coefficient L/C; hence, gears of a ratio such that the rotation of shaft 5 is made proportional to $(L/C)\int idT$ are introduced between shafts 4 and 5. Shafts 3 and 5 are now linked by an adding gear, with the result that the rotation of shaft 6 is proportional to $v(i) + (L/C)\int idT$.

Equation 102 shows that the quantity just obtained on shaft 6 must be subtracted from the electromotive force E, and the result is di/dT. To make this subtraction, shaft 6 is connected to the lead screw, shaft U, of a second integrator, the connection being made by means of a left-hand gearbox so that the sense of the rotation is reversed, and the rotation of the lead screw represents a negative quantity. In addition, the zero point of the integrator is offset in a positive sense by an amount proportional to the quantity E. The lead screw therefore has a position at all times proportional to $E - v(i) - (L/C)\int idT$, which from Eq. 102 is equal to di/dT.

If the shaft which drives the disk of the second integrator, shaft V, is connected to shaft 1, the rotations of output of the integrator shaft W, are proportional to

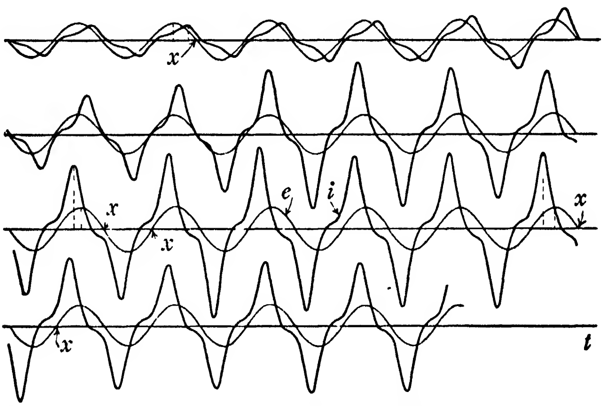
$$\int \frac{di}{dT}\,dT = i. \qquad [103]$$

As yet no drive is available for shaft 2, which is designated to represent i. However, an integrator producing i at its output shaft is now available. The final operation in the set-up is to *connect the output of the second integrator back to shaft* 2, so that shaft 2 is definitely turned in proportion to the variable i which it represents. In the diagram of Fig. 25, this important connection is shown dotted; at this point, the reader should fill in the connection with a solid line to indicate its completion.

In the interconnection of the mechanisms the various terms of the equation are produced one at a time, exactly as in the step-by-step

method. The final back-coupling of the machine by connecting the output of the second integrator back to shaft 2, corresponds to the point in the numerical calculation where a return is made to repeat the calculations for the next step. In the machine solution, however, there is the important difference that all the shafts move continuously when the independent variable shaft is turned, and the solution obtained is that which would be obtained if the step-by-step calculation could be carried out for a time interval Δt of zero.

Means to obtain the answer remain to be established. If the solution desired is the relation between i and T, the machine can be made to draw the curve automatically on an output table as the solution proceeds. For this purpose, shaft 1 can be connected to the abscissa drive, shaft R, of the output table, and shaft 2 connected to the ordinate drive, that is, shaft S. The ordinate and abscissa drive shafts are perpendicular to each



x denotes the time at which the voltage magnitude was changed

FIG. 26. Curves drawn automatically by a differential analyzer showing i and e as functions of time, in a ferroresonant series circuit.[4]

other and lie in the same plane. The abscissa shaft drives a mechanism that moves a curve-drawing stylus parallel to the axis of abscissas, and the ordinate shaft drives the stylus parallel to the axis of ordinates. If the point of the stylus is brought in contact with a sheet of paper mounted parallel to the plane of the shafts, a curve of i as a function of T is plotted as the machine is operated. Alternatively, any other two shafts could be coupled to shafts R and S of the output table, and a curve could be thus obtained showing the functional relationships between the variables or terms of the equations which the shafts represent.

The curves[4] of Fig. 26 are included to illustrate the form of a solution

[4] T. R. Smith, "The Study of an R-L-C Series Circuit with a Non-linear Inductance," S. B. Thesis, M. I. T., 1932.

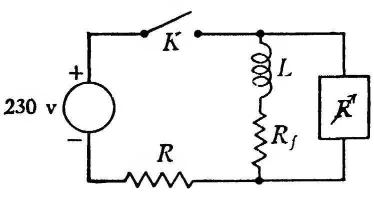
given by a differential analyzer. These curves show the voltage and current wave forms in a ferroresonant series circuit comprising an iron-core reactor, a linear resistor, and linear capacitor. The curves illustrate a condition known as voltage resonance;[5] that is, as the sinusoidal voltage is gradually increased a condition is encountered at which the current peaks increase much more than in direct proportion to the increase in the voltage. In the solution cited the effects of eddy-current and hysteresis loss are included, since the properties of the inductor were represented in terms of appropriate $\varphi(F)$ characteristics for the core material. The plots of the $\varphi(F)$ loops were placed on an input table and the dependent- and independent-variable shafts of the input table were connected to the appropriate shafts in the machine. The functional relationship between the rotation of the shafts was then maintained by the person operating the crank of the input table.

Although the examples of machine solutions given here concern relatively simple electric-circuit problems, the complexity of the problems that can be handled by the machine depend only on the number of integrators, input tables, multipliers, adders, and so on, available with the machine.

PROBLEMS

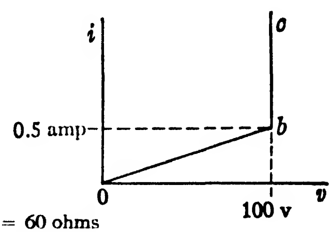
1. The circuit shown in Fig. 27 represents a direct-current generator circuit. The energy of the field circuit is discharged upon the opening of switch K through a thyrite resistor $\not{R}$. The volt-ampere characteristic of $\not{R}$ can be well approximated by the line 0bo shown in Fig. 28. On the assumptions that switch K opens in zero time and that no arc is formed at K,

(a) What is the current through $\not{R}$ as a function of time?
(b) What energy is dissipated in $\not{R}$?



$L = 120$ h $\qquad R = 100$ ohms

FIG. 27. Direct-current generator circuit, Prob. 1.



$R_f = 60$ ohms

FIG. 28. Idealized characteristic for Thyrite (Fig. 5a), Prob. 1. The characteristic, of course, is symmetrical about the origin.

2. In the circuit shown in Fig. 29, L is the inductance of a relay coil. The iR drop in the coil is negligible in comparison with the tube drop.

When the current in the coil reaches 20 ma the relay armature is pulled to the coil; when the current drops to 10 ma, the armature is released. If the time of travel of the armature is negligible, what is the maximum frequency at which the relay can be oper-

[5] C. G. Suits, "Studies in Non-Linear Circuits," *A.I.E.E. Trans.* L (1931), 724–736.

ated if the nonlinear device and coil in series are alternately connected by means of switch K to a 100-v battery and short-circuited (without breaking the circuit)?

The tube characteristic is that of Fig. 8.

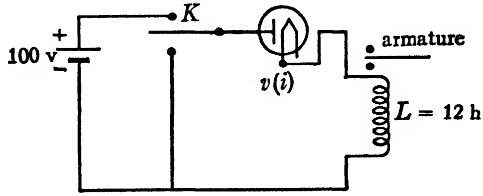


FIG. 29. Circuit of relay coil, Prob. 2.

3. In the circuit of Fig. 30, rectifiers 1 and 2 are identical. Their volt-ampere characteristic is given in Fig. 8.

(a) What are the currents, i, i_1, i_2, and i_3, after K is closed (assuming that there is no current in the circuit prior to closure of K)?

(b) After the steady-state conditions have been reached with K closed, K is opened again. For the resulting transient current through L,

1. What is the analytic form of the solution as far as it can be carried out?
2. How can a graphical solution be carried out from where the analytic form stops? This is to be illustrated by a number of sketches.
3. What is the numerical value of i_3 for the instant after K is opened? What length of time is required for i_3 to decrease to one-half of this value?

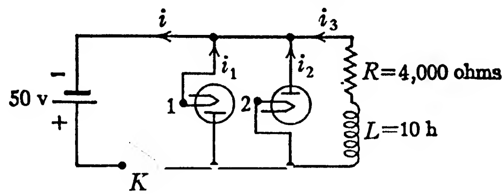
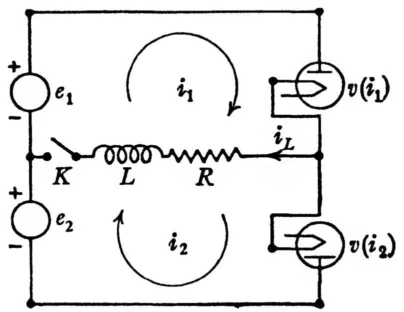


FIG. 30. Rectifier circuit for Prob. 3.

4. Figure 31 shows a full-wave rectifier circuit. The tubes have the characteristic of Fig. 8.

(a) What is i_L as a function of t from the instant of closing the switch K (when t is zero) to the time when the second tube starts to conduct?

(b) Is the first tube still conducting when the second tube starts to conduct?



$e_1 = e_2 = 162 \cos 377t$, v

t in sec

$R = 2{,}500$ ohms $\quad L = 10$ h

FIG. 31. Full-wave rectifier circuit for Prob. 4.

5. When an inductive circuit is opened, arcing occurs at the contacts of the switch. In investigating switching and circuit-breaker problems, it is frequently assumed that the contacts open instantaneously by their full amount. This simplifying assumption means that the arc is of constant length and, if other effects such as temperature change are neglected, that the arc may be considered as a nonlinear resistance rather than a time-varying nonlinear resistance. The following table gives the volt-ampere characteristic for such a switching arc in an air-break switch:

Volts	*Amperes*	*Volts*	*Amperes*
230	0	72.4	2.50
175	0.25	65.5	3.00
146	0.50	58.6	3.50
113	1.00	52.9	4.00
94.8	1.50	47.1	4.50
81.6	2.00	41.4	5.00

This switch is to be used on a 115-v, direct-current circuit for which R is 50.0 ohms and L is 26.6 h (for instance, the field circuit of a generator).

(a) What time is required to open the circuit?
(b) How much energy is dissipated at the switch?
(c) What voltage is across the R and L combination during the arcing period?
(d) What, if any, limits must be placed on R in order that this switch can interrupt this circuit?

6. To facilitate the interruption of current in circuits such as that in Prob. 5, the switch is frequently constructed so that it inserts a resistance in parallel with the cir-

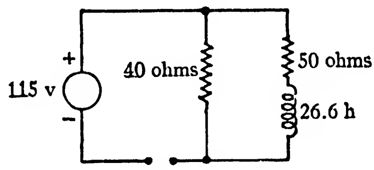


FIG. 32. Circuit for computation of switching transients, Prob. 6.

cuit just before opening. For a field circuit, for example, this inserted resistance is the so-called field-discharge resistance. If a 40-ohm resistance is placed in parallel with the field circuit, Fig. 32, just before opening the switch, what are the answers to parts (a), (b), and (c) of Prob. 5?

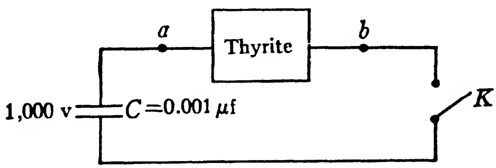


FIG. 33. Circuit for testing Thyrite, Prob. 7.

7. A circuit for testing the properties of thyrite is shown in Fig. 33. The condenser C is charged to 1,000 v from an external source and the source is then removed. Switch K is then closed. The characteristic of the thyrite is given by

$$v_{ab} = 580i^{0.28} \text{ v,} \tag{104}$$

when i is in amperes.

What is the transient current in the circuit, counting time from the instant the switch K is closed? This transient is to be plotted for the time interval required for the current to drop to one-tenth of its initial value, and compared with the characteristic obtained when the thyrite is replaced by a linear resistance of 145 ohms.

8. A circuit which can be used as a sweep circuit for a cathode-ray oscillograph or as an oscillator consists essentially of the elements shown in Fig. 34. A capacitance C is charged through a nonlinear resistance R (a vacuum tube) whenever switch K (actually a gas-filled tube) is open. When the condenser voltage reaches 120 v, switch K is suddenly closed and the condenser is discharged to 20 v in 1.0×10^{-5} sec. Switch K then opens and the charging process repeats. (The 20-v battery merely indicates that a residual voltage of 20 v exists on the condenser when K is opened.) Data relating the voltage v_t across the nonlinear element R to the current i through it are tabulated below. Condenser C has a capacitance of 5.0×10^{-9} farad.

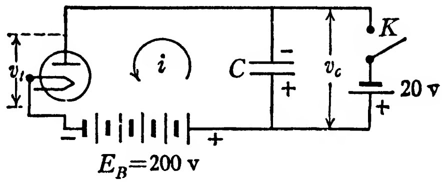


FIG. 34. Sweep circuit or oscillator, Prob. 8.

At what frequency does this circuit oscillate?

v_t (volts)	i (milliamperes)	v_t (volts)	i (milliamperes)
200	18.8	110	5.3
190	17.1	100	4.2
180	15.4	90	3.2
170	13.8	80	2.2
160	12.2	70	1.4
150	10.7	60	0.8
140	9.3	50	0.4
130	7.9	40	0.05
120	6.6		

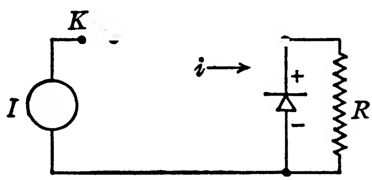


FIG. 35. Constant-current circuit, Prob. 9.

9. In the circuit of Fig. 35 a constant current I of 0.2 amp is maintained by means of a tetrode tube in series with a battery. Resistance R is 1,000 ohms, in parallel with a copper-oxide rectifier which has a characteristic curve described by

$$i = 10^{-4} v^{3/2} \text{ amp}, \qquad [105]$$

when v is in volts.

(a) What are the values of the currents through the rectifier and the resistance?
(b) If the resistance is replaced by a capacitance of 1.0 μf, what is the voltage across the copper-oxide rectifier 0.001 sec after the switch K is closed?

10. The tube whose volt-ampere characteristic is given in Fig. 8 is to be used to rectify a 60 ~ sinusoidal voltage of 100-v amplitude. The single-section filter shown in Fig. 36 is used to smooth out the fluctuations in the output of the tube.

If there is no load on the output of the filter, the input voltage is suddenly applied when at its maximum value and of such polarity that the tube conducts, and the condenser is initially uncharged, how does the output voltage vary as a function of time for the first $1\frac{1}{2}$ cycles of the input voltage?

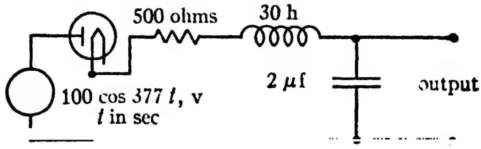


Fig. 36. Filter circuit for Prob. 10.

11. The magnetic circuit of a 1,000-kva transformer has a mean length of 95 in and an average cross section of 278 sq in. The high tension winding has 1522 turns and operates on a 66,000 v, 60 ~ circuit. The core is constructed of material having the magnetic characteristics given in the table which follows.

How does the current taken by the high-tension winding (when the low-tension winding is open-circuited) vary as a function of time after connecting the winding to its source when the voltage is one half of its negative maximum and increasing negatively? All effects of magnetic leakage and of the resistance of the high-tension winding may be neglected. The applied voltage is sinusoidal.

	$\mathcal{B}$ in gauss	$\mathcal{H}$ in oersteds
Ascending:		
	0	1.62
	1,000	1.73
	2,500	1.79
	5,000	2.00
	6,000	2.24
	7,500	3.00
	10,000	4.60
	12,500	11.1
	14,000	20.8
Descending:		
	14,000	20.8
	12,500	11.0
	10,000	4.40
	9,000	3.20
	7,500	1.85
	6,000	1.00
	5,000	0.50
	2,500	−1.02
	1,000	−1.55
	0	−1.62

Data for only the top half of the loop are given, the loop being symmetrical.

Tables: Copper and Aluminum Conductors; Resistivities of Metals and Alloys; Relative Permittivities

Data as determined by the Bureau of Standards for solid copper wires and concentric-lay cables of standard annealed copper are given in Tables I and II. Table III is for concentric-lay aluminum cables, steel reinforced (A.C.S.R.), as published by the Aluminum Company of America. Standard values for resistivity, temperature coefficient, and density of annealed copper and hard-drawn aluminum are given in Table IV.

The process of manufacturing copper conductors or wires consists of hot-rolling the cast bars into rods and then cold-drawing the rods through dies in successive steps to give the desired size. The cold-drawing in general tends to increase the resistivity, increase the tensile strength, reduce the ductility, and increase the hardness. The soft or annealed condition following cold-working may be restored by annealing at a temperature of about 600 degrees centigrade. Commercial copper conductors may be finished as either hard-drawn, medium hard-drawn, or annealed. The type used depends in general upon the application and desired qualities; for instance, hard-drawn wire is used for overhead circuits where maximum tensile strength is desired, and annealed wire is used for insulated cables for underground circuits.

The temperature coefficient of resistance (constant mass) of copper is in general dependent upon the conductivity and the temperature. However, the relation between conductivity and the temperature coefficient is practically independent of chemical constituency (slight impurities in the copper) and of physical constants (such as are produced by annealing or hard-drawing) and may be represented by a constant proportionality. The temperature coefficient α_{20} at 20 degrees centigrade for copper of 100 per cent conductivity is 0.00393 as given in Table IV. The temperature coefficient α_t for any other temperature t in degrees centigrade may be obtained as follows:

$$\alpha_t = \frac{1}{\dfrac{100}{n\alpha_{20}} + (t - 20)}, \qquad [1]$$

in which n is the per cent conductivity and is determined by expressing as a percentage the ratio of the resistivity of the International Annealed

Copper Standard (corresponding to 100 per cent conductivity) at 20 degrees centigrade to the resistivity of the sample under consideration at 20 degrees centigrade.

The per cent conductivity of commercial copper wires or conductors generally varies from approximately 100 per cent for annealed copper to 97 per cent for hard-drawn copper. The average value of the per cent conductivity of hard-drawn copper wires is usually taken as 97.3 per cent.

Aluminum wires are cold drawn in wire mills in the same way that copper is drawn, and also can be annealed. The hard-drawn wire is used for bare cable for outdoor stringing but ordinarily is reinforced by a core of steel wire, because the ultimate tensile strength of aluminum is only about 40 per cent that of copper. Aluminum is also rolled into flat bars or channel sections for use in bus structures.

Annealed insulated aluminum wire and cable are made, but it has been impossible for insulated aluminum to compete favorably with copper in the face of recent very low copper prices because the overall diameter necessary in aluminum and the volume of insulation required are substantially greater than for equivalent copper wires or cables. Annealed aluminum has a conductivity of approximately 62 per cent. In cables it possesses the advantage of burning clear on short circuit more quickly than copper.

For the details of the various relative technical advantages and disadvantages of copper and aluminum conductors questions of conductivity, skin effect, reactance, corona, thermal properties, weight, space occupied, mechanical strength, weathering, construction and maintenance difficulties — the student is referred to the literature of the manufacturers. In each installation these matters and the relative costs must be weighed in terms of the specific situation.

Commercial wire sizes are generally expressed in terms of gauge numbers or circular mil area. The most commonly used gauge for electrical conductors is the American wire gauge* (AWG) sometimes also termed the Brown & Sharpe wire gauge* (B & S). The diameters of wires are generally given in mils, the mil being equal to one-thousandth of an inch. The circular mil is used as a unit of area. The area of any circle in circular mils is equal to the square of the diameter in mils. Thus a cylindrical wire of one mil diameter has a cross-sectional area of one circular mil. The American wire gauge is based on a geometrical progression of diameters between specified diameters of 5 mils for 36 AWG and 460 mils for 0000 AWG. Since there are 38 gauge sizes between these two, the ratio of diameters of successive sizes is $\sqrt[39]{460/5}$ or 1.122932. Since the resistance, weight, and cross-sectional area are dependent upon the square of the

* Devised by J. R. Brown, 1857.

diameter and since the sixth power of the geometrical ratio between successive gauge diameters is approximately 2 (actually 2.0050), the resistance per unit length, weight per unit length, and cross-sectional area for every third gauge number are approximately halved or doubled. This factor is of interest in that it facilitates obtaining approximate values of the wire data from values applying to one particular gauge.

The term *cable*, from an electrical point of view, represents a conductor consisting of a number of individual strands or wires, and also is quite generally used to represent a completely finished insulated conductor. To increase their flexibility, conductors larger than 0000 AWG are practically always stranded, and conductors below 0000 AWG are also frequently stranded. A concentric-lay cable is a conductor in which closed helical layers of wires surround a straight central wire. The alternate helical layers of wire generally have a lay or twist in opposite directions as viewed from the axis of the conductors. These cables are generally made of individual wires or strands of the same size the first layer around the central wire being made up of six individual wires and each successive layer having six more wires than the preceding layer.

The relative permittivity K is given in Table VI for each of a representative group of gases, liquids, and solids. It is defined as $\varepsilon/\varepsilon_0$ in which ε is the permittivity of the medium, and ε_0 is the permittivity of free space and is also known as the dielectric constant or the specific electric inductive capacity. In mks units ε_0 is 1.113×10^{-10}. In cgs electrostatic units ε_0 is one.

Generally the relative permittivity of a given substance is a function of pressure, temperature, and frequency, since the number of molecules per unit volume is a function of temperature and pressure, and the amount of orientation of polar molecules is a function of both temperature (viscosity) and frequency. Accordingly, temperature, pressure, and frequency should be specified in giving values of relative permittivity for different substances. However, for many substances the dielectric constants are actually sensibly constant for wide ranges of both temperature and frequency.

In Table VII are given the relative permittivities for compound insulations that are used in the manufacture of insulated wires and cables for use in electric-power transmission and distribution. The values given are representative values in American practice. The particular values applying for insulated cables as made by a given manufacturer may vary slightly from these values because of the different compositions of the insulation as used by different manufacturers.

TABLE I

COMPLETE WIRE TABLE, STANDARD ANNEALED COPPER*

BASED ON A RESISTIVITY OF 10 376 ABS OHMS/MIL FT THE RESISTIVITY OF STANDARD COMMERCIAL ANNEALED WIRE DOES NOT EXCEED 10 570 ABS OHMS/MIL FT (1 85% EXCESS) AMERICAN WIRE GAUGE (BROWN & SHARPE) 1 ABS OHM = 0 9995 INTERNATIONAL OHM

Gauge No	Nominal† diam in mils at 20 C	Cross section at 20 C		Abs ohms for 1,000 ft 20 C or 68 F	Ft/lb	Ft for one abs ohm 20 C or 68 F	Nominal diam in mils over insulation						
		Cir mils	Sq in				Enamel	Single cotton	Double cotton	Single silk	Double silk	Cotton enamel	Silk enamel
0000	460 0	211 600	0 166 2	0 049 03	1 561	20 390							
000	409 6	167 800	0 131 8	0 061 83	1 968	16 170							
00	364 8	133 100	0 104 5	0 077 97	2 482	12 825							
0	324 9	105 500	0 082 89	0 098 32	3 130	10 175							
1	289 3	83 690	0 065 73	0 124 0	3 947	8 066							
2	257 6	66 370	0 052 13	0 156 4	4 977	6 397							
3	229 4	52 640	0 041 34	0 197 1	6 276	5 072							
4	204 3	41 740	0 032 78	0 248 6	7 914	4 023							
5	181 9	33 100	0 026 00	0 313 5	9 980	3 190							
6	162 0	26 250	0 020 62	0 395 3	12 58	2 530							
7	144 3	20 820	0 016 35	0 498 4	15 87	2 006							
8	128 5	16 510	0 012 97	0 628 5	20 01	1 591		136 2	142 0				
9	114 4	13 090	0 010 28	0 792 5	25 23	1 261	116 4	121 2	125 9			123 2	
10	101 9	10 380	0 008 155	0 999 4	31 82	1 000	103 9	107 7	112 4			109 7	
11	90 74	8 234	0 006 467	1 261	40 12	793 6	92 7	95 5	100 0			97 5	
12	80 81	6 530	0 005 129	1 589	50 59	629 3	82 8	85 6	90 1			87 6	
13	71 96	5 178	0 004 067	2 004	63 80	499 1	74 0	76 8	81 3			78 8	
14	64 08	4 107	0 003 225	2 526	80 44	395 8	66 1	68 9	73 4			70 9	

15	57.07	3 257.	0.002 558	3.186	101.4	313.8	59.1	61.9	66.4	...	. .	63.9	...
16	50.82	2 583.	0.002 028	4.018	127.9	248.9	52.4	55.6	60.1	52.6	54.4	56.7	53.7
17	45.26	2 048.	0.001 609	5.067	161.3	197.4	46.9	50.1	54.6	47.1	48.9	51.2	48.2
18	40.30	1 624.	0.001 276	6.388	203.4	156.5	41.9	45.1	49.6	42.1	43.9	46.2	43.2
19	35.89	1 288.	0.001 012	8.055	256.5	124.1	37.5	40.7	45.2	37.7	39.5	41.8	38.8
20	31.96	1 022.	0.000 802 3	10.16	323.4	98.45	33.4	36.8	41.3	33.8	35.6	38.2	35.2
21	28.46	810.1	0.000 636 3	12.81	407.8	78.07	29.9	33.3	37.8	30.3	32.1	34.7	31.7
22	25.35	642.4	0.000 504 6	16.15	514.2	61.92	26.7	29.6	33.6	27.1	28.9	31.0	28.5
23	22.57	509.5	0.000 400 2	20.37	648.4	49.11	23.8	26.9	30.9	24.4	26.2	28.1	25.6
24	20.10	404.0	0.000 317 3	25.68	817.7	38.94	21.3	24.4	28.4	21.9	23.7	25.6	23.1
25	17.90	320.4	0.000 251 7	32.39	1 031.	30.88	19.1	22.2	26.2	19.7	21.5	23.4	20.9
26	15.94	254.1	0.000 199 6	40.83	1 300.	24.49	16.85	20.2	24.3	17.7	19.5	21.15	18.65
27	14.20	201.5	0.000 158 3	51.50	1 639.	19.42	15.15	18.5	22.5	16.0	17.8	19.45	16.95
28	12.64	159.8	0.000 125 5	64.93	2 067.	15.40	13.55	16.9	20.9	14.4	16.2	17.85	15.35
29	11.26	126.7	0.000 099 53	81.87	2 607.	12.21	12.25	15.6	19.6	13.1	14.9	16.55	14.05
30	10.03	100.5	0.000 078 94	103.3	3 287.	9.686	10.95	14.3	18.3	11.8	13.6	15.25	12.75
31	8.928	79.70	0.000 062 60	130.2	4 145.	7.681	9.7	13.2	17.2	10.7	12.5	14.0	11.5
32	7.950	63.21	0.000 049 64	164.2	5 227.	6.092	8.8	12.3	16.3	9.8	11.6	13.1	10.6
33	7.080	50.13	0.000 039 37	207.0	6 591.	4.831	7.75	11.4	15.4	8.9	10.7	12.05	9.55
34	6.305	39.75	0.000 031 22	261.0	8 310.	3.831	6.95	10.6	14.6	8.1	9.9	11.25	8.75
35	5.615	31.52	0.000 024 76	329.2	10 480.	3.038	6.15	9.9	13.9	7.4	9.2	10.45	7.95
36	5.000	25.00	0.000 019 64	415.0	13 210.	2.410	5.55	3.8	12.8	6.8	8.6	9.35	7.35
37	4.453	19.83	0.000 015 57	523.4	16 660.	1.911	4.95	8.3	12.3	6.3	8.1	8.75	6.75
38	3.965	15.72	0.000 012 35	659.9	21 010.	1.515	4.45	7.8	11.8	5.8	7.6	8.25	6.25
39	3.531	12.47	0.000 009 793	832.2	26 500.	1.201	3.85	7.3	11.3	5.3	7.1	7.65	5.65
40	3.145	9.888	0.000 007 766	1050.	33 410.	0.9529	3.45	6.9	10.9	4.9	6.7	7.25	5.25

* Based on a table prepared by U. S. Bureau of Standards; specifications of the American Society for Testing Materials; specifications of the Belden Manufacturing Company.

† A.S.T.M. Standards permit 1% ± variation in diameter for sizes larger than No. 30; ±0.0001 in. variation for sizes smaller than No. 30.

TABLE II

BARE CONCENTRIC-LAY CABLES OF STANDARD ANNEALED COPPER*

BASED ON 100% CONDUCTIVITY (10.376 ABS OHMS/MIL FT)

Size of cable		D-c resistance abs ohms for 1,000 ft 25 C or 77 F	Lb for 1,000 ft	Standard concentric stranding		
Cir mils	AWG no.			No. of wires	Diam of wires in mils	Outside diam in mils
2 000 000	.	0 005 39	6180	127	125 5	1631.
1 900 000	.	0 005 68	5870	127	122 3	1590.
1 800 000	...	0 005 99	5560.	127	119 1	1548.
1 700 000	....	0 006 34	5250	127	115 7	1504.
1 600 000	....	0 006 74	4940	127	112 2	1459.
1 500 000	...	0 007 19	4630	91	128 4	1412.
1 400 000	....	0 007 70	4320	91	124 0	1364.
1 300 000	....	0 008 30	4010.	91	119 5	1315.
1 200 000	. .	0 008 99	3710	91	114 8	1263.
1 100 000	....	0 009 81	3400	91	109 9	1209.
1 000 000	.	0 010 8	3090	61	128 0	1152.
950 000	.	0 011 4	2930	61	124 8	1123.
900 000	....	0 012 0	2780	61	121 5	1093.
850 000	...	0 012 7	2620	61	118 0	1062.
800 000	....	0 013 5	2470	61	114 5	1031.
750 000	....	0 014 4	2320	61	110 9	998.
700 000	....	0 015 4	2160	61	107 1	964.
650 000	....	0 016 6	2010	61	103 2	929.
600 000	....	0 018 0	1850	61	99 2	893.
550 000	....	0 019 6	1700.	61	95 0	855.
500 000	....	0 021 6	1540	37	116 2	814.
450 000	....	0 024 0	1390	37	110 3	772.
400 000	....	0 027 0	1240	37	104 0	728.
350 000	....	0 030 8	1080	37	97 3	681.
300 000	..	0 036 0	926	37	90 0	630.
250 000	..	0 043 1	772	37	82 2	575.
212 000	0000	0 050 9	653.	19	105 5	528.
168 000	000	0 064 2	518	19	94 0	470.
133 000	00	0 081 1	411.	19	83 7	418.
106 000	0	0 102	326.	19	74 5	373.
83 700	1	0 129	258.	19	66 4	332.
66 400	2	0 162	205.	7	97 4	292.
52 600	3	0 205	163.	7	86.7	260.
41 700	4	0 259	129	7	77 2	232.
33 100	5	0 326	102	7	68.8	206.
26 300	6	0 410	81 0	7	61 2	184.
20 800	7	0 519	64 3	7	54 5	164.
16 500	8	0 654	51.0	7	48 6	146.

* U S [illegible]

TABLE III

BARE CONCENTRIC-LAY ALUMINUM CABLES* (STEEL REINFORCED)

CONDUCTIVITY OF ALUMINUM TAKEN AS 61%
(WHERE 100% CORRESPONDS TO 10 376 ABS OHMS/MIL FT)
CONDUCTIVITY OF EQUIVALENT COPPER CABLES IS TAKEN AS 97%
NO ALLOWANCE IS MADE FOR CONDUCTIVITY OF STEEL REINFORCING WIRES
TWO PER CENT ARE ADDED TO ALLOW FOR INCREASE IN LENGTH BY SPIRAL STRANDING

Size of conductor		*No of wires*		*Copper equivalent Cir mils or AWG number*	*Total lb for one mile*	*Abs ohms/mile at 5 C*		
							60 ~	
Cir mils	*AWG number*	*Al*	*Steel*			*D-c*	*0 amp/sq in*	*1,000 amp/sq in†*
1 590 000		54	19	1 000 000	10 735	0 0587	0 0591	0 0638
1 431 000		54	19	900 000	9 [illegible]	0 0652	0 0656	0 0703
1 272 000	....	54	19	800 000	8 588	0 0[illegible]	0 0738	0 0782
1 113 000	....	54	19	700 000	7 517	0 0839	0 0844	0 0885
954 000	...	54	7	600 000	6 481	0 0979	0 0982	0 102
795 000	....	26	7	500 000	5 776	0 117	0 117	0 117
795 000	...	54	7	500 000	5 402	0 117	0 119	0 121
715 500		54	7	450 000	4 860	0 131	0 132	0 134
636 000	....	26	7	400 000	4 620	0 147	0 147	0 147
636 000	...	54	7	400 000	4 321	0 147	0 148	0 152
556 500	..	30	7	350 000	4 600	0 168	0 168	0 168
477 000	....	30	7	300 000	3 943	0 196	0 196	0 196
397 500		30	7	250 000	3 286	0 235	0 235	0 235
336 400		30	7	0000	2 781	0 278	0 278	0 278
300 000	....	26	7	188 700	2 179	0 311	0 311	0 311
266 800	.	26	7	000	1 939	0 350	0 350	0 350
266 800		6	7	000	1 813	0 350	0 351	0 425
211 600	0000	6	1	00	1 549	0 441	0 445	0 510
167 806	000	6	1	0	1 227	0 556	0 560	0 611
133 077	00	6	1	1	974	0 702	0 706	0 739
105 535	0	6	1	2	773	0 885	0 888	0 907
83 693	1	6	1	3	614	1 12	1 12	1 13
66 371	2	6	1	4	486	1 41	1 41	1 42
52 635	3	6	1	5	386	1 78	1 78	1 78
41 741	4	6	1	6	306	2 24	2 24	2 24
33 102	5	6	1	7	242	2 82	2 82	2 82
26 251	6	6	1	8	192	3 56	3 56	3 56

* Abridged from tables published by the Aluminum Company of America. Many additional sizes and strandings not shown are available.

† The magnetic permeability of steel reinforcing wires of the core changes with current, causing a corresponding change in current distribution in the surrounding aluminum wires. A change in the value of effective resistance results. Several duplicate sizes with different strandings are shown to indicate that the magnitude of this change in resistance with current depends on the stranding.

TABLE IV

STANDARD VALUES AT 20 C FOR RESISTIVITY, CONDUCTIVITY, TEMPERATURE COEFFICIENT, AND DENSITY

1. ANNEALED COPPER* (100% CONDUCTIVITY)
2. HARD-DRAWN ALUMINUM† (61% CONDUCTIVITY)

	1. Copper	2. Aluminum
Mass resistivity (abs ohms)		
A uniform wire, mass 1 g and length 1 m .	0.15336	0.07636
A uniform wire, mass 1 lb and length 1 mile .	875.64	435.96
Volume resistivity (abs ohms)		
A uniform wire, length 1 m and cross section 1 sq mm	0.017250	0.028279
A uniform wire, length 1 ft and cross section 1 cir mil	10.376	17.010
Temperature coefficient		
Fractional increase /C	0.00393	0.00403
Density		
Grams /cu cm	8.89	2.70

* *International Annealed Copper Standards* adopted by the International Electrotechnical Commission, 1913.

† Based on data published by the Aluminum Company of America.

TABLE V

RESISTIVITY AND TEMPERATURE COEFFICIENTS FOR METALS AND ALLOYS*

Material		*Temperature C*	*Abs microhm-cm*	*Temperature coefficient* α_t *in* 0.1%/C
Aluminum . . .	Commercial hard drawn	20	2.829	+3.9
Constantin . . .	60% Cu 40% Ni	0	42.41	+0.167
Copper	annealed	20	1.72	+3.93
	hard drawn	20	1.78	+3.8
German silver .		20	17–41	+0.04 to +0.38
Gold	pure	0	2.19	+3.65
Iron . .		0	8.55	+7.257
Lead	average	0	19.8	+3.955
Magnesium . . .	chemically pure	0	4.27	+3.88
Manganin . .		20	34–100	+0.02 to −0.03
Mercury . . .		0	94.082	+0.9098
Molybdenum . .	very pure	0	5.14	+4.791
Monel metal . .		20	42.5–45	+0.02 to +0.2
Nichrome . . .		20	110	+0.03 to +0.4
Nickel	electrolytic	0	6.93	+5.44
Palladium . . .	average	0	10.00	+3.610
Platinum . . .		0	11.19	+3.52
Silver	pure	20	1.622	+3.61
Tin	drawn	0	10.48	+4.359
Tungsten . . .	aged filament	0	5.00	+5.238
Zinc		0	5.64	+3.468

* Values taken from International Critical Tables.

Values of K (Eq. 149, p. 75) for some of the above Materials

Material	K	*Material*	K
Aluminum	236 C	Nickel	184 C
Copper, annealed	234.5	Silver	247
Copper, hard drawn	242	Tungsten	191
Iron	138	Zinc	288

TABLE VI

RELATIVE PERMITTIVITIES*

$\left(\text{SPECIFIC ELECTRIC INDUCTIVE CAPACITY} = K = \frac{\varepsilon}{\varepsilon_0}\right)$

FOR FREQUENCIES LESS THAN $10^6 \sim$

Gases	*Temperature C*	*Pressure atm*	*K*
Air	0	1	1 000585
Air	0	20	1.0117
Carbon dioxide	0	1	1.00098
Hydrogen	0	1	1.00026

Liquids	*Temperature C*	*K*
Acetone	20	21.3
Alcohol,		
ethyl	20	25.7
methyl	20	31.2
propyl	20	21.8
Aniline	20	7.21
Benzene	20	2 28
Distilled water	20	80

Solids (20 C–100 C)	*K*
Asphalt	2.7
Gutta percha (refined)	2.78
Mica	4.5–7.5
Paper (kraft)	3.5
Paraffin	1.9–2.3
Porcelain	4.5–6.9
Shellac	2.7–3.7

* International Critical Tables.

TABLE VII

RELATIVE PERMITTIVITIES

$\left(\text{SPECIFIC INDUCTIVE CAPACITY} = K = \frac{\varepsilon}{\varepsilon_0}\right)$

COMPOUND INSULATIONS SUCH AS ARE USED FOR INSULATED WIRES AND CABLES FOR POWER PURPOSES. VALUES GIVEN ARE BASED ON REPRESENTATIVE AMERICAN PRACTICE AND ARE FOR POWER FREQUENCIES, AND TEMPERATURES IN THE OPERATING RANGE

Type	*K*
Oil-impregnated paper (solid types)	3.7
Oil-impregnated paper (oil-filled types)	3.5
Varnished cambric	5.5
Rubber	5.0

The Solution of Linear Algebraic Equations by Means of Determinants

1. The form of the equations, and the corresponding determinant

For the purpose of solving a set of linear simultaneous equations by means of the determinant method, it is essential that they be written in the following orderly form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \quad [1a]$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \quad [1b]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n. \quad [1n]$$

This represents a set of n equations with n unknowns. The unknowns are the quantities $x_1, x_2. \cdots x_n$; the numbers $y_1, y_2, \cdots y_n$, on the right-hand sides of the equations, are regarded as known; and the numerical coefficients associated with the unknowns are denoted symbolically by the letter " a " with subscripts which indicate their positions with regard to the row and column in which they appear. The rows are the equations themselves, and the columns are the vertical sets of terms involving separately the unknown x_2, and so on.

The first subscript on a coefficient " a " indicates the row in which it is located, and the second subscript indicates the column. Thus, for example, a_{35} is the coefficient of x_5 in the third equation. The second subscript always agrees with that of the unknown with which the coefficient is associated. In a numerical example the coefficients are simply numbers and, of course, have no subscripts, but for the analytical discussion of the determinant method of solution the subscript notation is essential, as may be appreciated in the following development.

The determinant of the set of equations (Eqs. 1a to 1n) is a function of *all* the coefficients $a_{11}, a_{12}, \cdots a_{21}, a_{22}, \cdots a_{nn}$. It is written symbolically in the form

$$A = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots & \cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}. \quad [2]$$

The capital letter A is used as an abbreviation. The row and column order of the coefficients is preserved in the writing of the determinant. The coefficients are also referred to as the *elements*.

2. The evaluation of a determinant

The function A of Eq. 2 has a *value* which may be obtained in a number of different ways.* One of these is based upon an expansion of the determinant in terms of the coefficients of a row or column. For the first row, the expansion reads

$$A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}. \qquad [3]$$

Any other row, or any column, may be chosen for the expansions; the final (fully expanded) form for the determinant is always the same. Thus for the third row

$$A = a_{31}A_{31} + a_{32}A_{32} + \cdots + a_{3n}A_{3n}, \qquad [3a]$$

while for the second column

$$A = a_{12}A_{12} + a_{22}A_{22} + \cdots + a_{n2}A_{n2}. \qquad [3b]$$

The quantities A_{11}, A_{12}, $\cdots$ A_{nn} also are determinants, but they have one row and one column fewer than the original determinant A. They are called the *cofactors* of A. The expansion may again be individually applied to these cofactors to reduce them still further. In this way the determinant A is eventually obtained in its fully expanded form in which only combinations of the coefficients a_{11}, a_{12}, $\cdots$ a_{nn} appear.

A cofactor is obtained from the given determinant by canceling the row and column indicated by its subscripts and then prefixing the resulting reduced determinant by a sign-controlling factor which is $+1$ if the sum of the subscripts is even and -1 if this sum is odd. Thus A_{13}, for example, is obtained from A, Eq. 2, by canceling the first row and third column. The algebraic sign of A_{13} is $+1$ because the sum of the subscripts $1 + 3$ or 4 is even. The result has the following appearance.

$$A_{13} = +\begin{vmatrix} a_{21} & a_{22} & a_{24} \cdots a_{2n} \\ a_{31} & a_{32} & a_{34} \cdots a_{3n} \\ \cdots & \cdots & \cdots\cdots \\ a_{n1} & a_{n2} & a_{n4} \cdots a_{nn} \end{vmatrix}. \qquad [4]$$

Other cofactors are formed in the same fashion.

In applying the expansion method to the cofactors, for the purpose of further reducing them, the same general procedure is followed. When this is done in an analytic example for which the coefficients are denoted symbolically, there may be some slight difficulty arising from the fact that in the cofactors, such as A_{13} of Eq. 4, the subscripts on the coefficients no longer identify the positions of these coefficients consistently with regard

* The method of evaluation by multiplication along diagonals, sometimes given in algebra texts, is not valid for determinants having more than nine elements, that is, three rows and three columns.

to the rows and columns. In applying the expansion method to these cofactors, one must be guided entirely by the row-and-column positions and not by the subscripts on the coefficients. A specific example makes this clear.

If the given determinant has three rows and three columns, thus:

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad [5]$$

expansion along the first row gives

$$A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \qquad [5a]$$

Now expansion of the individual cofactors along their first rows gives the separate results

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\,|\,a_{33}\,| - a_{23}\,|\,a_{32}\,|, = a_{22}a_{33} - a_{23}a_{32}, \qquad [6a]$$

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\,|\,a_{33}\,| - a_{23}\,|\,a_{31}\,|, = a_{21}a_{33} - a_{23}a_{31}, \qquad [6b]$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,|\,a_{32}\,| - a_{22}\,|\,a_{31}\,|, = a_{21}a_{32} - a_{22}a_{31}. \qquad [6c]$$

It is to be noted that a determinant consisting of just one element is equal to that element. Substituting Eqs. 6a, 6b, and 6c into Eq. 5a yields the fully expanded form for the determinant Eq. 5, thus

$$\left.\begin{aligned} A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\ + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned}\right\}. \qquad [5b]$$

3. A FUNDAMENTAL PROPERTY OF THE DETERMINANT

In the expansions indicated by Eqs. 3, 3a, and 3b, it should be particularly noted that each term consists of the product of an element and the cofactor formed by the cancellation of the row and column in which that element is located. This cofactor is said to *correspond* to the element in question. An element is always multiplied by its *corresponding* cofactor. Thus the element which belongs to row 3 and column 5, for example, is multiplied by the cofactor formed by canceling row 3 and column 5, and so on.

If the elements of any row or column are multiplied respectively by the cofactors of the elements of *another* row or column and the results are added, the value arrived at is *not* that of the determinant A but instead it is always zero: This fundamental property of a determinant may best

be illustrated by a numerical example. For simplicity, real numbers are used. If

$$A = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 6 & 3 \\ 1 & 5 & 4 \end{vmatrix}, \tag{7}$$

then

$$A_{11} = \begin{vmatrix} 6 & 3 \\ 5 & 4 \end{vmatrix} = 24 - 15 = 9; \tag{8a}$$

$$A_{12} = -\begin{vmatrix} 4 & 3 \\ 1 & 4 \end{vmatrix} = -16 + 3 = -13; \tag{8b}$$

$$A_{13} = \begin{vmatrix} 4 & 6 \\ 1 & 5 \end{vmatrix} = 20 - 6 = 14; \tag{8c}$$

$$A_{21} = -\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = -12 + 5 = -7; \tag{8d}$$

$$A_{22} = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7; \tag{8e}$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = -10 + 3 = -7; \tag{8f}$$

$$A_{31} = \begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix} = 9 - 6 = 3; \tag{8g}$$

$$A_{32} = -\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = -6 + 4 = -2; \tag{8h}$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0. \tag{8i}$$

The value of A is obtained by adding the products of the elements of any row or column and their corresponding cofactors. Thus for the first, second, or third row there results

$$A = 2 \times 9 - 3 \times 13 + 1 \times 14 = -7, \tag{7a}$$

$$A = -4 \times 7 + 6 \times 7 - 3 \times 7 = -7, \tag{7b}$$

$$A = 1 \times 3 - 5 \times 2 + 4 \times 0 = -7, \tag{7c}$$

and for the first, second, or third columns the expansions read

$$A = 2 \times 9 - 4 \times 7 + 1 \times 3 = -7, \tag{7d}$$

$$A = -3 \times 13 + 6 \times 7 - 5 \times 2 = -7, \tag{7e}$$

$$A = 1 \times 14 - 3 \times 7 + 4 \times 0 = -7. \tag{7f}$$

However, for the elements of the first row and the cofactors of the elements of the second row the sum of the products gives zero, thus

$$-2 \times 7 + 3 \times 7 - 1 \times 7 = 0; \qquad [9]$$

or for the elements of the second column and the cofactors of the elements of the third column there results

$$3 \times 14 - 6 \times 7 + 5 \times 0 = 0. \qquad [10]$$

Thus whenever the elements are combined with the cofactors of the wrong row or column, the result is always zero. This property suggests how the determinant may be used to solve Eqs. 1a to 1n for the unknowns.

4. Cramer's rule

Equations 1a to 1n are multiplied respectively by the cofactors of the elements of the first column of A, that is, by A_{11}, A_{21}, A_{31}, $\cdots$ A_{n1}, and the results are added. According to the expansion rule for determinants, the resulting factor of x_1 is equal to the value of the determinant A, while the resulting factors of x_2, x_3, $\cdots$ x_n are all zero because of the property just pointed out in Art. 3. Hence the result is

$$Ax_1 = A_{11}y_1 + A_{21}y_2 + \cdots + A_{n1}y_n, \qquad [11]$$

from which the unknown x_1 is found.

In the same manner x_2 may be found by multiplying the equations respectively by the cofactors of the elements of the second column of A and adding. The result is

$$Ax_2 = A_{12}y_1 + A_{22}y_2 + \cdots + A_{n2}y_2. \qquad [11a]$$

Once more, by recalling the form of the expansion of a determinant, the right-hand side of Eq. 11 is recognized as the expansion of a determinant obtained from A by replacing the elements of the first column by y_1, y_2, y_3, $\cdots$ y_n and expanding down this column. Similarly the right-hand side of Eq. 11a may be looked upon as a determinant formed from A by replacing its second column by the members y_1, y_2, $\cdots$ y_n. Thus the unknowns x_1 and x_2 can be written as

$$x_1 = \frac{1}{A}\begin{vmatrix} y_1 & a_{12} \cdots a_{1n} \\ y_2 & a_{22} \cdots a_{2n} \\ \cdots & \cdots\cdots \\ y_n & a_{n2} \cdots a_{nn} \end{vmatrix}. \qquad [12]$$

$$x_2 = \frac{1}{A}\begin{vmatrix} a_{11} & y_1 & a_{13} \cdots a_{1n} \\ a_{21} & y_2 & a_{23} \cdots a_{2n} \\ \cdots & \cdots & \cdots\cdots \\ a_{n1} & y_n & a_{n3} \cdots a_{nn} \end{vmatrix}. \qquad [12a]$$

The remaining unknowns are given by similar forms in which the members $y_1, y_2, \cdots y_n$ are placed in the third, fourth, . . . nth columns of A.

The form of solution made evident by the results, Eqs. 12 and 12a, is called *Cramer's rule*. As a numerical illustration a set of three equations with the determinant of Eq. 7 is presented:

$$2x_1 + 3x_2 + x_3 = 2, \tag{13a}$$

$$4x_1 + 6x_2 + 3x_3 = 4, \tag{13b}$$

$$x_1 + 5x_2 + 4x_3 = 6. \tag{13c}$$

The application of Cramer's rule gives

$$x_1 = \frac{1}{A}(2A_{11} + 4A_{21} + 6A_{31}), \tag{14a}$$

$$x_2 = \frac{1}{A}(2A_{12} + 4A_{22} + 6A_{32}), \tag{14b}$$

$$x_3 = \frac{1}{A}(2A_{13} + 4A_{23} + 6A_{33}). \tag{14c}$$

Substituting the numerical values found earlier for A and its cofactors, these become

$$x_1 = \frac{2 \times 9 - 4 \times 7 + 6 \times 3}{-7} = -\frac{8}{7}, \tag{14d}$$

$$x_2 = \frac{-2 \times 13 + 4 \times 7 - 6 \times 2}{-7} = \frac{10}{7}, \tag{14e}$$

$$x_3 = \frac{2 \times 14 - 4 \times 7 + 6 \times 0}{-7} = 0. \tag{14f}$$

5. Utility and applicability of the determinant method

The usefulness of the determinant method of solving simultaneous equations lies principally in the ease with which the solutions can be manipulated analytically. Thus, for example, the reciprocity theorem and Thévenin's theorem are readily proved in the text with the help of the determinant notation, while without this aid these proofs would become considerably more difficult to establish. Numerous other instances like these appear in the further study of circuit theory.

In solving numerical examples of simultaneous equations, the determinant method, although systematic, is generally recognized as requiring a larger total number of multiplications and additions than a systematic process of elimination. The use of determinants as compared with the elimination process may be made to appear in a somewhat more favorable

light by applying to the evaluation of the determinant and its cofactors a number of ingenious simplification methods based upon additional properties of determinants. The discussion of these methods requires a more detailed study of the theory of determinants, but one such method (which perhaps is the most useful) is illustrated below. This method rests upon the following theorem, stated at this juncture without proof:

▶If in a determinant the elements of a row or column are replaced by those elements plus the corresponding elements of another row or column, each multiplied by a constant, the value of the determinant is unchanged.◀

The method is equivalent to the scheme of elimination usually carried out when a number of equations are being solved simultaneously. This step is more readily accomplished with the determinant than with the set of simultaneous equations.

As an example, the determinant below is evaluated:

$$A = \begin{vmatrix} 1.2 & 4.9 & 3.1 & 0.7 \\ 9.3 & 2.6 & 8.4 & 5.0 \\ 7.1 & 6.2 & 5.5 & 2.1 \\ 3.0 & 2.4 & 7.1 & 2.6 \end{vmatrix}. \qquad [15]$$

According to the theorem stated, the first row can be multiplied by $-9.3/1.2$ and added to the second row; the first row then can be multiplied by $-7.1/1.2$ and added to the third row, and again by $-3.0/1.2$ and added to the fourth row. The value of the determinant is still unchanged and may be represented as:

$$A = \left| \begin{array}{c} \begin{array}{cccc} \quad 1.2 & \quad 4.9 & \quad 3.1 & \quad 0.7 \end{array} \\ \left[\begin{array}{cccc} (9.3 & 2.6 & 8.4 & 5.0) \\ -9.3 & -38.0 & -24.0 & -5.41 \end{array}\right] \\ \left[\begin{array}{cccc} (7.1 & 6.2 & 5.5 & 2.1) \\ -7.1 & -29.0 & -18.3 & -4.14 \end{array}\right] \\ \left[\begin{array}{cccc} (3.0 & 2.4 & 7.1 & 2.6) \\ -3.0 & -12.2 & -\ 7.75 & -1.75 \end{array}\right] \end{array} \right|. \qquad [15a]$$

The second, third, and fourth rows are each the algebraic sum of the terms within the brackets, which after adding become:

$$A = \begin{vmatrix} 1.2 & 4.9 & 3.1 & 0.7 \\ 0 & -35.4 & -15.6 & -0.41 \\ 0 & -22.8 & -12.8 & -2.04 \\ 0 & -9.8 & -0.65 & +0.85 \end{vmatrix}. \qquad [15b]$$

This simple process of multiplication by a constant (which permits the use of a slide rule set at a fixed slide position) and subtraction, has reduced the problem of evaluation of a fourth-order determinant to the

evaluation of a third-order one, the value of which is to be multiplied by a single factor, in this case 1.2, to give the desired result. It is a simple matter to repeat the operation on the third-order determinant and reduce its evaluation to a factor times a second-order one. This is done below.

$$A = \begin{vmatrix} 1.2 & 4.9 & 3.1 & 0.7 \\ 0 & -35.4 & -15.6 & -0.41 \\ & 0 & -2.7 & -1.78 \\ & 0 & 3.66 & 0.96 \end{vmatrix} = (1.2)(-35.4)\begin{vmatrix} -2.7 & -1.78 \\ 3.66 & 0.96 \end{vmatrix}. \quad [15c]$$

The value of the original fourth-order determinant is now simply:

$$\left.\begin{aligned} A &= (1.2)(-35.4)(-2.7 \times 0.96 + 1.78 \times 3.66) \\ &= -1.2 \times 35.4 \times 3.91 = -166. \end{aligned}\right\} \quad [15d]$$

The same procedure of simplification can be utilized to advantage when complex numbers are encountered. For example, the determinant below is evaluated:

$$\mathrm{A} = \begin{vmatrix} 3+j9 & 2+j7 & 4+j1 \\ 4-j6 & 3+j8 & 5+j4 \\ 7+j2 & 1-j4 & 3+j2 \end{vmatrix}. \quad [16]$$

The first row is multiplied by 4/3 and subtracted from the second row, and then by 7/3 and subtracted from the third row giving:

$$\mathrm{A} = \begin{vmatrix} 3+j\,9 & 2\quad +j\,7 & 4\quad +j1 \\ 0-j18 & 0.33-j\,1.35 & -0.35+j2.66 \\ 0-j19 & -3.66-j20.3 & -6.34-j0.33 \end{vmatrix}. \quad [16a]$$

Now the $j9$ is removed from the first row by adding to it one-half of the second row.

$$\mathrm{A} = \begin{vmatrix} 3+j\,0 & 2.18+j\,6.33 & 3.83+j2.33 \\ 0-j18 & 0.33-j\,1.35 & -0.35+j2.66 \\ 0-j19 & -3.66-j20.3 & -6.34-j0.33 \end{vmatrix}. \quad [16b]$$

Now the first row is multiplied by $j6$ and added to the second row; then the first row is multiplied by $j19/3$ and added to the third row giving,

$$\mathrm{A} = \begin{vmatrix} 3+j0 & 2.18+j\,6.33 & 3.83+j\,2.33 \\ 0 & -37.55+j11.63 & -14.33+j25.64 \\ 0 & -43.75-j\,6.5 & -21.09+j23.87 \end{vmatrix} \quad [16c]$$

which is evaluated as

$$\mathrm{A} = 3\begin{vmatrix} -37.55+j11.63 & -14.33+j25.64 \\ -43.75-j\,6.5 & -21.09+j23.87 \end{vmatrix}. \quad [16d]$$

$$\left.\begin{aligned} A &= 3[(-37.55 + j11.63)(-21.09 + j23.87) \\ &\qquad - (-43.75 - j6.5)(-14.33 + j25.64)] \\ &= 3[790 - 278 - j(245 + 895) - 627 - 167 - j(93 - 1{,}120)] \\ &= 3[-282 - j113] = -846 - j339. \end{aligned}\right\} \quad [16e]$$

Comparison of the labor involved in evaluating a determinant by the use of this method with that involved in the general expansion method shows that for determinants of order higher than the third, the general method requires a substantially greater number of operations.

For the solution of numerical equations in which the coefficients are real numbers, a systematic method of elimination involves fewer operations of multiplication and addition than does the solution by means of determinants. For example:

$$x + 2y + 3z = 4, \qquad [17a]$$

$$4x + 5y + 2z = 8, \qquad [17b]$$

$$6x + 7y + z = 9. \qquad [17c]$$

If Eq. 17a is multiplied by $-2/3$ and added to Eq. 17b,

$$\begin{array}{rl} -\frac{2}{3}x - \frac{4}{3}y - 2z = -\frac{8}{3}, & [17d] \\ 4x + 5y + 2z = 8, & [17b] \\ \hline \frac{10}{3}x + \frac{11}{3}y + 0 = \frac{16}{3}. & [18] \end{array}$$

If Eq. 17a is multiplied by $-1/3$ and added to Eq. 17c,

$$\begin{array}{rl} -\frac{1}{3}x - \frac{2}{3}y - z = -\frac{4}{3}, & [17e] \\ 6x + 7y + z = 9, & [17c] \\ \hline \frac{17}{3}x + \frac{19}{3}y + 0 = \frac{23}{3}. & [19] \end{array}$$

Equations 18 and 19 each can be multiplied by 3:

$$10x + 11y = 16, \qquad [18a]$$

$$17x + 19y = 23. \qquad [19a]$$

If Eq. 18a is multiplied by $-19/11$ and added to Eq. 19a,

$$\begin{array}{rl} -\frac{190}{11}x - 19y = -\frac{304}{11}, & [18b] \\ 17x + 19y = 23, & [19a] \\ \hline -\frac{3}{11}x + 0 = -\frac{51}{11}. & [20] \end{array}$$

Therefore

$$x = 17. \qquad [21]$$

If this value of x is substituted in Eq. 18a,

$$170 + 11y = 16, \qquad [18c]$$

whence,

$$y = -14. \qquad [22]$$

If these values of x and y are substituted in Eq. 17a,

$$17 - 28 + 3z = 4, \qquad [17d]$$

$$z = 5. \qquad [23]$$

This process requires 17 operations of multiplication and 11 operations of addition. If the solution is obtained by the method of determinants by using Cramer's rule, Art. 4, or by the method shown in this article for the solution of Eq. 15, the number of operations is substantially greater.

This systematic method of elimination can be used to advantage for numerical computations when the coefficients are complex numbers, particularly if one has a ready means of transferring from the rectangular to the polar form and back to the rectangular form. Comparison of the two methods is more difficult to make in this situation.

When the coefficients are real numbers, the number of multiplication and addition operations can be determined by the following formulae in which n is the number of unknowns and none of the coefficients is zero.

$$\text{Multiplication operations} = \sum_1^n (k^2 + k - 1) = \mathfrak{M}, \qquad [24]$$

$$\text{Addition operations} = \sum_1^n (k^2 - 1) = \mathfrak{A}, \qquad [25]$$

$$\mathfrak{M} - \mathfrak{A} = \sum_1^n k. \qquad [26]$$

n	$\mathfrak{M}$	$\mathfrak{A}$
3	17	11
4	36	26
5	65	50
6	106	85
10	430	375

Doubling the number of unknowns multiplies the number of operations by approximately seven if n is less than 10.

In applying Cramer's rule to the solution of an arbitrarily given set of equations, one should recognize that solutions may not always exist. For example, if the value of the determinant should happen to be zero, then the solutions evidently become infinite, unless the determinants formed by

replacing columns in A by the members $y_1, y_2, \cdots y_n$ are also zero. Then the Cramer rule yields indeterminate forms for the unknowns. A special instance of this sort occurs if $y_1, y_2, \cdots y_n$ each is zero and A is zero. The method of obtaining solutions in such cases, and their interpretation, evidently requires a more detailed study of the present subject. Such a study is worth while not only because the special cases not covered by Cramer's rule play an important part in circuit theory and other fields of engineering, but also because this branch of algebra is thereby recognized to have a much broader significance and a greater utility than can be seen from the more limited point of view.

APPENDIX C

Units, Dimensions, Standards

1. General considerations

"The quantitative measure of anything is expressed by two factors — one, a certain definite amount of the kind of physical quantity measured, called the unit, the other, the number of times this unit is taken."[1] Units or standard sizes can be chosen arbitrarily for all physical quantities independent of their relations to other physical quantities, and individual standards can be specified for all units, but then a numerical constant must be included as a factor in every equation expressing a relation among these quantities. Such a constant can be evaluated by an experiment or a measurement relating the different quantities involved in the equation. This unsystematic procedure results in a number of serious inconveniences and uncertainties. It is possible, however, by a systematic selection of units to avoid all numerical constants except those which are due to geometric relations. This selection is made by using the equations relating the physical quantities to define as many units as possible. Thus the equation

$$f = ma \qquad [1]$$

states that unit force is equal to unit mass times unit acceleration. Unit acceleration, in turn, is defined in terms of unit space and unit time. A certain minimum number of units must be arbitrarily defined, and these are called the fundamental units of the system. For translational motions, the units generally chosen are those for mass, length, and time.

In equations representing physical facts, only like quantities can be equated or added. Thus, in Eq. 1 the equals sign is understood to mean not only that the force f is equal in magnitude to the product of the magnitudes of a mass m and an acceleration a, but also that the product of mass and acceleration has the same physical nature as a force. This may be stated more explicitly by saying that the *dimensions* of a quantity show the exponents with which the different component quantities enter as factors. If dimensions alone without regard to magnitudes are considered, Eq. 1 is written

$$[F] = [M]\,[A]\,. \qquad [2]$$

[1] Frederick E. Fowle (ed.), *Smithsonian Physical Tables* (8th ed. [revised]; Washington: The Smithsonian Institution, 1933), p. xxxi.

Dimensionally in terms of mass $[M]$, length $[L]$, and time $[T]$,

$$[F] = [MLT^{-2}]\,, \tag{3}$$

$$[M] = [ML^0T^0]\,, \tag{4}$$

$$[A] = [M^0LT^{-2}]\,. \tag{5}$$

Recourse to this fact of necessary dimensional homogeneity is useful in checking for errors in writing equations.

The number of quantities which may be used in describing physical phenomena is somewhat arbitrary and a matter of convenience. For example, it is not necessary to define velocity as a separate quantity; the equivalent always can be expressed in terms of length and time. To express it thus, however, is more cumbersome. In more complex situations the tendency to utilize fewer quantities results in greater awkwardness. In the equations of electricity and magnetism the more commonly used quantities are some thirty in number. The number of independent equations usually considered as relating these quantities is less than the total number by four. Thus *any four* quantities may be designated as *fundamental,* and used as dimensions. By means of the independent equations, all other quantities may then be expressed in terms of the fundamental four; that is, they may be dimensioned in terms of the selected four quantities, and their units may be defined in terms of the units for the selected quantities.

The determination of which four quantities shall be chosen as fundamental is governed largely by the feasibility of specifying standards which are permanent, reproducible, available for use, and adaptable to precise comparison. In the study of electricity and magnetism, the four fundamental quantities usually selected are mass, length, time, and either permittivity or permeability, although resistance, current, length, and time also have been used.

2. Systems of electromagnetic units

In Table I enough of the relations of electricity and magnetism are assembled to establish in a systematic way units for each of the principal quantities. Reciprocal quantities and some others, the definitions of which are readily obvious, have been omitted. In the column headed *CGS Electrostatic Units* are given the definitions for centimeter-gram-second units relating to the electrostatic field and to conduction current. In the column headed *CGS Electromagnetic Units* are given the definitions for centimeter-gram-second units relating to the static magnetic field and to conduction current.

As mentioned in Art. 1, it is possible to establish units for dealing with a group of related quantities in an orderly way so as to avoid dimension-

less factors in the equations, except those which are due to inherent geometric relations.[2] Such a group of units is called a *self-consistent system* or merely a *system* of units. In any electromagnetic system of units, the only unavoidable dimensionless factor which appears because of the geometry is 4π. Systems of units which allow the 4π factors to remain in the places where they appear in Table I are called *unrationalized systems.* Systems which arbitrarily place a 4π in the denominator of the Coulomb's law expressions for electric and magnetic charges, thus removing them from the expressions where they appear in Table I are called *rationalized systems.* Systems which also change the size of the units of permeability and permittivity by a factor of 4π are called *subrationalized systems.* The advantage of either of these procedures lies in moving the 4π factors to relations which are found to be used somewhat less commonly than others. In this article are presented three unrationalized systems which have general adoption.

Any system which includes length, mass, and time in its fundamental dimensions is called an absolute system, and the units of the system are called absolute units. The first system discussed is the absolute cgs electrostatic system. In this system the abbreviation *aesu* followed by the symbol for the quantity is used to denote the cgs absolute electrostatic unit for that quantity if the unit has no name; otherwise the prefix *stat* is added to the name of the unit in the practical system.

The definitions of the various units can be traced by following down the electrostatic column of Table I. The *statcoulomb* is defined as that charge which when concentrated in free space one centimeter from a like charge experiences a repulsion of one dyne. The *unit of permittivity* is defined as the permittivity of free space. All the rest of the units follow successively from the defining equations. Since Coulomb's law, which is the only experimental relation involved, contains four dimensions, $[F]$, $[Q]$, $[\varepsilon]$, and $[L]$, three fundamental dimensions are needed. Solely for work in electrostatics, these could be force, length, and permittivity, there being no need for mass or time. If now the conduction-current equations are added, the equation for current in terms of time rate of flow of charge introduces two new dimensions, $[I]$ and $[T]$; so a total of four fundamental dimensions now is required, one of which may be time. In order to make the system an absolute one, mass must be introduced in place of force as a fundamental dimension by utilizing Newton's law, Eq. 1.

The second system described is the absolute electromagnetic system, in which the abbreviation *aemu* followed by the symbol for the quantity is used to denote the cgs absolute electromagnetic unit for that quantity if the unit has no name; otherwise the prefix *ab* is added to the name of the unit in the practical system, except for the units for F, $\mathcal{B}$, and φ

[2] William M. Hall, "The Formation of Systems of Units," *J.F.I.*, CCXXV (1938), 197–218.

which have names of their own. The definitions of the various units can be traced by following down the electromagnetic column of Table I. In this table the force relation

$$d\mathfrak{f}_1 = Id\ell_1 \times \left[\frac{\mu I d\ell_2}{r^2} \times \frac{r_{21}}{r}\right] \qquad [6]$$

is used in order that a definition for current may be obtained without involving any other magnetic quantity except permeability. The *abampere* is that current which, when in a current-carrying element of a long straight wire in free space parallel to a like current carrying element one centimeter distant and normal to the line connecting them, experiences an attraction force per unit length of two dynes per centimeter. The *unit of permeability* is the permeability of free space. The other units follow readily from the defining equations. Since there is only one experimental relation, which contains four dimensions, three fundamental dimensions are needed for work solely in magnetostatics. These could be force, length, and permeability, there being no need for mass or time. If now the conduction equations are added, an additional fundamental dimension is needed, as is true for the electrostatic system. This may be time. In order to make the system an absolute one, mass must be introduced in place of force, as for the absolute electrostatic system.

Since the units for electromagnetic quantities have been defined in two systems, the relation between these systems of units becomes important. Experimental work by Rowland[3] has shown that the ratio between the abampere (aemu) and the statampere (aesu) is $(2.99796 \pm 0.00004)10^{10}$, or one abampere equals $(2.99796 \pm 0.00004)10^{10}$ statamperes. This conversion factor is equal to the magnitude of the velocity of propagation of electromagnetic waves in free space, expressed in centimeters per second. This ratio is commonly denoted by the letter c. By the use of this ratio, the ratio between the absolute cgs electrostatic and the absolute cgs electromagnetic unit for any electromagnetic quantity can be determined.

As an example, the conversion factor for the units of permeability is derived. The force on a current element due to another current element, when written in terms of aemu, is

$$\mathfrak{f} \text{ in dynes} = (\mu \text{ in aemu})\ (I \text{ in abamperes})^2\, d\ell_1 \times \left[\frac{d\ell_2}{r^2} \times \frac{r_{21}}{r}\right]. \qquad [6a]$$

When this is written in terms of statamperes, it is

$$\mathfrak{f} \text{ in dynes} = \frac{(\mu \text{ in aemu})\ (I \text{ in statamperes})^2\, d\ell_1}{c^2} \times \left[\frac{d\ell_2}{r^2} \times \frac{r_{21}}{r}\right]. \qquad [6b]$$

[3] Henry A. Rowland, *Physical Papers* (Baltimore: Johns Hopkins Press, 1902).

From this it follows that the aesu of μ is very much larger than that of free space, being equal to c^2 times the aemu of μ. This means that the permeability of a given medium is numerically much smaller when expressed in aesu than it is when expressed in aemu.

By extension of this procedure it is possible to extend the aesu to include all of the magnetostatic quantities, and so to form a complete system of electromagnetic units. Likewise it is possible to extend the aemu system to include all of the electrostatic quantities and so to form a complete system.

The cgs electrostatic units are not commonly used except in problems relating to the electric field, and the cgs electromagnetic units are not commonly used except in problems relating to the magnetic field, because the units of one system for quantities outside the realm for which the system was primarily developed are generally either inconveniently large or inconveniently small. For problems in which both electric- and magnetic-field relations must be considered simultaneously, common practice in the past has been to mix the two systems, using proper conversion factors. The result, called the Gaussian system, is, however, also awkward.

In use for many years in engineering has been an incomplete system of units of more convenient sizes termed the practical system. In this system the units commonly used are defined as follows:

$$1 \text{ coulomb} = \frac{1}{10} \text{ abcoulomb} = \frac{c}{10} \text{ statcoulombs,} \tag{7}$$

$$1 \text{ ampere} = \frac{1}{10} \text{ abampere} = \frac{c}{10} \text{ statamperes,} \tag{8}$$

$$1 \text{ volt} = 10^8 \text{ abvolts} = \frac{10^8}{c} \text{ statvolt,} \tag{9}$$

$$1 \text{ ohm} = 10^9 \text{ abohms} = \frac{10^9}{c^2} \text{ statohm,} \tag{10}$$

$$1 \text{ farad} = \frac{1}{10^9} \text{ abfarad} = \frac{c^2}{10^9} \text{ statfarads,} \tag{11}$$

$$1 \text{ henry} = 10^9 \text{ abhenries} = \frac{10^9}{c^2} \text{ stathenry,} \tag{12}$$

$$1 \text{ watt} = 1 \text{ joule/second} = 10^7 \text{ ergs/second.} \tag{13}$$

This system is incomplete because the units for many electrical quantities have never been defined in it. There is an unlimited number of ways in which it can be extended to include the rest of the electrical quantities,

depending on the sizes chosen for certain of the units. An example of such a system used in the past is one described by Maxwell, in which the unit of length is a quadrant of the earth's circumference, or 10^9 centimeters, the unit of mass is 10^{-11} gram, and the unit of time is the second.

In order to have many units of convenient size, preferably the practical ones in common engineering use, a self-consistent absolute system of units complete for electricity and magnetism, called the meter-kilogram-second practical system (abbreviated mks), has been devised and advocated by Giorgi. The system is built around the meter, kilogram, second, and the practical electrical units, the size of other units being forced to conform. One result of this plan is that neither the permittivity nor the permeability of free space expressed in mks practical units is unity. The definitions of the various units can be traced by following down the mks practical column of Table I.

The use of this system[4] has the advantage that it makes available a self-consistent absolute system of units for both electricity and magnetism. The standards for the mks system are the International Prototype Meter Bar, the International Prototype Kilogram, the second as defined by the motion of the earth, and the permeability of free space. In the past it has been necessary to specify arbitrary standards for additional electrical quantities, simply because comparisons with the absolute standards could not be made with sufficient precision. Since these standards were chosen so as to make the practical units come as close as possible to multiples of the absolute electromagnetic units, the change does not affect the calibration of any ordinary meters or apparatus. Further discussion of this question of standards and practical units is in references given in the bibliography.

In Table II are shown the units of the three systems discussed, with conversion factors which relate corresponding units, and the dimensions of each electrical quantity in terms of (a) mass, length, time, and permittivity, (b) mass, length, time, and permeability, and (c) resistance, current, length, and time. The dimensions of the electrical quantities in terms of current, resistance, length, and time are included in this table because of the greater ease with which these dimensions can be used in electrical problems involving dimensional considerations.

[4] A. E. Kennelly, "I.E.C. Adopts MKS System of Units," *E.E.*, LIV (1935), 1373–1384.

TABLE I

Relation	CGS Electrostatic Units	CGS Electromagnetic Units	MKS Practical Units
$f = \frac{Q_1Q_2}{\epsilon r^2}\,r$	dyne $= \frac{(\text{statcoulomb})^2}{(\text{aesu}\epsilon)(\text{centimeter})^2}$		dyne-five $= \frac{(\text{coulomb})^2}{(\text{mksu}\epsilon)(\text{meter})^2}$
$\mathcal{E} = \frac{f}{Q}$	aesu$\mathcal{E}$ $= \frac{(\text{dyne})}{(\text{statcoulomb})} = \frac{(\text{statvolt})}{(\text{centimeter})}$		aprau$\mathcal{E}$ $= \frac{\text{dyne-five}}{\text{coulomb}} = \frac{\text{volt}}{\text{meter}}$
$V = \int \mathcal{E} \cdot d\ell$	statvolt $= (\text{aesu}\mathcal{E})(\text{centimeter}) = \frac{(\text{erg})}{(\text{statcoulomb})}$		volt $= (\text{aprau}\mathcal{E})(\text{meter}) = \frac{\text{joule}}{\text{coulomb}}$
$\mathcal{D} = \epsilon\mathcal{E}$	aesu$\mathcal{D}$ $= (\text{aesu}\epsilon)\,\frac{(\text{statvolt})}{(\text{centimeter})}$		aprau$\mathcal{D}$ $= \frac{(\text{aprau}\epsilon)(\text{volt})}{(\text{meter})}$
$\psi = \int_s \mathcal{D} \cdot ds$	aesuψ $= (\text{aesu}\epsilon)(\text{statvolt})(\text{centimeter})$		aprauψ $= (\text{aprau}\epsilon)(\text{volt})(\text{meter})$
$q = \frac{dQ}{d\ell}$	aesu$q$ $= \frac{(\text{statcoulomb})}{(\text{centimeter})}$		aprau$q$ $= \frac{(\text{coulomb})}{(\text{meter})}$
$\sigma = \frac{dQ}{ds}$	aesuσ $= \frac{(\text{statcoulomb})}{(\text{centimeter})^2}$		aprauσ $= \frac{(\text{coulomb})}{(\text{meter})^2}$
$\rho = \frac{dQ}{dv}$	aesuρ $= \frac{(\text{statcoulomb})}{(\text{centimeter})^3}$		aprauρ $= \frac{(\text{coulomb})}{(\text{meter})^3}$
$C = \frac{Q}{V}$	statfarad $= \frac{(\text{statcoulomb})}{(\text{statvolt})}$		farad $= \frac{(\text{coulomb})}{(\text{volt})}$
$W = VQ$	erg $= (\text{statvolt})(\text{statcoulomb})$		joule $= (\text{volt})(\text{coulomb})$
$I = \frac{dQ}{dt}$	statampere $= \frac{(\text{statcoulomb})}{(\text{second})}$	abampere $= \frac{(\text{abcoulomb})}{(\text{second})}$	ampere $= \frac{(\text{coulomb})}{(\text{second})}$

$\mathcal{J} = \frac{dI}{ds}\,\frac{ds}{ds}$	aesu$\mathcal{J}$ $= \frac{\text{(statampere)}}{\text{(centimeter)}^2}$	aemu$\mathcal{J}$ $= \frac{\text{(abampere)}}{\text{(centimeter)}^2}$	aprau$\mathcal{J}$ $= \frac{\text{(ampere)}}{\text{(meter)}^2}$
$R = \frac{V}{I}$	statohm $= \frac{\text{(statvolt)}}{\text{(statampere)}}$	abohm $= \frac{\text{(abvolt)}}{\text{(abampere)}}$	ohm $= \frac{\text{(volt)}}{\text{(ampere)}}$
$\rho = \frac{\mathcal{E}}{\mathcal{J}}$	aesuρ = (statohm)(centimeter)	aemuρ = (abohm)(centimeter)	aprauρ = (ohm)(meter)
$P = VI$	aesu$P$ $= \frac{\text{(erg)}}{\text{(second)}}$ = (statvolt)(statampere)	aemu$P$ $= \frac{\text{(erg)}}{\text{(second)}}$ = (abvolt)(abampere)	watt $= \frac{\text{(joule)}}{\text{(second)}}$
$\begin{cases} d\mathcal{F} = \mu I_1 d\ell_1 \times \left[\frac{I_2 d\ell_2 \times \mathbf{r}}{r^3}\right] \\ \quad = \mu I_1 d\ell_1 \times d\mathcal{H} \end{cases}$		dyne = (aemuμ)(abampere)^2 oersted $= \frac{\text{(dyne)}}{(\text{aemu}\mu)\text{(abampere)(centimeter)}}$ $= \frac{\text{(gilbert)}}{\text{(centimeter)}}$	dyne-five = (aprauμ)(ampere)^2 praoersted $= \frac{\text{(dyne-five)}}{(\text{aprau}\mu)\text{(ampere)(meter)}}$ $= \frac{\text{(pragilbert)}}{\text{(meter)}}$ $= 4\pi\,\frac{\text{(ampere-turn)}}{\text{(meter)}}$
$F = \oint \mathcal{H} \cdot d\ell$		gilbert = (oersted)(centimeter)	pragilbert = (praoersted)(meter) = 4π(ampere-turn)
$\mathcal{B} = \mu\mathcal{H}$		gauss = (aemuμ)(oersted) $= \frac{\text{(maxwell)}}{\text{(centimeter)}^2}$	aprau$\mathcal{B}$ = (aprauμ)(praoersted) $= \frac{\text{(weber)}}{\text{(meter)}^2}$
$\phi = \int_s \mathcal{B} \cdot ds$		maxwell = (gauss)(centimeter)^2	weber = (aprau$\mathcal{B}$)(meter)^2
$\mathcal{R} = \frac{F}{\phi}$		aemu$\mathcal{R}$ $= \frac{\text{(gilbert)}}{\text{(maxwell)}}$	aprau$\mathcal{R}$ $= \frac{\text{(pragilbert)}}{\text{(weber)}}$
$L = \frac{d\phi}{dI}$		abhenry $= \frac{\text{(maxwell)}}{\text{(abampere)}}$	henry $= \frac{\text{(weber)}}{\text{(ampere)}}$
$\lambda = N\phi$		aemuλ = (maxwell)(turn)	aprauλ = (weber)(turn)

TABLE II

UNITS, CONVERSION FACTORS, AND DIMENSIONS

Quantity	Symbol	MKS Practical Unit	Conversion Factor F^* to obtain esu	Conversion Factor F^* to obtain emu	CGS esu	Conversion Factor F^* to obtain emu	Conversion Factor F^* to obtain mks	CGS emu	Conversion Factor F^* to obtain esu	Conversion Factor F^* to obtain mks	Dimensions $[M]$	$[L]$	$[T]$	$[\varepsilon]$	$[M]$	$[L]$	$[T]$	$[\mu]$	$[R]$	$[I]$	$[L]$	$[T]$
Permittivity	ε		$10^{-11}c^2$	10^{-11}		c^{-2}	$10^{11}c^{-2}$		c^2	10^{11}	0	0	0	1	0	−2	2	−1	−1	0	−1	1
Charge	Q	coulomb	$10^{-1}c$	10^{-1}	statcoulomb	c^{-1}	$10c^{-1}$	abcoulomb	c	10	$\frac{1}{2}$	$\frac{3}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	0	1
Charge Density: Linear	q	coulomb/meter	$10^{-3}c$	10^{-3}	statcoulomb/centimeter	c^{-1}	$10^{3}c^{-1}$	abcoulomb/centimeter	c	10^{3}	$\frac{1}{2}$	$\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	−1	1
Charge Density: Surface	σ	coulomb/meter^2	$10^{-5}c$	10^{-5}	statcoulomb/centimeter^2	c^{-1}	$10^{5}c^{-1}$	abcoulomb/centimeter^2	c	10^{5}	$\frac{1}{2}$	$-\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	0	1	−2	1
Charge Density: Volume	ρ	coulomb/meter^3	$10^{-7}c$	10^{-7}	statcoulomb/centimeter^3	c^{-1}	$10^{7}c^{-1}$	abcoulomb/centimeter^3	c	10^{7}	$\frac{1}{2}$	$-\frac{3}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	0	1	−3	1
Electric Field Intensity	$\mathcal{E}$	volt/meter	$10^{6}c^{-1}$	10^{6}	statvolt/centimeter	c	$10^{-6}c$	abvolt/centimeter	c^{-1}	10^{-6}	$\frac{1}{2}$	$-\frac{1}{2}$	−1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−2	$\frac{1}{2}$	1	1	−1	0
Potential Difference	V	volt	$10^{8}c^{-1}$	10^{8}	statvolt	c	$10^{-8}c$	abvolt	c^{-1}	10^{-8}	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	−2	$\frac{1}{2}$	1	1	0	0
Electric Flux	ψ		$10^{-1}c$	10^{-1}		c^{-1}	$10c^{-1}$		c	10	$\frac{1}{2}$	$\frac{3}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	0	1
Displacement (electric flux density)	$\mathcal{D}$		$10^{-5}c$	10^{-5}		c^{-1}	$10^{5}c^{-1}$		c	10^{5}	$\frac{1}{2}$	$-\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	0	1	−2	1
Polarization	$\mathcal{P}$		$10^{-5}c^2$	10^{-5}		c^{-2}	$10^{5}c^{-2}$		c^2	10^{5}	$\frac{1}{2}$	$-\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	0	1	−2	1
Capacitance	C	farad	$10^{-9}c^2$	10^{-9}	statfarad	c^{-2}	$10^{9}c^{-2}$	abfarad	c^2	10^{9}	0	1	0	1	0	−1	2	−1	−1	0	0	1
Elastance	S	daraf	$10^{9}c^{-2}$	10^{9}	statdaraf	c^2	$10^{-9}c^2$	abdaraf	c^{-2}	10^{-9}	0	−1	0	−1	0	1	−2	1	1	0	0	−1

Current	I	ampere	$10^{-1}c$	10^{-1}	statampere	c^{-1}	$10c^{-1}$	abampere	c	10	$\frac{1}{2}$	$\frac{3}{2}$	-2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	1	0	0
Current Density	$\mathcal{J}$	$\frac{\text{ampere}}{\text{meter}^2}$	$10^{-5}c$	10^{-5}	$\frac{\text{statampere}}{\text{centimeter}^2}$	c^{-1}	$10^{5}c^{-1}$	$\frac{\text{abampere}}{\text{centimeter}^2}$	c	10^{5}	$\frac{1}{2}$	$-\frac{1}{2}$	-2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	1	-2	0
Resistance	R	ohm	$10^{9}c^{-2}$	10^{9}	statohm	c^{2}	$10^{-9}c^{2}$	abohm	c^{-2}	10^{-9}	0	-1	1	-1	0	1	-1	1	1	0	0	0
Conductance	G	mho	$10^{-9}c^{2}$	10^{-9}	statmho	c^{-2}	$10^{9}c^{-2}$	abmho	c^{2}	10^{9}	0	1	-1	1	0	-1	1	-1	-1	0	0	0
Resistivity	ρ	ohm-meter	$10^{11}c^{-2}$	10^{11}	statohm cm	c^{2}	$10^{-11}c^{2}$	abohm cm	c^{-2}	10^{-11}	0	0	1	-1	0	2	-1	1	1	0	1	0
Conductivity	γ	$\frac{\text{mho}}{\text{meter}}$	$10^{-11}c^{2}$	10^{-11}	$\frac{\text{statmho}}{\text{centimeter}}$	c^{-2}	$10^{11}c^{-2}$	$\frac{\text{abmho}}{\text{centimeter}}$	c^{2}	10^{11}	0	0	-1	1	0	-2	1	-1	-1	0	-1	0
Permeability	μ		$10^{7}c^{-2}$	10^{7}		c^{2}	$10^{-7}c^{2}$	$\frac{\text{gauss}}{\text{oersted}}$	c^{-2}	10^{-7}	0	-2	2	-1	0	0	0	1	1	0	-1	1
Reluctivity	ν		$10^{-7}c^{2}$	10^{-7}		c^{-2}	$10^{7}c^{-2}$	$\frac{\text{oersted}}{\text{gauss}}$	c^{2}	10^{7}	0	2	-2	1	0	0	0	-1	-1	0	1	-1
Magnetic Field Intensity	$\mathcal{H}$	praoersted	$10^{-3}c$	10^{-3}		c^{-1}	$10^{3}c^{-1}$	oersted	c	10^{3}	$\frac{1}{2}$	$\frac{1}{2}$	-2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	1	-1	0
Magnetomotive Force	F	pragilbert	$10^{-1}c$	10^{-1}		c^{-1}	$10c^{-1}$	gilbert	c	10	$\frac{1}{2}$	$\frac{3}{2}$	-2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	1	0	0
Magnetic Flux	ϕ	weber	$10^{8}c^{-1}$	10^{8}		c	$10^{-8}c$	maxwell	c^{-1}	10^{-8}	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	-1	$\frac{1}{2}$	1	1	0	1
Magnetic Flux Density	$\mathcal{B}$	$\frac{\text{weber}}{\text{meter}^2}$	$10^{4}c^{-1}$	10^{4}		c	$10^{-4}c$	gauss	c^{-1}	10^{-4}	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	1	1	-2	1

* The Conversion Factor F is the factor by which the number of units which express a particular amount of a quantity in one system must be multiplied to obtain the number of units which express the same amount of that quantity in another system of units. **For example, to convert 10 coulombs to abcoulombs:**

$$10 \text{ coulombs} = 10F \text{ abcoulombs} = 10 \times 10^{-1} = 1 \text{ abcoulomb.}$$

In order to obtain the ratio of sizes of units:

$$\frac{\text{size of coulomb}}{\text{size of abcoulomb}} = F = \frac{1}{10}; \quad \frac{\text{size of abcoulomb}}{\text{size of coulomb}} = \frac{1}{F} = 10.$$

TABLE II (CONTINUED)

Quantity	Symbol	MKS Practical Units	Conversion Factor F* to obtain esu	Conversion Factor F* to obtain emu	CGS esu	Conversion Factor F* to obtain emu	Conversion Factor F* to obtain mks	CGS emu	Conversion Factor F* to obtain esu	Conversion Factor F* to obtain mks	Dimensions [M]	[L]	[T]	[ε]	[M]	[L]	[T]	[μ]	[R]	[I]	[L]	[T]
Intensity of Magnetization	$\mathfrak{M}$	weber / meter²	10^4c^{-1}	10^4		c	$10^{-4}c$	gauss	c^{-1}	10^{-4}	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	1	$\frac{1}{2}$	1	1	−2
Reluctance	$\mathcal{R}$	pragilbert / weber	$10^{-9}c^2$	10^{-9}		c^{-2}	10^9c^{-2}	gilbert / maxwell	c^2	10^9	0	1	−2	1	0	−1	0	−1	−1	0	0	−1
Permeance	$\mathcal{P}$	weber / pragilbert	10^9c^{-2}	10^9		c^2	$10^{-9}c^2$	maxwell / gilbert	c^{-2}	10^{-9}	0	−1	2	−1	0	1	0	1	1	0	0	1
Inductance	L,M	henry	10^9c^{-2}	10^9	stathenry	c^2	$10^{-9}c^2$	abhenry	c^{-2}	10^{-9}	0	−1	2	−1	0	1	0	1	1	0	0	1
Magnetic Pole Strength	m		10^8c^{-1}	10^8		c	$10^{-8}c$	pole	c^{-1}	10^{-8}	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	−1	$\frac{1}{2}$	1	1	0	1
Force	f	{dyne-five newton	10^5	10^5	dyne	1	10^{-5}	dyne	1	10^{-5}	1	1	−2	0	1	1	−2	0	1	2	−1	1
Energy (work)	W	joule	10^7	10^7	erg	1	10^{-7}	erg	1	10^{-7}	1	2	−2	0	1	2	−2	0	1	2	0	1
Power	P	watt	10^7	10^7	erg / second	1	10^{-7}	erg / second	1	10^{-7}	1	2	−3	0	1	2	−3	0	1	2	0	0

Numerical Values for Free Space

	ε_0	μ_0
MKS Practical	$10^{11}c^{-2}$	10^{-7}
CGS emu	c^{-2}	1
CGS esu	1	c^{-2}

$c = (2.997\,96 \pm 0.000\,04)\ 10^{10} \approx 3 \times 10^{10}$

$c^2 = (8.987\,8 \pm 0.000\,2)\ 10^{20} \approx 9 \times 10^{20}$

$c^{-1} = (0.333\,560 \pm 0.000\,004)10^{-10} \approx \frac{1}{3} \times 10^{-10}$

$c^{-2} = (0\,111\,262 \pm 0.000\,003)10^{-20} \approx \frac{1}{9} \times 10^{-20}$

c is a numeric equal in magnitude to the velocity of electromagnetic wave propagation in free space when expressed in centimeters per second.

Bibliography

The study of a single textbook, however complete the book and however thorough and diligent the pursuit, does not in itself lead to a broad and effective knowledge and appreciation of a portion of an art or science. Perspective and a sense of values are developed through appraisal of independent points of view. Familiarity with thoroughgoing textbooks and original sources is necessary for the development and application of the theoretical principles of any field. This wider and more independent study leads, among other things, to contact with diverse applications so essential to development of effective methods of thought. The primary object of the bibliography of this book is to aid in the cultivation of these habits of studying sources other than the specified textbook and of consulting the literature.

The bibliography is presented in two parts: the general list which follows, and the series of footnote references inserted in the text. References in the general bibliography pertain to large portions of this book and parallel or supplement it. Those in the footnotes pertain specifically to smaller portions and generally present more advanced treatments or applications, or cite original sources. Of the two parts, that which follows is evidently the more comprehensive, although there is naturally some overlapping between them.

The order of presentation in the general bibliography is by chapters, though certain chapters which are commonly spanned within a single general reference are grouped. When a reference contains material pertaining to more than one chapter grouping in the bibliography, it is repeated in each group. Within each group, the order of presentation approximates the sequence of text material in the corresponding chapter or chapters.

This bibliography is in no way to be taken as an exhaustive and final collection. At the same time, it is sufficiently comprehensive and varied to allow the acquisition of an extensive knowledge of the subject. Many of the works cited contain further references, so that an exhaustive bibliography can readily be compiled with the listed material as a basis.

The following abbreviations for periodicals and circulars are used in footnotes and in this bibliography.

A.I.E.E. Trans.	American Institute of Electrical Engineers *Transactions*
Am. Math. Mo.	*American Mathematical Monthly*
Ann. d. Chem.	*Annalen der Chemie*
Ann. d. Phys.	*Annalen der Physik*
Arch. f. Elek.	*Archiv fur Elektro-technik*
B.S.T.J.	*Bell System Technical Journal*
Bell Monograph	Bell Telephone System, Technical Publications
Circ. Nat. Bur. Stand.	National Bureau of Standards Circular
E.E.	*Electrical Engineering*
Elec. J.	*Electric Journal*
Elec. W.	*Electrical World*
G.E. Rev.	*General Electric Review*
Gen. Rad. Exp.	*General Radio Experimenter*
I.E.E.J.	Institution of Electrical Engineers *Journal*
I.R.E. Proc.	Institute of Radio Engineers *Proceedings*

J.A.S.A.	*Journal* of the Acoustical Society of America
J. App. Phys.	*Journal of Applied Physics*
J.F.I.	*Journal* of the Franklin Institute
J. Math. Phys.	*Journal of Mathematics and Physics*
J. Res. Nat. Bur. Stand.	National Bureau of Standards *Journal of Research*
L'Éclairage Élec.	*L'Éclairage Électrique*
N.R.C.	National Research Council (various publications)
Personnel J.	*The Personnel Journal*
Phil. Mag.	*Philosophical Magazine*
Phys. Rev.	*Physical Review*
Proc. Lond. Math. Soc.	*Proceedings* of the London Mathematical Society
Sci. Paper Nat. Bur. Stand.	National Bureau of Standards Scientific Paper

Engineering Education and the Engineering Profession (General)

G. F. Swain, *How to Study* (New York: McGraw-Hill Book Company, Inc., 1917). This booklet contains much excellent advice concerning the mental attitude, the system, and the habits to be developed for effective study.

Walter S. Gifford, "Does Business Want Scholars?" *Harper's Magazine*, CLVI (1928), 669–674; Donald S. Bridgman, "Success in College and Business," *Personnel J.*, IX (1930), 1–19. These two articles give data relating success in college studies, in campus activities, and in earning college expenses to success in the business of the Bell Telephone System.

Engineering — A Career — A Culture (New York: The Engineering Foundation, 1932). Chapter i outlines engineering requirements and opportunities; Ch. v describes the field of electrical engineering; Ch. vi deals with personal qualifications and education.

J. A. L. Waddell, Frank W. Skinner, and Harold E. Wessman (eds.), *Vocational Guidance in Engineering Lines* (Easton, Pa.: The Mack Printing Company, 1933). The contents of this book were elicited from many contributors by the American Association of Engineers and have the endorsement of very prominent engineers. Chapter x, on electrical engineering, starts at a very juvenile level but subsequently (as is true of most of the book) merits reading by students already pursuing engineering courses.

Alfred P. Morgan, *The Pageant of Electricity* (New York: D. Appleton-Century Company, 1939). This book gives a comprehensive summary of the history of progress in electrical research and application with considerable attention to the more recent past as well as a review of the early discoveries. Though written for popular consumption, it is well worth the consideration of electrical engineers and students of electrical engineering.

Determination of Circuit Parameters (Chapter I)

N. H. Frank, *Introduction to Electricity and Optics* (New York: McGraw-Hill Book Company, Inc., 1940). This textbook presents an exposition of the fundamental principles of electricity and magnetism with special emphasis on field theory, and offers the elementary applications of these principles to electric circuits and to the electrical, magnetic, and optical properties of matter. The treatment is quantitative throughout Attention is paid to modern atomic ideas as well as to the more classical modes of description.

W. H. Timbie and Vannevar Bush, *Principles of Electrical Engineering* (2d ed.; New York: John Wiley & Sons, 1930). Chapters iv, xi, and xiii discuss the resistance, inductance, and capacitance parameters respectively.

S. S. Attwood, *Electric and Magnetic Fields* (New York: John Wiley & Sons, 1932). Methods of computation of capacitance and inductance are presented. The method of images and field mapping are included.

Edward Bennett and H. M. Crothers, *Introductory Electrodynamics for Engineers* (New York: McGraw-Hill Book Company, Inc., 1926). Chapter iv treats the calculation of capacitance, including an elementary treatment of the relations among *pairs* of conductors; Chs. viii and ix, the calculation of resistance; and Ch. xi, the calculation of inductance.

Alexander Russell, *A Treatise on the Theory of Alternating Currents* (2d ed.; Cambridge: at the University Press, 1914), Vol. I. Chapters i, ii, iii, v, vi, viii, and xix relate to the calculation of inductance and capacitance. Chapters v and vi give a moderately advanced comprehensive treatment of capacitance coefficients, partial capacitances are used (though not named as such) in a model of a cable.

J. L. LaCour and O. S. Bragstad, *Theory and Calculation of Electric Currents* (tr. Stanley P. Smith; London: Longmans, Green and Company, 1913). Chapter xxi deals with the resistance, inductance, and capacitance parameters.

James Clerk Maxwell, *A Treatise on Electricity and Magnetism* (3d ed.; Oxford: The Clarendon Press, 1892), Vols. I and II. Volume I, Part I: Chapter iii introduces the ideas of elastance and capacitance coefficients for multiconductor systems and develops several theorems involving the coefficients, including the expression for energy stored in the electrostatic field; Chs. vi and vii relate to equipotential lines and surfaces, and flow lines in the electrostatic field; Ch. xi deals with the method of images. Volume I, Part II: Chapters vii, viii, and ix deal with conductance and general three-dimensional current problems. Volume II, Part IV: Chapters xii, xiii, and xiv deal with magnetic problems by using current sheets, Ch. xiii containing the method of computing inductances by use of geometric mean distances.

E. B. Rosa and F. W. Grover, "Formulas and Tables for the Calculation of Mutual and Self Inductance," *Sci. Paper Nat. Bur. Stand.*, No. 169 (3d ed. [revised]; Washington: Government Printing Office, 1916). This paper contains formulas for the static self- and mutual inductances of many configurations, formulas for geometrical and arithmetical mean distances, and some high-frequency formulas for inductance and resistance.

L. F. Woodruff, *Principles of Electric Power Transmission* (2d ed.; New York: John Wiley & Sons, 1938). Chapters ii and iv treat the calculation of inductance and capacitance of certain multiwire transmission lines and cables. The method of geometric mean distances is used for inductance calculations. Chapter vii treats skin effect.

"Inductance of Steel Reënforced Aluminum Cable," *E.E.*, LIV (1935), 296–299. This article uses the method of geometric mean distances to compute the inductance of transmission lines made of cables consisting of aluminum strands spiraled around a core of spiraled steel strands.

C. Dannatt and J. W. Dalgleisch, *Electric Power Transmission and Interconnection* (London: Sir Isaac Pitman & Sons, Ltd., 1930). Chapter iii deals with the determination of the capacitance and inductance of systems of n parallel conductors with applications to a number of specific problems. The effects of ground and ground wires on capacitance are included.

W. V. Lyon, *Problems in Direct Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1931). Chapter i contains problems on resistance; Chs. v and vi contain problems on the magnetic and electrostatic fields respectively.

RESISTANCE NETWORKS (CHAPTER II)

W. H. Timbie and Vannevar Bush, *Principles of Electrical Engineering* (2d ed.; New York: John Wiley & Sons, 1930). Chapters ii and iii discuss Kirchhoff's laws and the

solution of direct-current network problems. Numerical solutions to illustrative examples are given.

Edward Bennett and H. M. Crothers, *Introductory Electrodynamics for Electrical Engineers* (New York: McGraw-Hill Book Company, Inc., 1926). Chapter ix deals with resistance networks.

C. V. Christie, *Electrical Engineering* (4th ed.; New York: McGraw-Hill Book Company, Inc., 1931). Chapter iii deals with resistance and resistance networks.

E. A. Guillemin, *Communication Networks.* Vol. I: *The Classical Theory of Lumped Constant Networks* (New York: John Wiley & Sons, 1931). Chapter iv deals with the steady-state solution of the general network, including a presentation of the method of determinants. The treatment applies to alternating currents but can be reduced to the direct-current case by substituting resistance where impedance appears.

John B. Wilbur, " The Mechanical Solution of Simultaneous Equations," *J.F.I.*, CCXXII (1936), 715–724. This article describes a machine capable of solving nine (or fewer) simultaneous equations having real coefficients.

F. A. Laws, *Electrical Measurements* (2d ed.; McGraw-Hill Book Company, Inc., 1938). Chapters i to v describe various direct-current bridges, potentiometers, and other measuring circuits.

National Electric Light Association, Serial Report of the Electrical Apparatus Subcommittee: *D. C. Substation Design Practice* (New York: National Electric Light Association, 1929). This report shows how various electric power companies and electric railway companies obtain and distribute direct current.

W. V. Lyon, *Problems in Direct Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1931). Chapters ii to iv constitute a collection of problems dealing with Ohm's and Kirchhoff's laws, and with energy and power.

Direct-Current Transients (Chapter III)

E. A. Guillemin, *Communication Networks.* Vol. I: *The Classical Theory of Lumped Constant Networks* (New York: John Wiley & Sons, 1931). Chapter i discusses the philosophy of linear electrical networks. Chapter ii treats the single-loop network having constant excitation.

R. R. Lawrence, *Principles of Alternating Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1935). Chapter v presents analyses of *RL*, *RS*, and *RLS* series circuits having constant excitation.

E. B. Kurtz and G. F. Corcoran, *Introduction to Electric Transients* (New York: John Wiley & Sons, 1935). Section I is devoted to direct-current transients, presenting physical consideration, mathematical analyses, and oscillographic verifications. Power transients also are presented.

G. W. Pierce, *Electric Oscillations and Electric Waves* (New York: McGraw-Hill Book Company, Inc., 1920). Chapters i to iv present fundamental laws and equations, transients, and energy transformations in the series *RLS* circuit, and the geometry of complex quantities.

C. E. Magnusson, *Electric Transients* (New York: McGraw-Hill Book Company, Inc., 1926). This presentation of methods of solution of transient problems is valuable principally because of the many oscillograms of varied transients which it contains.

Philip Franklin, *Differential Equations for Electrical Engineers* (New York: John Wiley & Sons, 1933). This book gives a mathematical background which aids greatly in obtaining a thorough grasp of methods of transient analysis. Chapters i, iii, and vii present, respectively, complex functions, linear differential equations having constant coefficients, and analytic functions.

W. F. Osgood, *Advanced Calculus* (New York: The Macmillan Company, 1925). Chapter xx is a very readable treatment of complex numbers.

T. C. Fry, "Differential Equations as a Foundation for Electric Circuit Theory," *Am. Math. Mo.*, XXXVI (1929), 499–504. This article gives a clear and concise statement showing why use of complex numbers is peculiarly appropriate in circuit theory in which physical quantities are so obviously real.

G. A. Campbell, "Cisoidal Oscillations," *A.I.E.E. Trans.*, XXX (1911), 873–913; V. Bush, "Oscillating Current Circuits by the Method of Generalized Angular Velocities," *A.I.E.E. Trans.*, XXXVI (1917), 207–234; A. E. Kennelly, "The Impedances, Angular Velocities, and Frequencies of Oscillating Current Circuits," *I.R.E. Proc.*, IV (1916), 47–94. These three articles present a number of additional points of view on the general method of solving for transients. Footnote, p. 223.

W. V. Lyon, *Problems in Direct Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1931). Problems dealing with transients are in Chs. vii and viii.

PRINCIPLES OF ALTERNATING-CURRENT CIRCUITS (CHAPTERS IV, V, VI, VII, VIII)

E. A. Guillemin, *Communication Networks*. Vol. I: *The Classical Theory of Lumped Constant Networks* (New York: John Wiley & Sons, 1931). Chapters iii and iv deal with alternating-current circuits in the steady state, Chs. v, vi, and vii deal with transients; Ch. viii contains applications to coupled circuits.

R. R. Lawrence, *Principles of Alternating Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1935). This book presents a very complete treatment of alternating-current circuits largely from the point of view of the power engineer.

C. V. Christie, *Electrical Engineering* (4th ed.; New York: McGraw-Hill Book Company, Inc., 1931). Chapter iv deals with the principles of alternating current circuits.

Russell M. Kerchner and G. F. Corcoran, *Alternating Current Circuits* (New York: John Wiley & Sons, 1938). Chapters i to viii treat the principles of steady-state alternating-current circuits; Ch. xvi treats the transient state.

J. M. Bryant, J. A. Correll, and E. W. Johnson, *Alternating Current Circuits* (3d ed.; New York: McGraw-Hill Book Company, Inc., 1939). Chapters i to v deal with elementary steady-state alternating-current circuits, Ch. vi treats somewhat more complicated circuits.

Alexander Russell, *A Treatise on the Theory of Alternating Currents* (2d ed.; Cambridge: at the University Press, 1914). Chapters ii, iii, ix, x, xi, xiii, and xiv deal with the fundamental principles; Ch. vii deals with high-frequency currents; Ch. xxi treats the method of duality.

J. L. LaCour and O. S. Bragstad, *Theory and Calculation of Electric Currents* (tr. Stanley P. Smith; London: Longmans, Green and Company, 1913). The introduction and Chs. i, ii, iv, and v deal with elementary alternating-current principles and circuits; Chs. vi, vii, and viii deal with more complicated circuits.

A. E. Kennelly, "Impedance," *A.I.E.E. Trans.*, X (1893), 175–216; C. P. Steinmetz, "Reactance," *A.I.E.E. Trans.*, XI (1894), 640–648. These articles introduced the methods of complex alegbra to engineers.

E. B. Kurtz and G. F. Corcoran, *Introduction to Electric Transients* (New York: John Wiley & Sons, 1935). Chapters vi to ix deal with transients in simple and multibranch alternating-current circuits.

R. D. Evans and H. K. Sels, "Transmission-Line Constants and Resonance," *Elec. J.*, XVIII (1921), 306–309. "Transmission Lines and Transformers," *Elec. J.*, XVIII (1921), 356–359. These articles deal with the use of equivalent circuits and general circuit constants in transmission-line problems.

William Nesbit, *Electrical Characteristics of Transmission Circuits* (East Pittsburgh Westinghouse Technical Night School Press, 1926.) This book contains extensive tables of parameters of transmission circuits, and useful relations in terms of general circuit constants.

C. Dannatt and J. W. Dalgleisch, *Electric Power Transmission and Interconnection* (London: Sir Isaac Pitman & Sons, Ltd., 1930). Chapter v deals with representation of electric power networks in terms of equivalent π and T circuits and in terms of general circuit constants.

O. G. C. Dahl, *Electric Circuits.* Vol. I: *Theory and Applications* (New York: McGraw-Hill Book Company, Inc., 1928). Chapter ix deals with general circuit constants.

H. L. Hazen, O. R. Schurig, and M. F. Gardner, "The M.I.T. Network Analyzer, Design and Application to Power System Problems," *A.I.E.E. Trans.*, XLIX (1930), 1102–1113; H. A. Travers and W. W. Parker, "An Alternating-Current Calculating Board," *Elec. J.*, XXVII (1930), 266–270. These papers describe laboratory networks which can be arranged to represent in miniature actual alternating-current power systems.

Vannevar Bush, "Recent Progress in Analysing Machines," in *Proceedings of the Fourth International Congress for Applied Mechanics* (Cambridge: at the University Press, 1935), pp. 3–24. This is a review of recent progress in the development of mathematical machines, especially equation solvers.

W. V. Lyon, *Problems in Alternating Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1931). The problems in Chs. i and ii are on single-phase circuits in the steady state; those in Ch. iv are on transients in single-phase circuits; those in Ch. vii involve mutual inductance.

Loci of Complex Functions (Chapter IX)

J. L. LaCour and O. S. Bragstad, *Theory and Calculation of Electric Currents* (tr. Stanley P. Smith; London: Longmans, Green and Company, 1913). Chapter iii treats graphical representation in the complex plane; Chs. ix, x, and xv make much use of graphical methods.

W. O. Shumann, "Zur Theorie der Kreisdiagramme," *Arch. f. Elek.*, XI (1922), 140–146; R. Richter, *Elektrische Maschinen* (Berlin: J. Springer, 1930), Vol. II, Ch. i. These references give the general principles of representation and inversion in the complex plane, show the general relation which circular loci must satisfy, and present means for constructing scales which relate the independent variable to the final locus.

George B. Hoadley, "The Science of Balancing an Impedance Bridge," *J.F.I.*, CCVIII (1939), 733–754. By means of locus diagrams which show the potential difference between the detector terminals when the various impedances in the bridge arms are adjusted, this article analyzes the balancing procedure for impedance bridges.

A. C. Seletsky, "Cross Potential of a 4-Arm Network," *E.E.*, LII (1933), 861–867. This article gives loci of voltage across the midpoints of a four-arm network having constant parameters and fixed applied alternating voltage, as functions of impedance of one arm or of the frequency of the source.

L. F. Woodruff, *Principles of Electric Power Transmission* (2d ed.; New York: John Wiley & Sons, 1938), Ch. vi; O. G. C. Dahl, *Electric Circuits.* Vol. I: *Theory and Applications* (New York: McGraw-Hill Book Company, Inc., 1928). These references deal with circle diagrams as applied to the electric power transmission line; the derivations are largely in terms of general circuit constants.

Polyphase Systems, Symmetrical Components (Chapters X and XI)

R. R. Lawrence, *Principles of Alternating Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1935). Chapters viii to xiv deal with polyphase systems; Ch. xii, with the method of symmetrical components.

C. V. Christie, *Electrical Engineering* (4th ed.; New York: McGraw-Hill Book Company, Inc., 1931), Ch. vi; J. L. LaCour and O. S. Bragstad, *Theory and Calculation of Electric Currents* (tr. Stanley P. Smith; London: Longmans, Green and Co., 1913), Chs. xiii and xiv; Alexander Russell, *A Treatise on the Theory of Alternating Currents* (2d ed.; Cambridge: at the University Press, 1914), Vol. I, Chs. xv to xix. These references deal with polyphase circuits and related problems.

Russell M. Kerchner and G. F. Corcoran, *Alternating-Current Circuits* (New York: John Wiley & Sons, 1938). Chapters ix to xi deal with polyphase systems; Chs. xiv and xv, with symmetrical components.

J. M. Bryant, J. A. Correll, and E. W. Johnson, *Alternating-Current Circuits* (3d ed.; New York: McGraw-Hill Book Company, Inc., 1939). Chapters vii to ix deal with polyphase systems; Ch. x, with symmetrical components.

O. G. C. Dahl, *Electric Circuits*. Vol. I: *Theory and Applications* (New York: McGraw-Hill Book Company, Inc., 1928). Chapter iii deals with the method of symmetrical components.

W. V. Lyon, *Application of the Method of Symmetrical Components* (New York: McGraw-Hill Book Company, Inc., 1937). The entire volume is devoted to the method of symmetrical components, including an introductory historical development, Ch. i; general principles for use of the method for three phase circuits, Ch. ii; for four-phase circuits, Ch. iii; and numerous applications in subsequent chapters.

C. F. Wagner and R. D. Evans, *Symmetrical Components* (New York: McGraw-Hill Book Company, Inc., 1933). The entire volume is devoted to the method of symmetrical components including the historical development and general principles — Chs. i and ii — and numerous applications in subsequent chapters. There is a large bibliography.

C. L. Fortescue, "Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks," *A.I.E.E. Trans.*, XXXVII (1918), 1027–1140. This is the first presentation of the general method of symmetrical components.

W. V. Lyon, *Problems in Alternating Currents* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1931). Chapter v contains general problems on polyphase circuits; Ch. vii contains some problems involving mutual inductance in polyphase circuits.

Electromechanically Coupled Systems (Chapter XII)

R. D. Fay and W. M. Hall, "The Determination of the Acoustic Output of a Telephone Receiver from Input Measurements," *J.A.S.A.*, V (1933), 46–56. This article gives an analysis of the operation of a telephone receiver in terms of measured terminal electrical impedance as a function of length of closed tube into which the receiver works.

F. A. Firestone, "The Mobility Method of Computing the Vibration of Linear Mechanical and Acoustical Systems: Mechanical-Electrical Analogies," *J. App. Phys.*, IX (1938), 373–387. The methods of the electrical engineer for solving circuit problems are used, but the problem is kept in purely mechanical terms.

A. E. Kennelly, "The Measurement of Acoustic Impedance with the Aid of the Telephone Receiver," *J.F.I.*, CC (1925), 467–488. The method makes use of the analogy between electric-line conductors and acoustic-tube conductors.

J. P. Maxfield and H. C. Harrison, "Method of High Quality Recording and Reproducing of Music and Speech Based on Telephone Research," *B.S.T.J.*, V (1926),

493–523. This article gives an analysis of the general requirements for sound recording and reproducing without distortion, and demonstrates the application of telephone transmission theory to the solution of mechanical transmission systems viewed as analogues of electric circuits.

C. S. Draper and G. V. Schliestett, "General Principles of Instrument Analysis," *Instruments*, XII (1939), 137–142. This article presents a method of attack which systematizes and simplifies instrument problems. Essential fundamental information is condensed and generalized in tables and charts.

G. F. Gardner, "Simple Mathematical Operations Performed by Electrical Instruments," *G.E. Rev.*, XXXVII (1934), 148–154. This article shows how various operations of common algebra, complex algebra, and calculus can be performed with the aid of ordinary electrical instruments.

E. C. Wente, "The Sensitivity and Precision of the Electrostatic Transmitter for Measuring Sound Intensities," *Phys. Rev.*, XIX (1922), 498–503. Construction and test characteristics of the transmitter are given.

W. P. Mason, "An Electromechanical Representation of a Piezoelectric Crystal," *I.R.E. Proc.*, XXIII (1935), 1252–1263. An equivalent electrical network is developed which represents a piezoelectric crystal interacting with an electric circuit.

G. W. Pierce, "Magnetostriction Oscillators," *I.R.E. Proc.*, XVII (1929), 42–88. A method is described for using magnetostriction to produce and control electrical and mechanical frequencies of oscillation by the interaction of mechanical vibration of a magnetostrictive rod and the electric oscillations of an electric circuit.

Analysis of Nonlinear Circuits (Chapter XIII)

Joseph Lipka, *Graphical and Mechanical Computation* (New York: John Wiley & Sons, 1921). Part I deals largely with alignment charts; Part II, largely with empirical formulas for representing experimental data.

H. Von Sanden, *Practical Mathematical Analysis* (tr. H. Levy; New York: E. P. Dutton and Company, 1923), pp. 171–181; C. Runge and H. Koenig, *Vorlesungen uber Numerisches Rechnen* (Berlin: J. Springer, 1924), Ch. x. Von Sanden gives a method of numerical integration to obtain a particular solution to $dy/dx = f(x,y)$. Runge and Koenig treat the same problem from a more advanced standpoint.

Reinhold Rüdenberg, *Elektrische Schaltvorgange* (3d ed.; Berlin: J. Springer, 1933). This book gives an extended discussion of the subject of transients in nonlinear systems

V. Bush, F. D. Gage, and H. R. Stewart, "A Continuous Integraph," *J.F.I.*, CCIII (1927), 63–84. An integrating machine is described which uses a direct-current watt-hour meter as an integrating element. The average precision is about 1 per cent.

V. Bush, "The Differential Analyser. A New Machine for Solving Differential Equations," *J.F.I.*, CCXII (1931), 447–448. This machine operates by means of mechanical interconnection of Kelvin integrators and gear-type multipliers and adders. The average precision is about 0.1 per cent. "Recent Progress in Analysing Machines" in *Proceedings of the Fourth International Congress for Applied Mechanics* (Cambridge: at the University Press, 1935), pp. 3–24. This is a review of recent progress in the development of mathematical machines, especially equation solvers.

G. F. Gardner, "Simple Mathematical Operations Performed by Electrical Instruments," *G.E. Rev.*, XXXVII (1934), 148–154. This article shows how various operations of common algebra, complex algebra, and calculus can be performed with the aid of ordinary electrical instruments.

T. Brownlee, "The Calculation of Circuits Containing Thyrite," *G.E. Rev.*, XXXVII (1934), 175–179. This article gives data on Thyrite and shows several

methods for calculation of current and voltage in circuits containing Thyrite, inductance, and capacitance.

Ernst Weber, "Field Transients in Magnetic Systems," *A.I.E.E. Trans.*, L (1931), 1234–1246. This article gives an analysis of the effect of eddy currents on flux build-up.

P. H. Odessey and Ernst Weber, "Critical Conditions in Ferroresonance," *A.I.E.E. Trans.*, LVII (1938), 444–452. This article gives a general analysis of the series *RLS* circuit with nonlinear inductance when alternating voltage is applied. Graphical methods are employed, and the results of experimental studies are shown.

William T. Thomson, "Resonant Nonlinear Control Circuits," *A.I.E.E. Trans.*, LVII (1938), 469–476. This article attacks the same circuit as does the Odessey and Weber article but from a purely analytical point of view. It includes also the parallel condenser-coil circuit and the use of superposed direct current for control.

E. G. Keller, "Resonance Theory of Series Non-linear Control Circuits," *J.F.I.*, CCXXV (1938), 561–577. Theory and experimental checks are presented for simple series circuits having nonlinear elements which are important in industrial control circuits. The analysis is given in detail only for variable inductance.

H. T. Mayne, "An Alternator-Voltage Regulator Using a Nonlinear Circuit," *A.I.E.E. Trans.*, LVI (1937), 462–464. This paper describes a regulator which uses mercury-vapor control tubes and saturable-core inductance.

F. R. Longley, "The Calculation of Alternator Swing Curves," *A.I.E.E., Trans.*, XLIX (1930), 1129–1151. This article makes extensive use of the step-by step method.

C. G. Suits, "Non-linear Circuits for Relay Applications," *E.E.*, L (1931), 963 964; "A Voltage Selective Nonlinear Bridge," *Physics*, I (1931), 171 181; "New Applications of Non-linear Circuits to Relay and Control Problems," American Institute of Electrical Engineers, Paper 32–85 (1932); "Non-linear Circuits Applied to Relays," *E.E.*, LII (1933), 244–246; "Flashing Lamps without Moving Parts," *Elec. W.*, XCIX (1932), 1098–1099. These articles show how simple series and parallel circuits involving nonlinear inductance can be adapted for relay and control applications.

"Reactance Amplifiers," *Electronics*, X (October, 1937), 28 30. This article cites recent developments in circuits using iron-core reactors saturated by direct current from copper oxide rectifiers. Various uses as magnetic amplifiers and in control circuits for industrial purposes are mentioned.

PHYSICAL CONSTANTS, WIRE TABLES (APPENDIX A), ENGINEERING HANDBOOKS

"Copper Wire Tables," *Circ. Nat. Bur. Stand.*, No. 31 (2d ed.; Washington: Government Printing Office, 1914). Physical constants of copper and extensive tables for copper wire are presented, as prepared at the request and with the co-operation of the standards committee of the American Institute of Electrical Engineers.

"American Institute of Electrical Engineers Standards," No. 30: *Definitions and General Standards for Wires and Cables* (New York: American Institute of Electrical Engineers, 1937). Definitions of various kinds of wires and cables are given, and standards for conductivity, voltage, insulation, resistance, capacitance, and temperature tests are designated.

Electrical Characteristics of A.C.S.R. [Revised] (Pittsburgh: Aluminum Company of America, 1934). This publication contains tables of resistance, inductance, reactance, susceptance, weight, dimensions, and temperature rise for aluminum cables, steel reinforced, with various combinations of steel and aluminum stranding.

Frederick E. Fowle (ed.), *Smithsonian Physical Tables* (8th ed. [revised]; Washington: The Smithsonian Institution, 1933); National Research Council, *International Critical Tables* (New York: McGraw-Hill Book Company, Inc., 1926), Vols. I–VII and index; Charles D. Hodgman (ed.), *Handbook of Chemistry and Physics* (23d ed.; Cleveland: Chemical Rubber Publishing Company, 1939). These three items contain

a mass of information on the physical and chemical properties of the elements and of many materials. The *Handbook of Chemistry and Physics* contains also mathematical tables.

R. G. Hudson, *The Engineers' Manual* (2d ed.; New York: John Wiley & Sons, 1939). This manual contains fundamental formulas of mathematics, hydraulics, mechanics, heat and electricity, tables of physical constants, properties of materials, mathematical tables, and conversion factors.

"Wiley Engineering Handbook Series," Vol. I: *Handbook of Engineering Fundamentals*, Ovid W. Eshbach, ed. (New York: John Wiley & Sons, 1936). This book contains fundamental scientific laws basic to mathematics, physics, and chemistry, properties and uses of engineering materials, and mathematical and physical tables.

"Wiley Engineering Handbook Series," Vol. IV: *Electrical Engineers' Handbook: Electric Power*, Harold Pender, William A. Del Mar, and Knox McIlwain, eds.; Vol. V: *Electrical Engineers' Handbook: Electric Communication and Electronics*. Harold Pender and Knox McIlwain, eds. (3d ed. [rewritten]; New York: John Wiley & Sons, 1936). These are general handbooks giving properties of materials, fundamental theory, and practical information useful in the major branches of electrical engineering named.

Frank F. Fowle (ed.), *Standard Handbook for Electrical Engineers* (6th ed.; New York: McGraw-Hill Book Company, Inc., 1933). This is a general handbook giving properties of materials, fundamental principles, and practical information useful in the field of electrical engineering.

Keith Henney (ed.), *The Radio Engineering Handbook* (2d ed.; New York: McGraw-Hill Book Company, Inc., 1935). This book contains mathematical and electrical tables, fundamental principles of electricity and magnetism, and special information useful to radio engineers.

H. B. Dwight, *Tables of Integrals and Other Mathematical Data* (New York: The Macmillan Company, 1934). This is an especially usable set of integral tables; the other tables include trigonometric functions, logarithms, exponentials, conversion between radians and degrees, and numerical constants.

Determinants (Appendix B)

R. S. Burington and C. C. Torrance, *Higher Mathematics with Applications to Science and Engineering* (New York: McGraw-Hill Book Company, Inc., 1939), Part A, Ch. vi; L. E. Dickson, *First Course in the Theory of Equations* (New York: John Wiley & Sons, 1922), Ch. viii; H. E. Hawkes, *Advanced Algebra* (Boston: Ginn & Company, 1905), Ch. xviii; H. E. Hawkes, *Higher Algebra* (Boston: Ginn & Company, 1913), Ch. viii; Maxime Bôcher, *Introduction to Higher Algebra* (New York: The Macmillan Company, 1929), Ch. ii; I. S. and E. S. Sokolnikoff, *Higher Mathematics for Engineers and Physicists* (New York: McGraw-Hill Book Company, Inc., 1934), Ch. iii. These books show briefly how a determinant arises from a group of algebraic equations and give numerous theorems concerning the properties and manipulation of determinants.

R. E. Doherty and E. G. Keller, *Mathematics of Modern Engineering* (New York: John Wiley & Sons, 1936). Volume I, Part II, Ch. ii, gives theorems pertaining to the manipulation of determinants and examples of the application of determinants to problems which arise in electrical engineering.

E. Pascal, *Die Determinanten* (Leipsig: B. G. Teubner, 1900). This is a complete advanced treatise on determinants and matrices.

Paul B. Fischer, *Determinanten* (Sammlung Göschen, ed.; Berlin: Walter de Gruyter & Company, 1928). This is a thorough treatise on determinants and includes a comprehensive bibliography.

Units, Dimensions, and Standards (Appendix C)

G. E. M. Jauncey and A. S. Langsdorf, *M.K.S. Units and Dimensions and a Proposed M.K.O.S. System*, (New York, The Macmillan Company, 1940). This booklet contains an elementary approach to the general problem of units and dimensions, and traces the development of the mks system of units to their present status.

William M. Hall, "The Formation of Systems of Units," *J.F.I.*, CCXXV (1938), 197–218. The paper presents a method of formulating consistent systems of units for all branches of physics and engineering.

Frederick E. Fowle (ed.), *Smithsonian Physical Tables* (8th ed. [revised]; Washington: The Smithsonian Institution, 1933). The introduction gives an excellent discussion of units, standards, and dimensions.

"Electric Units and Standards," *Circ. Nat. Bur. Stand.*, No. 60 (2d ed.; Washington: Government Printing Office, 1920). This circular gives an excellent review of the subject and contains an extensive bibliography.

"Systems of Electrical and Magnetic Units," *N.R.C.*, Bulletin No. 93 (Washington: National Academy of Science, 1933). This is a series of papers giving important contributions by Glazebrook, Abraham, Page, Campbell, Curtis, and Kennelly.

A. E. Kennelly, "I.E.C. Adopts MKS System of Units," *A.I.E.E. Trans.*, LIV (1935), 1373–1384. This is a historical sketch of the problems of establishing units and standards, including the recent adoption of the mks system for use beginning January 1, 1940. There is an extensive bibliography.

James Clerk Maxwell, *A Treatise on Electricity and Magnetism* (3d ed.; Oxford: The Clarendon Press, 1892), Vols. I and III. Volume I, Part I, Arts. 1–6; Vol. I, Part II, Ch. xi; and Vol. III, Part IV, Chs. x, xviii, and xix, deal with units, dimensions, and standards.

Harvey L. Curtis, Charles Moon, and C. Matilda Sparks, "A Determination of the Absolute Ohm, Using an Improved Self Inductor," *J. Res. Nat. Bur. Stand.*, XXI (1938), 375–423. This article describes the determination of resistance in terms of the inductance of a solenoid computed from its measured dimensions. The result gives the international ohm as 1.000479 absolute ohms.

John William Strutt Rayleigh, *Scientific Papers* (Cambridge: at the University Press, 1900), Vol. II. This collection contains many papers relating to the measurement of the absolute values of standards of resistance and electromotive force.

H. A Rowland, *Physical Papers* (Baltimore: The Johns Hopkins Press, 1902). Papers 15, 34, and 37 relate to the determination of the ohm; paper 44 relates to the determination of the ratio of the electrostatic to the electromagnetic unit of charge.

Index

Asterisks after folios indicate pictorial illustrations.

D

E